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Numerical Integration of Products of Fourier and Ordinary Polynomials*

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Abstract. Sets of coefficients for four finite difference methods of numerical integration are presented that will integrate without truncation error products of fourier and ordinary polynomials. These sets are formulated such that they are free from computational difficulties.

I. Introduction

In a previous paper [4] sets of modified integration coefficients for the Cowell method of numerical integration of order six were given which had the property that they would integrate without truncation error products of a fourier polynomial and an ordinary polynomial. These sets of coefficients were characterized by their explicit formulation. The purpose of this paper is the extension of the set of modified Cowell coefficients to any order of the integration method, as well as the development of similar sets of modified integration coefficients for the Störmer, Adams-Moulton, and the Adams-Bashforth methods of numerical integration. These modified coefficients will be given explicitly in a form such that they can be computed from a simple algorithm which does not suffer from numerical difficulties.

The four methods of numerical integration based upon backward differences will be referred to as follows [2]: 1) Cowell — the implicit method for differential equations of the second order in which the first derivatives are absent; 2) Störmer — the explicit method for these second order differential equations; 3) Adams-Moulton — the implicit method for differential equations of the first order; and 4) Adams-Bashforth — the explicit method for differential equations of the first order.

II. Integration Formulae

For the integration of a function $f(t)$, tabulated at equally spaced values of the independent variable t with the step-length h, any ascending diagonal of a difference table can be used for the development of a method of numerical integration [4]. In such a table, let m be an integer, $t_m = mh$, $f_m = f(t_m)$; and let $\Delta^0 f(m) = f_m$, $\Delta f(m-\frac{1}{2}) = f_m - f_{m-1}$, $\Delta^2 f(m) = f_{m+1} - 2f_m + f_{m-1}$, ect. For a differential equation of the form

$$\frac{d^2 x(t)}{dt} = f(x, t)$$

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the double integration may be obtained by

$$(1.1)\qquad x_1 - 2x_0 + x_{-1} = h^2 \sum_{k=0}^{\infty} \alpha_k \Delta^k f\left(p - \frac{k}{2}\right)$$

where $\alpha_0 = 1$, $p = 1$ corresponds to the classical Cowell formula, and $p = 0$ represents the Störmer formula.

For the integration of a differential equation of the first order,

$$\frac{dx(t)}{dt} = f(x, t)$$

a suitable integration formula is

$$(1.2)\qquad x_1 - x_0 = h \sum_{k=0}^{\infty} \alpha_k \Delta^k f\left(p - \frac{k}{2}\right).$$

When $p = 1$ the formula is the classical Adams-Moulton method, and when $p = 0$ it is the classical Adams-Bashforth method. Since the sets of α's are assumed to be constant in these classical methods, the solution of the differential equations is expressed as a linear combination of the differences of a particular diagonal of the difference table. The coefficients α are different for the four methods of integration. However, with this distinction well in mind, no ambiguity will result if the same symbol is used for all four of the integration methods.

In order to determine the four sets of coefficients α, assume that

$$(2)\qquad f(t) = Z^{t/h}$$

where Z represents any fixed complex number. Then it follows that for the double integration

$$\int_0^h \int_{\tau-h}^{\tau} Z^{\tau/h}\, d\tau\, dt = Z\left[\frac{h(1-Z^{-1})}{\log Z}\right]^2,$$

and for the single integration

$$\int_0^h Z^{t/h} = \frac{h}{\log Z}(1-Z^{-1})Z.$$

Letting

$$\zeta = 1 - Z^{-1}$$

the integration formulae (1) become

$$(3.1)\qquad \left[\frac{\zeta}{\log(1-\zeta)}\right]^2 = P(\zeta),$$

$$(3.2)\qquad \frac{1}{1-\zeta}\left[\frac{\zeta}{\log(1-\zeta)}\right]^2 = P(\zeta),$$

$$(3.3)\qquad \frac{\zeta}{\log(1-\zeta)} = P(\zeta),$$

$$(3.4)\qquad \frac{\zeta}{(1-\zeta)\log(1-\zeta)} = P(\zeta),$$

where $P(\zeta)$ is defined as

$$P(\zeta)=\sum_{k=0}^{\infty}\alpha_k\zeta^k, \tag{4}$$

and where the integers 1, 2, 3, 4 denote the Cowell, Störmer, Adams-Moulton, and the Adams-Bashforth methods, respectively.

