Numerische Mathematik 9, 236—249 (1966)

A Second Order Finite Difference Analog of the First Biharmonic Boundary Value Problem*

J. H. BRAMBLE

Received December 10, 1965

1. Introduction

In recent years there has been much work on difference methods for approximating solutions to boundary value problems for elliptic partial differential equations. Most of this work has centered on second order equations and strong use has been made of an analogous maximum principle for the difference equations in order to obtain asymptotic estimates for the error (c.f. [1—8]).

As early as 1928 COURANT, FRIEDRICHS and LEWY [9] posed a difference analog for the first boundary value problem for the biharmonic equation and proved that the approximate solutions converge to the exact solution as the mesh is refined. They gave, however, no estimates for the error. Recently, THOMÉE [12] treated the Dirichlet problem for a class of elliptic equations of order $2m$ with constant coefficients. Among the problems treated by THOMÉE was that of [9]. THOMÉE, however, proved that in a certain norm the error is $O(h^{\frac{1}{2}})$ where h is the mesh size.

Still more recently ZLÁMAL [13] posed a different difference analog for a fourth order elliptic operator with variable coefficients which includes the biharmonic operator in two dimensions. He proved that for his problem the error is $O(h^{\frac{3}{8}})$.

This paper is concerned with the first boundary value problem for the biharmonic operator in the plane:

$$(1.1)\qquad \begin{aligned} \Delta^2 u &= F \quad \text{in } R \\ u = \frac{\partial u}{\partial n} &= 0 \quad \text{on } \dot{R} \end{aligned}$$

where Δ is the Laplace operator, R is a bounded region with boundary $\dot{R}$, F is a given function in R and $\partial u/\partial n$ is the outward normal derivative on $\dot{R}$. The boundary conditions are taken to be homogeneous for convenience, this restriction being removed in the appendix.

Section 2 simply gives some notation and definitions needed for the later sections.

In Section 3 some basic lemmas are proved which provide some a priori estimates needed later. The third of these lemmas, Lemma 3.3, is proved by using difference inequalities closely related to inequalities used by MIRANDA [11] in proving his biharmonic maximum principle in the plane. This inequality did not, however, lead to a discrete analog of MIRANDA's maximum principle but

* This research was supported in part by the National Science Foundation under Grant NSF GP-3666.

was used in conjunction with a method related to one of FICHERA, [10], in order to deduce inequalities for certain discrete L_2 norms.

Section 4 makes use of the lemmas of the previous section to show that in the case of the biharmonic operator in the plane the results of THOMÉE [12] and ZLÁMAL [13] can be improved.

In the final section a difference analog of problem (1.1) is constructed for regions with boundaries of arbitrary shape (only piecewise smooth). It is shown that the error is $O(h^2)$ in certain norms and $O(h^2|\log 1/h|^{\frac{1}{2}})$ in maximum norm. Strong use is made of the lemmas of Section 3. Furthermore, the matrix of the resulting linear system is symmetric and positive definite and a general formula is given for the construction of the modified matrix near the boundary. It should be pointed out that, while in many second order problems difference methods were often formulated and later estimates given, this represents the first time an $O(h^2)$ method has even been formulated for the first boundary problem for any higher order elliptic partial differential equation in a general domain.

Although the results of this paper are special in that only the biharmonic operator in two dimensions is treated it is hoped that some of the ideas will lead to similar results for more general elliptic operators in regions of general shape.

2. Notation and Definitions

Let R be a bounded, open, connected set in the (x, y) plane. We denote by $\dot{R}$ the boundary of R and $\bar{R} = R \cup \dot{R}$.

We shall be concerned with difference approximations of problem (1.1). In order to study such problems we cover the (x, y) plane with a square mesh of width h, whose lines are parallel to the x and y axes. The intersection of these lines will be called mesh (or grid) points and the set of all such mesh points will be denoted by S_h.