Upon expanding the left-hand sides of (3) in powers of ζ, with $|\zeta|<1$, the coefficients of similar terms of ζ may be compared, yielding, for the first six coefficients the following sets [2]:

Cowell method —

$$\begin{aligned} &\alpha_1=-1, && \alpha_2=\frac{1}{12}, && \alpha_3=0,\\ &\alpha_4=-\frac{1}{240}, && \alpha_5=-\frac{1}{240}, && \alpha_6=\frac{-221}{60\,480}; \end{aligned} \tag{5.1}$$

Störmer method —

$$\begin{aligned} &\alpha_1=0, && \alpha_2=\frac{1}{12}, && \alpha_3=\frac{1}{12},\\ &\alpha_4=\frac{19}{240}, && \alpha_5=\frac{3}{40}, && \alpha_6=\frac{863}{12\,096}; \end{aligned} \tag{5.2}$$

Adams-Moulton method —

$$\begin{aligned} &\alpha_1=-\frac{1}{2}, && \alpha_2=-\frac{1}{12}, && \alpha_3=-\frac{1}{24},\\ &\alpha_4=\frac{-19}{720}, && \alpha_5=\frac{-27}{1440}, && \alpha_6=\frac{-863}{60\,480}; \end{aligned} \tag{5.3}$$

Adams-Bashforth method —

$$\begin{aligned} &\alpha_1=\frac{1}{2}, && \alpha_2=\frac{5}{12}, && \alpha_3=\frac{9}{24},\\ &\alpha_4=\frac{251}{720}, && \alpha_5=\frac{475}{1440}, && \alpha_6=\frac{19\,087}{60\,480}. \end{aligned} \tag{5.4}$$

The order, n, of the integration method is defined by the subscript of the highest coefficient α_n retained in (1). It is important to remember that the classical sets of integration coefficients are based upon the expansion of the left-hand sides of (3) in an infinite power series in terms of the variable ζ. If, in application, (1) are used with the order chosen such that either the highest coefficients are zero, or the difference Δ^n of the function being integrated vanishes identically, then (1) will integrate the function exactly. However, in practice these two conditions are usually not satisfied. In effect, the integration of the function is being approximated by the truncated polynomial of the right-hand side of (3),

$$P_n(\zeta)=\sum_{k=0}^{n}\alpha_k\zeta^k. \tag{6}$$

Therefore, the integration of a function by the use of (1) with a finite set of coefficients will result in an error because the formulae are based upon an approximating polynomial instead of an exact polynomial.

While formulating a method to reduce these truncation errors for the four classical methods of numerical integration when they are applied to functions which are products of fourier polynomials and ordinary polynomials, the basic formulae (1) will not be altered, but the coefficients α will be modified.

III. Formulation of the Problem

Before deriving the modified coefficients which will integrate without truncation error the products of fourier and ordinary polynomials, the modified coefficients will be obtained which will integrate only fourier polynomials exactly. The final sets of coefficients which will integrate the products of the fourier and the ordinary polynomials will be obtained as limiting cases of these first sets of coefficients.

Much insight into the general problem of deriving the modified coefficients results from the determination of the coefficients which will integrate the functions $\sin\omega t$ and $\cos\omega t$. Thus, for (2) let $f(t) = \exp(i\omega t)$, and introducing the notation $2\sigma = \omega h$, Z in (2) becomes

$$Z = \exp(2i\sigma),$$

and ζ becomes

$$\zeta = 1 - \exp(-2i\sigma).$$

Therefore, Eqs. (3) reduce to:

$$L_i(\sigma) = P_n(\zeta), \quad i = 1, 2, 3, 4, \tag{7}$$

where

$$L_1(\sigma) = \left(\frac{\sin\sigma}{\sigma}\right)^2 \exp(-2i\sigma),$$

$$L_2(\sigma) = \left(\frac{\sin\sigma}{\sigma}\right)^2,$$

$$L_3(\sigma) = \frac{\sin\sigma}{\sigma}\exp(-i\sigma),$$

$$L_4(\sigma) = \frac{\sin\sigma}{\sigma}\exp(i\sigma).$$

The coefficients α must be chosen such that (7) are satisfied for the given frequency ω, as well as for $-\omega$, if the four integration methods of order n are to integrate the functions $f(t) = \sin\omega t$ and $f(t) = \cos\omega t$ exactly.

For the integration methods of order two the coefficients α_1 and α_2 must be chosen such that the $L_i(\sigma)$ are identical to

$$1 + \alpha_1\zeta + \alpha_2\zeta^2,$$

or

$$1 + \alpha_1[1 - \exp(-2i\sigma)] + \alpha_2[1 - \exp(-2i\sigma)]^2.$$

For the methods of order two, the modified coefficients can be determined by solving the two equations which result from (7) when the two frequencies ω and $-\omega$ are considered. If σ is not zero, or if, for the two Adams methods, σ is not an odd integer multiple of $\pi/2$, then the solution of the two equations yields for the

Cowell method —

(8.1) $$\alpha_1 = -1, \qquad \alpha_2 = \lambda(u);$$

Störmer method —

(8.2) $$\alpha_1 = \lambda^*(u) - 1, \qquad \alpha_2 = \lambda(u);$$

Adams-Moulton method —

(8.3) $$\alpha_1 = \mu^*(u) - 1, \qquad \alpha_2 = \mu(u);$$

Adams-Bashforth method —

(8.4) $$\alpha_1 = (3 - u)\,\mu^* - 1, \quad \alpha_2 = \mu(u) + \mu^*(u);$$

where

$$\sigma = \frac{\omega h}{2}, \qquad u = 4\sin^2\sigma,$$

and where

$$\lambda^*(u) = \left(\frac{\sin\sigma}{\sigma}\right)^2, \qquad \mu^*(u) = \frac{\sin\sigma}{2\sigma\cos\sigma},$$

$$\lambda(u) = \frac{1}{4}\left(\frac{1}{\sin^2\sigma} - \frac{1}{\sigma^2}\right),$$

$$\mu(u) = \frac{1}{4}\left(\frac{1}{\sin^2\sigma} - \frac{1}{\sigma\sin\sigma\cos\sigma}\right).$$

With these choices of α_1 and α_2 the four integration methods of second order will integrate the functions $\sin\omega t$ and $\cos\omega t$ exactly, except for any round-off error which accumulates in the algorithm.

Henceforth, the coefficients which have been adjusted in the above manner, will be referred to as modified coefficients, and the coefficients of (5) will be referred to as the classical coefficients. For the modified coefficients two subscripts, i and n, will denote the i-th coefficient of the set and the order of the method, respectively. As σ becomes very small, the above coefficients approach the corresponding classical coefficients.

If the order of the integration method is greater than two, then the $\alpha_{i,n}$ $(i = 1, 2, \ldots, n)$ coefficients can be determined such that the functions $\cos\omega t$ and $\sin\omega t$ are integrated exactly. It will be assumed that n is even and equal to 2ν. Thus, there will exist ν distinct frequencies $\omega_1, \omega_2, \ldots, \omega_\nu$ such that the 2ν coefficients will integrate the functions $\cos\omega_k t$ and $\sin\omega_k t$ $(k = 1, 2, \ldots, \nu)$ exactly, if (7) are satisfied for the corresponding σ_k and for $-\sigma_k$. Since the right-hand sides of (7) are polynomials in the variable ζ, the above condition is equivalent to the following problem of polynomial interpolation:

Given $2\nu + 1$ points ζ_k, $k = -\nu, \ldots, -1, 0, +1, \ldots, +\nu$, where $\zeta_k = 1 - \exp(-2i\sigma_k)$, and where the points are on a circle which has a unit radius in the complex ζ-plane, construct a polynomial $P_n(\zeta)$, $n = 2\nu$, such that $L_i(\sigma_k) = P_n(\zeta)$, $i = 1, 2, 3, 4$, and where $\sigma_{-k} = -\sigma_k$.

Once the coefficients α are determined such that the above conditions are satisfied, the integration methods will integrate without truncation error not only the functions $\sin \omega_k t$ and $\cos \omega_k t$, but also the linear combination

$$a_0 + \sum_{k=1}^{\nu} (a_k \cos \omega_k t + b_k \sin \omega_k t). \tag{9}$$

The constant a_0 is integrated exactly because the polynomial $P_n(\zeta)$ begins with unity.

IV. A Special Case

When only two of the n coefficients are modified, the remainder of the coefficients retaining their classical values, the resulting set of coefficients will integrate exactly the function

$$Q_{n-2}(t) + a \cos \omega t + b \sin \omega t, \tag{10}$$

where $Q_m(t)$ is an ordinary polynomial of degree m in the variable t. In this special case there are given two points $\zeta_k = 1 - \exp(-2i\sigma_k)$, $k = -1, +1$, on a unit circle in the complex ζ-plane passing through the orgin, where the remaining $n-2$ points are located. The problem is the construction of a polynomial $P_n(\zeta)$ such that the relations (7) are satisfied for the two values of k. This problem can be solved by the technique which was used for the modification of the coefficients for the integration methods of order two. Thus, the problem reduces to the solution of the two equations ($k = -1, +1$):

$$L_i(\sigma_k) = P_n(\zeta_k), \qquad i = 1, 2, 3, 4, \tag{11}$$

for two of the coefficients, for example $\alpha_{n,n}$ and $\alpha_{n-1,n}$.