For any function $V(x, y)$ defined at the mesh points we define in the usual way the following difference operators:

$$V_x(x,y) = \frac{1}{h}[V(x+h,y) - V(x,y)] \tag{2.1}$$
$$V_{\bar{x}}(x,y) = \frac{1}{h}[V(x,y) - V(x-h,y)]$$

and analogously for y. Further

$$\Delta_h V(x,y) = V_{x\bar{x}}(x,y) + V_{y\bar{y}}(x,y) \tag{2.2}$$

where

$$V_{x\bar{x}}(x,y) = (V_x)_{\bar{x}}(x,y).$$

The analog of the biharmonic operator is the usual 13 point operator

$$\Delta_h^2 V(P) = \Delta_h(\Delta_h V(P)) \tag{2.3}$$

with $P = (x, y)$.

We also need to define certain norms. Let Q_h be an arbitrary bounded subset of S_h defined for each h and let V be any mesh function such that $V(P) = 0$,

$P \notin Q_h$. Then we define

$$\|V\| = \Big(h^2 \sum_{P \in S_h} V^2(P)\Big)^{\frac{1}{2}} \tag{2.4}$$

and for any integer p

$$\|V\|_{(2p)} = \Big(h^2 \sum_{P \in S_h} V^{2p}(p)\Big)^{1/2p}. \tag{2.5}$$

Now if $V(P)$ is defined on Q_h (otherwise arbitrary) we define

$$\|V\|_{Q_h} = \Big(h^N \sum_{P \in Q_h} V^2(P)\Big)^{\frac{1}{2}} \tag{2.6}$$

where N is the smallest integer such that the number of points in Q_h is $O(h^{-N})$ for $h \to 0$. Again if $V(P)=0$, $P \notin Q_h$ we define

$$\|\delta V\| = \Big(h^2 \sum_{P \in S_h} [V_x^2(P) + V_y^2(P)]\Big)^{\frac{1}{2}}. \tag{2.7}$$

Finally the maximum norm is given by

$$|V|_{Q_h} = \max_{P \in Q_h} |V(P)| \tag{2.8}$$

for any V defined on Q_h.

We shall need some names for neighborhoods of a point P relative to the operators Δ_h and Δ_h^2. Thus let $\varphi(P) = \delta(P, P_0)$ (Kronecker's delta) and define

$$N_1(P_0) = \{P \mid \Delta_h \varphi(P) \neq 0\} \tag{2.9}$$

and

$$N_2(P_0) = \{P \mid \Delta_h^2 \varphi(P) \neq 0\}. \tag{2.10}$$

If we take Q_h to be an arbitrary subset of S_h then

$$N_i(Q_h) = \bigcup_{P \in Q_h} N_i(P), \qquad i = 1, 2. \tag{2.11}$$

Throughout this paper, we shall use the symbol, C, to denote a generic constant which does not depend on h. In two different places C will not necessarily refer to the same constant.

3. Some Discrete a Priori Inequalities

In this section we shall prove some lemmas which will be used in obtaining our error estimates in the later sections.

The following lemma is the discrete analog of a well known inequality valid in two dimensions.

Lemma 3.1. Let $V(P)$ be any function defined at the mesh points, which vanishes outside a bounded set of mesh points, R_h. Then for any integer $p \geqq 1$,

$$\|V\|_{(2p)} \leqq C_p \|\delta V\|, \tag{3.1}$$

where C_p is a constant which depends on p and R, but not on h.

Proof. For any mesh point $(\bar{x}, \bar{y}) \in R_h$

$$|V^p(\bar{x}, \bar{y})| \leqq \tfrac{1}{2} h \sum_x |(V^p)_x|, \tag{3.2}$$
$$|V^p(\bar{x}, \bar{y})| \leqq \tfrac{1}{2} h \sum_y |(V^p)_y|$$

where the $\sum_x$ is taken over the mesh points along the line $y = \bar{y}$ and similarly for $\sum_y$.