Before solving for the two unknown coefficients it is advantageous to introduce two sequences of polynomials $R_m(u)$ and $S_m(u)$ defined by the recurrence relations [4]:

$$\begin{aligned} R_{m+1} &= u(R_m - R_{m-1}), \qquad R_0 = 2, \qquad R_1 = u, \\ S_{m+1} &= u(S_m - S_{m-1}), \qquad S_0 = 0, \qquad S_1 = u. \end{aligned} \tag{12}$$

For $m = 0, 1, 2, \ldots$, the following relations can be proved by induction:

$$\begin{aligned} \sin 2m\sigma &= \frac{(-1)^{m-1} \sin 2\sigma\, S_{2m}}{u^{m+1}}, \\ \sin(2m+1)\sigma &= \frac{(-1)^m \sin\sigma\, R_{2m+1}}{u^{m+1}}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} S_{2m} &= (-1)^{m-1} u^{m+1} (m + Q^*_{m-1}), \\ R_{2m+1} &= (-1)^m u^{m+1} (2m + 1 + Q^*_m), \end{aligned} \tag{14}$$

where $Q^*_p(u)$ is a polynomial with constant coefficients of degree p in the variable u with the term of degree zero absent. Also the following preliminary relations will be needed:

$$\begin{aligned} \zeta_k^m - \zeta_{-k}^m &= \frac{\tau S_m(u_k)}{u_k}, \\ \zeta_k^m + \zeta_{-k}^m &= R_m(u_k), \end{aligned} \tag{15}$$

where $\tau = 2i \sin 2\sigma$.

Solving (11) for $\alpha_{n,n}$ and $\alpha_{n-1,n}$ gives

$$\alpha_{p,n} = \frac{1}{u^{n-1}} \left[\frac{L_{i,1}\zeta_{-1}^{q} - L_{i,-1}\zeta_{1}^{q}}{\tau} + \sum_{k=0}^{n-2} \alpha_{k,n}\, u^{k-1} \left(\frac{\zeta_{1}^{q-k} - \zeta_{-1}^{q-k}}{\tau} \right) \right],$$

where $\alpha_{0,n} = 1$, $q = n-1$ when $p = n$, $q = n$ when $p = n-1$, and $L_{i,k} = L_i(\sigma_k)$.

Using (15), the definition of ζ_k, and the relation,

$$\exp(i\sigma_k) = (\zeta_k)^{-1}\, 2i \sin \sigma_k,$$

and defining the first term in the bracket as $F^*_{p,i}(u)$, then

$$\begin{aligned} F^*_{p,1} &= \lambda^*(u)\, S_{q-2}, \\ F^*_{p,2} &= -\lambda^*(u)\, S_q/u, \\ F^*_{p,3} &= -\mu^*(u)\, R_{q-1}, \\ F^*_{p,4} &= \mu^*(u)\, R_{q+1}/u. \end{aligned}$$

Therefore, if the classical coefficients are used for $\alpha_{l,n}$, $l = 1, 2, \ldots, n-2$, then the last two coefficients are

$$\alpha_{p,n} = \frac{(-1)^{n-p}}{u^{n-1}} \left[F^*_{p,i}(u) + \sum_{k=0}^{n-2} \alpha_{k,n}\, u^{k-1} S_{q-k} \right], \tag{16}$$
$$p = n,\ q = n-1;\ p = n-1,\ q = n.$$

With these coefficients the integration methods will integrate the function (10) exactly.

V. Modified Coefficients for Distinct Frequencies

To integrate the function (9) without any truncation error requires the solution of the problem of interpolation given in Section III. Any of the classical interpolation formulae are available for this purpose. Here the technique will be to obtain the two highest coefficients by an application of the Lagrangian interpolation method:

$$L_i(\sigma) = \sum_{k=-\nu}^{\nu} L_{i,k} \prod_{\substack{m=-\nu \\ m \neq k}}^{\nu} \frac{\zeta - \zeta_m}{\zeta_k - \zeta_m}, \qquad n = 2\nu. \tag{17}$$

The lower coefficients could be obtained by the same technique, but this results in a rather tedious procedure that involves computation in the complex variable ζ. Therefore, a more efficient algorithm in the real field is developed in which the lower coefficients are determined by recurrence with respect to the coefficients of lower order. The final goal is the determination of the two highest coefficients in explicit form for any order. Thus, after the remaining coefficients are found by the recurrence method, the complete set of modified coefficients for the four methods will be readily available.

The derivation of the expression for the highest coefficient for the four methods of any given even order will be given. The modified coefficient $\alpha_{n,n}$

equals the coefficient of the ζ^n term of the Lagrangian interpolation formula:

$$L_{i,0}\prod_{k=1}^{\nu}\frac{1}{u_k}+\sum_{k=1}^{\nu}\frac{L_{i,k}\,\zeta_{-k}(\zeta_{-k}-1)^{\nu-1}-L_{i,-k}\,\zeta_k(\zeta_k-1)^{\nu-1}}{(\zeta_k-\zeta_{-k})\,u_k}\,\Pi^*, \tag{18}$$

where

$$\Pi^*=\prod_{\substack{m=1\\ m\neq k}}^{\nu}\frac{1}{u_k-u_m}.$$

The expression (18) can be expressed in terms of u, $\lambda(u)$, and $\mu(u)$ for the four integration methods.