Thus

$$\|V\|_{(2p)}^{2p} \leqq \tfrac{1}{4} \Big[h^2 \sum_{S_h} |(V^p)_x|\Big] \Big[h^2 \sum_{S_h} |(V^p)_y|\Big]. \tag{3.3}$$

By factoring the differences $(V^p)_x$ and using Schwarz's inequality we see that there is a constant C depending only on p and R such that

$$h^2 \sum_{S_h} |(V^p)_x| \leqq C \|V_x\| \, \|V\|_{(2(p-1))}^{p-1}, \tag{3.4}$$

with a similar inequality for the other factor in (3.3). Thus

$$\|V\|_{(2p)}^{2p} \leqq C \|\delta V\|^2 \|V\|_{(2(p-1))}^{2(p-1)}. \tag{3.5}$$

Iterating this inequality p times we obtain (3.1).

We note that the constant C_p in (3.1) tends to infinity as $p \to \infty$ so that we do not obtain a maximum norm estimate. The next lemma, however, enables us to obtain an estimate for the maximum norm.

Lemma 3.2. Let $V(p)$ be any function defined at the mesh points which vanishes outside R_h. Then

$$|V|_{R_h} \leqq C |\log 1/h|^{\frac{1}{2}} \|\delta V\|. \tag{3.6}$$

Proof. Let $G(P, Q)$ be the Green's function defined by

$$\Delta_{h,p} G(P, Q) = -h^{-2} \delta(P, Q), \qquad P \in R_h$$
$$G(P, Q) = 0, \qquad P \notin R_h$$

for $Q \in S_h$. Here again $\delta(P, Q)$ is the Kronecker delta. Now since $V(P) = 0$, $P \notin R_h$ we have the well known relation

$$\begin{aligned} V(P) &= -h^2 \sum_{Q \in S_h} G(P, Q) \, \Delta_h V(Q) \\ &= h^2 \sum_{Q \in S_h} [G_x(P, Q) \, V_x(Q) + G_y(P, Q) \, V_y(Q)]. \end{aligned} \tag{3.7}$$

Using Schwarz's inequality we have

$$|V(P)| \leqq \Big(h^2 \sum_{Q \in S_h} [G_x^2 + G_y^2]\Big)^{\frac{1}{2}} \|\delta V\|. \tag{3.8}$$

If we note that $G(P, Q) = G(Q, P)$ and set $V(S) = G(S, P)$ in (3.7) we see that

$$G(P, P) = h^2 \sum_{Q \in S_h} [G_x^2(P, Q) + G_y^2(P, Q)]. \tag{3.9}$$

But it was shown in [3] that there is a constant C independent of h such that

$$G(P, P) \leqq C |\log 1/h|. \tag{3.10}$$

Inequality (3.6) follows now from (3.8)–(3.10).

The next lemma is an inequality specifically involving the discrete biharmonic operator Δ_h^2. We shall need to define some sets of mesh points. Let $\dot{R}_h$ be a subset of those mesh points whose distance to the boundary is less than h. The set R_h will denote those mesh points of R not in $\dot{R}_h$ and R_h' is the set of points $P \in R_h$ such that $N_2(P) \subset R_h$. Finally $R_h^* = R_h - R_h'$ with these sets defined we shall prove

Lemma 3.3. Let $V(P)$ be any mesh function vanishing for $P \notin R_h$. Suppose that the function Φ, defined by $\Delta_h \Phi(P) = -1$, $P \in R_h$ $\Phi(P) = 0$, $P \notin R_h$, satisfies $\Phi(P) \leqq Kh$ for $P \in R_h^*$, ($K =$ constant).

Then there exists a constant C independent of h, for h sufficiently small, such that

$$\|V\| + \|\delta V\| \leqq C\{h^{-1}\|V\|_{R_h^*} + \|\Delta_h^2 V\|_{R_h'}\}. \tag{3.11}$$

Proof. By a direct calculation we have

$$\begin{aligned} &\Delta_h[\tfrac{1}{2}(V_x^2 + V_{\bar{x}}^2 + V_y^2 + V_{\bar{y}}^2) - V\Delta_h V] \\ &= -V\Delta_h^2 V + \tfrac{1}{2}[V_{xx}^2 + 2V_{x\bar{x}}^2 + V_{\bar{x}\bar{x}}^2] + \tfrac{1}{2}[V_{yy}^2 + 2V_{y\bar{y}}^2 + V_{\bar{y}\bar{y}}^2] - \\ &\quad - (\Delta_h V)^2 + V_{xy}^2 + V_{\bar{x}y}^2 + V_{x\bar{y}}^2 + V_{\bar{x}\bar{y}}^2. \end{aligned} \tag{3.12}$$

Since

$$-(\Delta_h V)^2 \geqq -2[V_{x\bar{x}}^2 + V_{y\bar{y}}^2] \tag{3.13}$$

it follows that

$$\begin{aligned} &\Delta_h[\tfrac{1}{2}(V_x^2 + V_{\bar{x}}^2 + V_y^2 + V_{\bar{y}}^2) - V\Delta_h V] \\ &\geqq -V\Delta_h^2 V + \tfrac{1}{2}[V_{xx}^2 - 2V_{x\bar{x}}^2 + V_{\bar{x}\bar{x}}^2] + \tfrac{1}{2}[V_{yy}^2 - 2V_{y\bar{y}}^2 + V_{\bar{y}\bar{y}}^2] \\ &= -V\Delta_h^2 V + h^2/2\left((V_{x\bar{x}}^2)_{x\bar{x}} + (V_{y\bar{y}}^2)_{y\bar{y}}\right). \end{aligned} \tag{3.14}$$

Now let

$$\chi = \tfrac{1}{2}V_x^2 + V_{\bar{x}}^2 + V_y^2 + V_{\bar{y}}^2 - V\Delta_h V \tag{3.15}$$

so that (3.14) is simply

$$-\Delta_h \chi \leqq V\Delta_h^2 V - h^2/2\left((V_{x\bar{x}}^2)_{x\bar{x}} + (V_{y\bar{y}}^2)_{y\bar{y}}\right). \tag{3.16}$$

Now by hypothesis the mesh function Φ defined by

$$\begin{aligned} \Delta_h \Phi(P) &= -1, \qquad P \in R_h \\ \Phi(P) &= 0, \qquad P \notin R_h, \end{aligned} \tag{3.17}$$

satisfies

$$\Phi(P) \leqq Kh, \qquad P \in R_h^*. \tag{3.18}$$

(We note here that it is easily shown that if R has a piecewise smooth boundary with no reentrant corners such a mesh function will exist. The next lemma is applicable in the more general case, allowing reentrant corners. The behaviour near the boundary of the discrete "torsion function" in this case is discussed in [*8*].)

Now we have

$$-h^2 \sum_{S_h} \chi \Delta_h \Phi = -h^2 \sum_{S_h} \Phi \Delta_h \chi. \tag{3.19}$$

Using (3.16)

$$-h^2\sum_{S_h}\chi\Delta_h\Phi \leqq h^2\sum_{R_h}(\Phi V\Delta_h^2 V) - \frac{h^4}{2}\sum_{S_h}[\Phi\{(V_{x\bar{x}}^2)_{x\bar{x}} + (V_{y\bar{y}}^2)_{y\bar{y}}\}]$$
$$(3.20) \qquad = h^2\sum_{R_h}[\Phi V\Delta_h^2 V] + \frac{h^4}{2}\sum_{S_h}[\Phi_x(V_{x\bar{x}}^2)_x + \Phi_y(V_{y\bar{y}}^2)_y].$$