Since, for the Cowell method,

$$L_{1,k}=\frac{u_k}{4\sigma_k^2}(1-\zeta_k),$$

the right-hand side of (18) becomes:

$$\sum_{k=1}^{\nu}\frac{(1-\zeta_k)\,\zeta_{-k}(\zeta_{-k}-1)^{\nu-1}-(1-\zeta_{-k})\,\zeta_k(\zeta_k-1)^{\nu-1}}{4\sigma_k^2(\zeta_k-\zeta_{-k})}\,\Pi^*. \tag{19}$$

Since

$$1-\zeta_k=-(\zeta_{-k}-1)^{-1},$$

the numerator of (19) becomes

$$-\zeta_{-k}(\zeta_{-k}-1)^{\nu-2}+\zeta_k(\zeta_k-1)^{\nu-2}$$

or, because $\zeta_k=1-\exp(2i\,\sigma_k)$, the numerator reduces to

$$(-1)^{\nu-1}[2i\sin 2(\nu-2)\,\sigma_k]+(-1)^{\nu}[2i\sin 2(\nu-1)\,\sigma_k].$$

Using the definitions (13), the above expression simplifies to

$$\tau\left[\frac{S_{2(\nu-2)}}{u^{\nu-1}}+\frac{S_{2(\nu-1)}}{u^{\nu}}\right],$$

and, since the denominator is equal to $4\sigma_k^2\tau$, (18) becomes

$$\sum_{k=1}^{\nu}\left[\frac{S_{2(\nu-2)}}{u_k^{\nu-1}}+\frac{S_{2(\nu-1)}}{u_k^{\nu}}\right]\frac{\Pi^*}{4\sigma_k^2}. \tag{20}$$

With the aid of the first formula of (14), the expression in brackets can be shown to be of the form

$$(-1)^{\nu}[1+Q_{\nu-2}^*(u_k)],$$

where

$$Q_m^*(u)=q_1u+q_2u^2+\cdots+q_mu^m.$$

Therefore, by using the finite expansion of $1/(u_1u_2\ldots u_\nu)$ from Section VII, (19) becomes of the form

$$(-1)^{\nu-1}\sum_{k=1}^{\nu}\left[\frac{1+c_1u_k+\cdots+c_{\nu-2}u_k^{\nu-2}}{u_k}-\frac{1+q_1u_k+\cdots+q_{\nu-2}u_k^{\nu-2}}{4\sigma_k^2}\right]\Pi^*. \tag{21}$$

Since the constants c are arbitrary, they can be chosen such that the numerators of the two terms in the brackets of (21) are identical, i.e., the coefficients of the polynomial

$$1 + c_1 u_k + \cdots + c_{\nu-2} u_k^{\nu-2}$$

are selected so that the polynomial equals

$$\frac{S_{2(\nu-2)}}{u_k^{\nu-1}} + \frac{S_{2(\nu-1)}}{u_k^{\nu}} = \frac{S_{2\nu-3}}{u_k^{\nu-1}}.$$

Thus, (21) can be expressed as

$$-\sum_{k=1}^{\nu} \frac{S_{2\nu-3}(u_k)}{u_k^{\nu-1}} \lambda(u_k) \Pi^*,$$

which is the $\alpha_{n,n}$ coefficient for the case of distinct frequencies for the Cowell method.

Defining $F_{n,1}(u)$ as

$$F_{n,1}(u) = -\frac{S_{2\nu-3}(u)}{u^{\nu-1}} \lambda(u), \tag{22.1}$$

the $\alpha_{n,n}$ coefficient for the Cowell method becomes the $(\nu-1)$-th divided difference of the function $F_{n,1}(u)$ at the nodes $u_1, u_2, \ldots, u_\nu$. In a similar manner it follows that the $\alpha_{n,n}$ coefficient for the Störmer $(i=2)$ and the Adams-Moulton $(i=3)$ methods is the $(\nu-1)$-th divided difference of the function $F_{n,i}(u)$, where

$$F_{n,2}(u) = \frac{S_{2\nu-1}(u)}{u^{\nu}} \lambda(u), \tag{22.2}$$

$$F_{n,3}(u) = \frac{R_{2\nu-2}(u)}{2u^{\nu-1}} \mu(u). \tag{22.3}$$

Likewise, for the Adams-Bashforth method $(\alpha_{n,n} - \frac{1}{2})$ is the $(\nu-1)$-th divided difference of $F_{n,4}(u)$, where

$$F_{n,4}(u) = -\frac{R_{2\nu}(u)}{2u^{\nu}} \mu(u). \tag{22.4}$$

The factor one-half appears in this method because a term of degree $(\nu-1)$ is necessary in the finite expansion of the first term of (18).