In order to obtain our results from (3.20) we need to show that the difference quotients Φ_x and Φ_y are uniformly bounded. This is clear since any first difference quotient, say Φ_x, satisfies

$$(3.21) \qquad \begin{aligned} \Delta_h\Phi_x(p) &= 0, & P &\in R_h' \\ |\Delta_h\Phi_x(p)| &\leqq C/h^2, & P &\in [N_1(R_h^*) - R_h'] \\ \Phi_x(p) &= 0, & P &\notin N_1(\bar{R}_h) \end{aligned}$$

and hence by the results of [2]

$$(3.22) \qquad |\Phi_x|_{S_h} \leqq C, \qquad |\Phi_y|_{S_h} \leqq C.$$

We note now that,

$$(3.23) \qquad h^2\sum_{S_h}\chi = 2\|\delta V\|^2.$$

Returning to (3.20) and using (3.23) we have

$$2\|\delta V\|^2 = h^2\sum_{S_h}\chi \leqq \frac{h^2}{2}\sum_{S_h - R_h}(1 + \Delta_h\Phi)(V_x^2 + V_{\bar{x}}^2 + V_y^2 + V_{\bar{y}}^2) +$$
$$(3.24) \qquad + \frac{h^4}{2}\sum_{S_h}[\Phi_x(V_{x\bar{x}}^2)_x + \Phi_y(V_{y\bar{y}}^2)_y] +$$
$$+ h^2\sum_{R_h}[\Phi V\Delta_h^2 V].$$

Using (3.22) we have, since $\sum_{S_h}|(V_{x\bar{x}}^2)_x| \leqq \frac{2}{h}\sum_{S_h} V_{x\bar{x}}^2$ etc.,

$$h^4\sum_{S_h}[\Phi_x(V_{x\bar{x}}^2)_x + \Phi_y(V_{y\bar{y}}^2)_y]$$
$$\leqq C h h^2\sum_{S_h}[V_{x\bar{x}}^2 + V_{y\bar{y}}^2]$$
$$(3.25) \qquad \leqq C h h^2\sum_{S_h}[V_{x\bar{x}}^2 + 2V_{xy}^2 + V_{y\bar{y}}^2]$$
$$= C h h^2\sum_{R_h} V\Delta_h^2 V.$$

Also from (3.18) the first term on the right of (3.24) is bounded in terms of $h^{-2}\|V\|_{N_2(R_h^*)}^2$. Thus we have

$$\|\delta V\|^2 \leqq C\left\{h^{-2}\|V\|_{N_2(R_h^*)}^2 + h h^2\sum_{R_h^*}|V\Delta_h^2 V| + h^2\sum_{R_h'}|V\Delta_h^2 V|\right\}$$
$$(3.26) \qquad \leqq C\left\{h^{-2}\|V\|_{N_2(R_h^*)}^2 + h^2\sum_{R_h'}|V\Delta_h^2 V|\right\}.$$

From the well known inequality

$$(3.27) \qquad \|V\| \leqq C\|\delta V\|$$

we have, using the arithmetic-geometric mean inequality

(3.28) $$\|\delta V\| \leqq C\{h^{-1}\|V\|_{N_2(R_h^*)} + \|\Delta_h^2 V\|_{R_h'}\}$$

for some constant C independent of h. The estimate of Thomée

$$\bar{C}\, h^{-2}\|V\|_{N_2(R_h^*)} \leqq \|\Delta_h V\| \leqq C\left(\|\Delta_h^2 V\|_{R_h'} + h^{-2}\|V\|_{R_h^*}\right)$$

together with (3.27) and (3.28) yields the desired result.

Lemma 3.3 is not quite general enough for domains with reentrant corners in that the inequality $\Phi(P) \leqq Kh$ in the hypothesis will not be satisfied. However, if we weaken this hypothesis we can include such regions. It is clear, following the proof of Lemma 3.3, that the following lemma, which includes lemma 3.3, is true.

Lemma 3.4. Let $V(P)$ be any mesh function vanishing for $P \notin R_h$. Suppose that the function Φ defined by $\Delta_h \Phi(P) = -1$, $P \in R_h$, $\Phi(P) = 0$, $P \notin R_h$ satisfies $\Phi(P) \leqq K h^\alpha$, for $0 < \alpha \leqq 1$, $P \in R_h^*$. Then there exists a constant C independent of h, for h sufficiently small, such that

(3.29) $$\|V\| + \|\delta V\| \leqq C h^{\frac{\alpha-1}{2}}\{h^{-1}\|V\|_{R_h^*} + \|\Delta_h^2 V\|_{R_h'}\}.$$

4. Application of Lemmas to the Results of Thomée and Zlámal

A particular case of those problems studied by Thomée [*12*] was problem (1.1). The difference problem in that case which he posed was that studied by Courant, Friedrichs and Lewy [*9*] who showed convergence only. Thomée essentially gave the following result. Let $U(P)$ satisfy

(4.1) $$\begin{aligned} \Delta_h^2 U(P) &= F(P), \qquad P \in R_h \\ U(P) &= 0, \qquad P \notin R_h. \end{aligned}$$

Then if $e(P) = u(P) - U(P)$, $P \in R_h$, $e(P) = 0$, $P \notin R_h$ and if $\dot{R}$ and u are sufficiently smooth, e satisfies

(4.2) $$\|\Delta_h e\| \leqq C h^{\frac{1}{2}},$$

where C is independent of h. Now it follows at once from (4.2) and the fact that $e(P) = 0$, $P \notin R_h$, that

(4.3) $$\|e\|_{R_h^*} \leqq C h^2.$$

Thus in the case that Lemma 3.3 is applicable ($\dot{R}$ piecewise smooth with no reentrant corners) we have

(4.4) $$\|e\| + \|\delta e\| \leqq C h$$

and from Lemmas 3.1 and 3.2

(4.5) $$\|e\|_{2p} \leqq C h$$

and

(4.6) $$|e|_{R_h} \leqq C h\, |\log 1/h|^{\frac{1}{2}}.$$

In case Lemma 3.4 holds but not Lemma 3.3 the factor h on the right hand sides of (4.4)—(4.6) will simply be replaced by $h^{\frac{\alpha+1}{2}}$.

Recently Zlámal [*13*] has posed a difference analog of (1.1) (and more general fourth order equations) in the case that R is composed of a finite number of rectangles and $\dot{R}$ lies on mesh lines for a sequence of meshes with mesh width h_n, $h_n \to 0$, $n \to \infty$.

In that the more general formulation in the next section includes his formulation as a special case, it will not be explicitly given here. We state, however his result and show how it may be extended by means of the lemmas.

Again let e be the error in ZLÁMAL's problem and take $e(P)=0$ $P \notin R_h$. ZLÁMAL showed that

$$\|\Delta_h e\| \leqq C h^{\frac{3}{2}} \tag{4.7}$$

and

$$|e|_{R_h} \leqq C h^{\frac{3}{2}}. \tag{4.8}$$

Now in the case of a rectangle Lemma 3.3 applies and we obtain

$$\|e\| + \|\delta e\| \leqq C h^2 \tag{4.9}$$

and from Lemmas 3.1 and 3.2

$$\|e\|_{2p} \leqq C h^2 \tag{4.10}$$

and

$$|e|^*_{R_h} \leqq C h^2 |\log 1/h|^{\frac{1}{2}}. \tag{4.11}$$

Clearly (4.11) is a sharper result than (4.8). Now if R has reentrant corners then the interior angles will be $\vartheta = \frac{3\pi}{2}$, and it can be shown that we may take $\alpha = \frac{3}{2} - \varepsilon$ for any $\varepsilon > 0$, in Lemma 3.4. We then obtain instead of (4.9)—(4.11)

$$\|e\| + \|\delta e\| \leqq C h^{11/6-\varepsilon}, \tag{4.12}$$

$$\|e\|_{2p} \leqq C h^{11/6-\varepsilon}, \tag{4.13}$$

$$|e|_{R_h} \leqq C h^{11/6-\varepsilon} \tag{4.14}$$

for any fixed $\varepsilon > 0$. Hence for any $\varepsilon < \frac{1}{3}$ (4.14) is a sharper estimate than (4.8).