The second highest coefficient equals the coefficient of the ζ^{n-1} term of the Lagrangian interpolation formula:

$$\Sigma^* - L_{i,0} \prod_{k=1}^{\nu} \frac{1}{u_k} + \sum_{k=1}^{\nu} \frac{L_{i,k}\zeta_{-k}(\zeta_{-k}-1)^{\nu-2} - L_{i,-k}\zeta_k(\zeta_k-1)^{\nu-2}}{(\zeta_k-\zeta_{-k})\,u_k} \Pi^*, \qquad i = 1, 2, 3, 4, \tag{23}$$

where

$$\Sigma^* = \left(1 - \sum_{k=1}^{\nu} u_k\right) \alpha_{n,n}.$$

With the aid of (12), (13), and (14), and with a development analogous to that for the highest coefficient, $(\alpha_{n-1,n} - \Sigma^*)$ becomes the $(\nu-1)$-th divided difference of the function $F_{n-1,i}(u)$, $i=1, 2, 3, 4$, where

$$
(24)\qquad
\begin{aligned}
F_{n-1,1}(u) &= -\frac{S_{2\nu-5}(u)}{u^{\nu-2}}\,\lambda(u),\\
F_{n-1,2}(u) &= \frac{S_{2\nu-3}(u)}{u^{\nu-1}}\,\lambda(u),\\
F_{n-1,3}(u) &= \frac{R_{2\nu-4}(u)}{2u^{\nu-2}}\,\mu(u),\\
F_{n-1,4}(u) &= -\frac{R_{2\nu-2}(u)}{2u^{\nu-1}}\,\mu(u).
\end{aligned}
$$

In order to obtain the remaining coefficients $\alpha_{m,n}$, $m=1, 2, \ldots, n-2$, assume that the set of coefficients of order $n-2$ are known and that the two highest coefficients have been computed. Consider two polynomials $P_n(\zeta)$ and $P_{n-2}(\zeta)$ interpolated at the nodes

$$
(25)\qquad \zeta_0, \zeta_{\pm 1}, \ldots, \zeta_{\pm\nu},
$$

and

$$
(26)\qquad \zeta_0, \zeta_{\pm 1}, \ldots, \zeta_{\pm l}, \qquad l=\nu-1,
$$

respectively. The polynomial having the points (26) as zeros is

$$
\zeta(\zeta^2-u_1\zeta+u_1)(\zeta^2-u_2\zeta+u_2)\ldots(\zeta^2-u_l\zeta+u_l).
$$

This polynomial which vanishes at the $n-2$ zeros, multiplied by a linear factor, is the difference of the two polynomials $P_n(\zeta)$ and $P_{n-2}(\zeta)$,

$$
(27)\quad P_n(\zeta)-P_{n-2}(\zeta)=\zeta(\zeta^2-u_1\zeta+u_1)(\zeta^2-u_2\zeta+u_2)\ldots(\zeta^2-u_l\zeta+u_l)(r\zeta-s).
$$

The parameters r and s can be obtained as a function of the two highest coefficients by first expressing the left-hand sides of (27) in terms of the modified coefficients of order n and $n-2$, respectively, and by then comparing the coefficients of ζ^n and ζ^{n-1}, yielding

$$
r=\alpha_{n,n}, \qquad s=\alpha_{n-1,n}+\alpha_{n,n}\sum_{k=1}^{l}u_k.
$$

With these values for r and s, (27) becomes

$$
(28)\qquad P_n(\zeta)=P_{n-2}(\zeta)+\left[\alpha_{n,n}\zeta+\alpha_{n-1,n}+\alpha_{n,n}\sum_{k=1}^{l}u_k\right]\prod_{k=1}^{l}(\zeta^3-u_k\zeta^2+u_k\zeta).
$$

The lower coefficients can now be obtained by comparing the coefficients of similar terms of ζ in (28). This technique is easily adaptable to a computer subroutine.

VI. Modified Coefficients for Confluent Frequencies

The modified coefficients previously developed will integrate without truncation error the function (9). However, if two or more of the frequencies ω_k approach a common limit, for example if $\omega_1 \to \omega_2 \to \cdots \to \omega_m$, the modified coefficients for this case of confluent frequencies will integrate exactly the function

$$a_0 + Q_{m-1}(t)\left[\sum_{k=m+1}^{\nu} (a_k \cos \omega_k t + b_k \sin \omega_k t)\right], \tag{29}$$

i.e., the product of an ordinary polynomial and a fourier polynomial.

In order to obtain the sets of modified coefficients which will integrate the above function, the limits must be obtained of the modified coefficients for distinct frequencies as the m frequencies approach the value of confluency. Since the modified coefficients for distinct frequencies are the divided differences of the function $F_{p,i}(u)$, $p=n, n-1$, and $i=1, 2, 3, 4$, at the nodes $u_1, u_2, \ldots, u_\nu$, the desired limits are available from the well established theory of divided differences [3].

Sets of these modified coefficients will be given without proof for three of the more important limiting cases: a) two, b) $\nu-1$, and c) all of the ν frequencies approach a common limit. The first case is applicable when the two highest coefficients for distinct frequencies suffer from a loss of significant digits because the differences of u_1 and u_2 in their denominators become small. As the confluent frequencies approach zero in the second case, the coefficients approach those of the special case of Section IV. The third case is characterized by the lack of the troublesome factors of u in the denominators of the two highest coefficients. Only the two highest coefficients will be given since the lower coefficients may be computed by the recurrence technique of Section V.

In the first case assume that ω_1 and ω_2 approach the common limit ω. If $G_n(u)$ is defined as

$$G_n(u) = \prod_{k=3}^{\nu} (u - u_k), \qquad n \geqq 6,$$

then

$$\alpha_{p,n} = K_{p,i} + D\left[\frac{F_{p,i}(u)}{G_n(u)}\right] + \sum_{k=3}^{\nu} F_{p,i}(u_k)\Pi^*, \qquad p=n, n-1, \tag{30}$$

with the definitions

$$K_{n,i} = 0 \quad \text{if} \quad i=1, 2, 3, \qquad K_{n,4} = \tfrac{1}{2},$$

and

$$K_{n-1,i} = \Sigma^* \quad \text{for} \quad i=1, 2, 3, 4,$$

and where $D = d/du$. For the second case let the first $(\nu-1)$ frequencies approach the common limit ω, then

$$\alpha_{p,n} = K_{p,i} + \frac{(-1)^\nu}{(u-u_\nu)^{\nu-1}} \sum_{k=2}^{\nu-2} \frac{(u_\nu - u)^k D^k F_{p,i}(u)}{k!} + \frac{F_{p,i}(u_\nu)}{(u_\nu - u)^{\nu-1}}, \quad p=n, n-1, \tag{31}$$

where the operator D is defined as

$$D^0 = 1; \qquad D^k = \frac{d^k}{du^k}, \qquad k=1, 2, \ldots.$$

For the third case all the frequencies approach the limit ω. Here the highest coefficients become

$$\alpha_{p,n} = K_{p,i} + \frac{D^{\nu-1} F_{p,i}(u)}{(\nu-1)!}, \qquad p=n, n-1. \tag{32}$$

VII. A Finite Expansion

The development of the two highest coefficients for the case of distinct frequencies requires an expansion of the function $1/(u_1 u_2 \ldots u_\nu)$.

The $(\nu-1)$-th divided difference of the function $g(u)=1/u$ is [3]:

$$[u_1 u_2 \ldots u_\nu] = \frac{(-1)^{\nu-1}}{\prod_{k=1}^{\nu} u_k}, \tag{33}$$

where the $(\nu-1)$-th divided difference of a function $g(u)$ is defined by

$$[u_1 u_2 \ldots u_\nu] = \sum_{k=1}^{\nu} g(u_k)\, \Pi^*, \qquad \Pi^* = \prod_{\substack{m=1\\ m\neq k}}^{\nu} \frac{1}{(u_k - u_m)}. \tag{34}$$

Combining (33) and (34) gives

$$\frac{1}{\prod_{k=1}^{\nu} u_k} = (-1)^{\nu-1} \sum_{k=1}^{\nu} \frac{1}{u_k} \Pi^*. \tag{35}$$

Since the $(\nu-1)$-th divided difference of a polynomial of degree $(\nu-2)$ is zero, then

$$\sum_{k=1}^{\nu} \frac{c_1 u_k + c_2 u_k^2 + \cdots + c_{\nu-1} u_k^{\nu-1}}{u_k} \Pi^* = 0. \tag{36}$$

Furthermore, the $(\nu-1)$-th divided difference of a polynomial of degree $\nu-1$ equals the coefficient of the term of the polynomial of degree $\nu-1$, or

$$\sum_{k=1}^{\nu} \frac{c_\nu u_k^\nu}{u_k} \Pi^* = c_\nu. \tag{37}$$

Adding (35), (36), and (37) yields the desired finite expansion of the function $1/(u_1 u_2 \ldots u_\nu)$:

$$\frac{1}{\prod_{k=1}^{\nu} u_k} = -c_\nu + \sum_{k=1}^{\nu} \frac{(-1)^{\nu-1}(1 + c_1 u_k + \cdots + c_{\nu-1} u_k^{\nu-1}) + c_\nu u_k^\nu}{u_k} \Pi^*, \tag{38}$$

where the constants c are arbitrary.

VIII. Some Power Expansions

During computation the functions $\lambda^*(u)$, $\lambda(u)$, $\mu^*(u)$, and $\mu(u)$, and their derivatives will suffer from a loss of significant digits if the variable σ becomes small. This loss of precision may be avoided by expanding the functions in a power series in terms of the variable u.