5. Second Order Approximation

In order to simplify the presentation and proof we shall treat in detail only the case of simply connected regions with smooth boundaries. The modifications needed to deal with regions whose boundaries have piecewise continuous curvature (possibly corners) are technical and will not be of concern here. It will be evident from the development that the method may, in fact, be applied equally well to this more general class of domains.

We start by defining a set of mesh points which will be analogous to the boundary $\dot{R}$. Let $\dot{R}_{1h}$ be the set of grid points not in R whose horizontal or vertical distance to $\dot{R}$ is less than or equal to $2h/3$, and let $\dot{R}_{2h}$ be those mesh points of R whose horizontal or vertical distance to $\dot{R}$ is less that $h/3$. We set $\dot{R}_h = \dot{R}_{1h} \cup \dot{R}_{2h}$. The set of mesh points of R but not in $\dot{R}_h$ will be called R_h. Further let us denote by R_h^* the subset of points $P \in R_h$ such that $\dot{R}_h \cap N_2(P)$ is not empty. Finally let $R_h' = R_h - R_h^*$ and $\overline{R}_h = R_h \cup \dot{R}_h$.

Now on R_h' we take the usual thirteen point operator

$$\Delta_h^2 V(P) = \Delta_h \Delta_h V(P) \tag{5.1}$$

for any V defined in R_h.

We want yet to define a difference operator $\bar{\Delta}_h^2$ on the set R_h^*. This operator should have the property that

$$\bar{\Delta}_h^2 v(P) = \Delta^2 v(P) + O(h^{-1}), \qquad P \in R_h^* \tag{5.2}$$

for any $v \in C^{(4)}(\bar{R})$ which vanishes with its gradient on $\dot{R}$. As will be seen, this is a property which will be used in obtaining $O(h^2)$ estimates for the error in our problem. For simplicity we are treating only the case of homogeneous boundary conditions, however the necessary modification for inhomogeneous boundary conditions will be stated in the appendix.

In order to define the difference operator at a point $P \in R_{h_1}^*$ it is convenient to introduce the following sets:

$$\begin{aligned} J_0(P) &= N_1(P) \cap R_h - P \\ J_1(P) &= N_1(P) \cap \dot{R}_h \\ J_2(P) &= N_1(J_1(P)) \cap R_h - P. \end{aligned} \tag{5.3}$$

Now let V be any function, defined at the mesh points, which vanishes outside R_h. Further let $\alpha(P)h$ be the distance from any point P to the boundary $\dot{R}$. At an arbitrary point $P \in R_h^*$ we define a mesh function $U_P(Q)$ for each point $Q \in N_1(P)$ as

$$\begin{aligned} U_P(Q) &= \Delta_h V(Q) + h^{-2} \sum_{S \in J_1(Q)} \left(\frac{\alpha(S)}{\alpha(P)}\right)^2 V(P), \qquad Q \in J_0(P) \cup P \\ U_P(Q) &= \frac{2}{\alpha^2(P)\, h^2} V(P), \qquad Q \in J_1(P). \end{aligned} \tag{5.4}$$

In terms of $U_P(Q)$ we define the difference operator

$$\bar{\Delta}_h^2 V(P) = \Delta_h U_P(P), \tag{5.5}$$

for each $P \in R_h^*$. By looking at the Taylor expansion it may be directly verified that the difference operator defined by (5.4), (5.5) satisfies (5.2) in case the curvature of $\dot{R}$ is piecewise continuous and $\dot{R}$ has only "convex" corners. The extension of the results of this section to non-convex corners is technical and is omitted here for simplicity.