The expansions of $\lambda^*(u)$ and $\lambda(u)$ are presented in [4]:

$$\lambda^*(u) = 1 - \frac{1}{12} u - \frac{1}{240} u^2 - \frac{31}{60\,480} u^3 - \cdots,$$

$$\lambda(u) = \frac{1}{12} + \frac{1}{240} u + \frac{31}{60\,480} u^2 + \frac{289}{3\,628\,800} u^3 + \cdots.$$

In order to obtain similar expansions for $\mu^*(u)$ and $\mu(u)$ as a function of the variable u consider the following integration formula [1]:

$$\begin{aligned}\Delta x\left(\tfrac{1}{2}\right)=\int_0^h f(t)\,dt=h\Big[1+\tfrac{1}{2}\Delta f\left(\tfrac{1}{2}\right)-\tfrac{1}{12}\Delta^2 f(0)-\tfrac{1}{24}\Delta^3 f\left(\tfrac{1}{2}\right)\\ +\tfrac{11}{120}\Delta^4 f(0)+\tfrac{11}{1440}\Delta^5 f\left(\tfrac{1}{2}\right)-\tfrac{191}{60\,480}\Delta^6 f(0)-\tfrac{191}{120\,960}\Delta^7 f\left(\tfrac{1}{2}\right)+\cdots\Big].\end{aligned}\tag{39}$$

The function to which this formula will be applied is $f(t)=\cos\omega t$. Since $t_m=mh$, and since $2\sigma=\omega h$, it follows that $f_m=\cos 2m\sigma$. Also, since

$$\cos 2(m+1)\sigma-\cos 2m\sigma=-2\sin(2m+1)\sigma\sin\sigma,$$

then

$$\Delta f(m)=-2\sin 2\sigma\sin 2m\sigma-\frac{u}{2}\cos 2m\sigma.$$

Furthermore,

$$\Delta^{2k-1} f(m)=(-1)^k\left[u^{k-1}\sin 2\sigma\sin 2m\sigma+\frac{u^k}{2}\cos 2m\sigma\right],$$

or, the odd first differences of $\cos\omega t$ are

$$\Delta^{2k-1} f(0)=\frac{1}{2}(-u)^k.$$

Likewise, since

$$\cos 2(m+1)\sigma+\cos 2(m-1)\sigma=2\cos 2\sigma\cos 2m\sigma,$$

then

$$\Delta^2 f(m)=-u\cos 2m\sigma,$$

and, in general,

$$\Delta^{2k} f(m)=(-u)^k\cos 2m\sigma.$$

Thus, the even central differences of $\cos\omega t$ are

$$\Delta^{2k} f(0)=(-u)^k.$$

From the integral

$$\int_0^h\cos\omega t\,dt=\frac{1}{\omega}\sin\omega h,$$

it follows that

$$\Delta x\left(\frac{1}{2}\right)=\frac{\sin\omega h}{\omega}=\frac{h\sin\sigma\cos\sigma}{\sigma}.$$

With these result (39) becomes

$$\frac{\sin\sigma\cos\sigma}{\sigma}=1-\frac{1}{6}u-\frac{1}{180}u^2-\frac{1}{1512}u^3-\frac{23}{226\,800}u^4-\cdots.$$

Using this relation the expansions of $\mu^*(u)$ and $\mu(u)$ are:

$$\mu^*(u)=\frac{1}{2}+\frac{1}{24}u+\frac{11}{1440}u^2+\frac{191}{120\,960}u^3+\cdots,$$

$$\mu(u)=-\frac{1}{12}-\frac{11}{720}u-\frac{191}{60\,480}u^2-\frac{2497}{3\,628\,800}u^3-\cdots.$$

IX. Elimination of Numerical Difficulties

The two highest coefficients suffer from a loss of significant digits during their computation when the differences of $u_1, u_2, \ldots, u_m$, $m \geqq 2$, in the denominators become small. This occurs when m of the frequencies approach a common limit. This difficulty may be avoided by using the set of coefficients for $l+m$ confluent frequencies, where l denotes the number of confluent frequencies of the original set of coefficients. For example, if this situation appears because ω_1 and ω_2 approach a common value when the modified coefficients for distinct frequencies are computed, the set of coefficients (30) should be used.

When the variable σ becomes small the functions $\lambda^*(u)$, $\lambda(u)$, $\mu^*(u)$, $\mu(u)$ and their derivatives become indeterminant. If this occurs the power expansions from Section VIII will eliminate the difficulty.

If $\sigma=(2m+1)\frac{\pi}{2}$, $m=0, 1, 2, \ldots$, the functions $\mu^*(u)$ and $\mu(u)$ and their derivatives, and the derivatives of the functions $\lambda^*(u)$ and $\lambda(u)$ become infinite. For a given problem the value of ω is specified, but the value of h can be varied. Therefore, since $2\sigma=\omega h$, a suitable choice of the step-length will avoid the problem of σ being an odd integer multiple of $\pi/2$.

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