Intuitively, however, the construction is based on the following considerations. If W is a sufficiently smooth function defined in the whole plane, which vanishes with its gradient on $\dot{R}$, then it is easily verified that for $V(P) = W(P)$, $P \in R_h$,

$$U_P(Q) = \Delta W(Q) + O(h) \tag{5.6}$$

for $Q \in N_1(P)$. Thus (5.2) will obviously be satisfied for $V = W$.

It will be useful to compare this operator with the operator Δ_h^2 for functions $V(P) = 0$, $P \notin R_h$. The relationship is easily seen to be

$$\bar{\Delta}_h^2 V(P) = \Delta_h^2 V(P) + h^{-4} \gamma(P) V(P) - h^{-4} \sum_{S \in J_2(P)} V(S) \tag{5.7}$$

where

$$\gamma(P) = \sum_{Q \in J_0(P)} \left\{ \sum_{S \in J_1(Q)} \left(\frac{\alpha(S)}{\alpha(P)}\right)^2 \right\} + \sum_{Q \in J_1(P)} \left\{ 2\left[\frac{1}{\alpha^2(P)} - 2\left(\frac{\alpha(Q)}{\alpha(P)}\right)^2\right] - 1 \right\}. \tag{5.8}$$

Thus the modification of $\Delta_h^2 V(P)$ is simply a change in the coefficient of $V(P)$ itself and possibly a change of some other coefficients from 2 to 1. The number $\gamma(P)$ is easily calculated from the α's.

The difference analog of (1.1) which we take is

$$
\begin{aligned}
\Delta_h^2 U(P) &= F(P), && P \in R_h' \\
\bar{\Delta}_h^2 U(P) &= F(P), && P \in R_h^* \\
U(P) &= 0, && P \notin R_h.
\end{aligned} \tag{5.9}
$$

We have then the following theorem.

Theorem 5.1. There exists a unique solution of (5.9). Furthermore if $u \in C^{(6)}(\bar{R})$ is the solution to (1.1) and if $e(P) = U(P) - u(P)$, $P \in R_h$, $e(P) = 0$, $P \notin R_h$ then for h sufficiently small

$$\|e\| + \|\delta e\| \leq C h^2 \tag{5.10}$$

where C is a constant independent of h.

Proof. In order to simplify the argument we shall assume that the set $\dot{R}_h$ has the property that each of its points has at least one horizontal and vertical neighbor not in R_h. We also suppose that h is chosen small enough that for each pair of points P and Q such that $P \in J_2(Q)$, the set $J_2(P) \cap J_2(Q)$ is a single point.

Let V be any mesh function which vanishes outside R_h. Then

$$h^2 \sum_{S_h} (\Delta_h V)^2 = h^2 \sum_{R_h} V \Delta_h^2 V.$$

By (5.7) we have

$$
\begin{aligned}
h^2 \sum_{S_h} (\Delta_h V)^2 + h^{-2} \sum_{P \in R_h^*} \Big[\gamma(P)\, V(P) - \sum_{S \in J_2(P)} V(S)\Big] V(P) \\
= h^2 \sum_{R_h'} V \Delta_h^2 V + h^2 \sum_{R_h^*} V \bar{\Delta}_h^2 V.
\end{aligned} \tag{5.11}
$$

Clearly from (5.7) and the definition of $J_2(P)$ the matrix of the system (5.9) is symmetric. By examining the left hand side of (5.11) we shall show that it is also positive definite.

Now let R_{0h}^* be the subset of points $P \in R_h^*$ for which $J_1(P)$ is empty. Then for $P \in R_{0h}^*$

$$\gamma(P) = \sum_{Q \in J_0(P)} \Big\{ \sum_{S \in J_1(Q)} \Big(\frac{\alpha(S)}{\alpha(P)} \Big)^2 \Big\} \geq 0 \tag{5.12}$$

and $J_2(P)$ is also empty. Further let R_{1h}^* be that subset of R_h^* where $J_1(P)$ is not empty but $J_2(P)$ is empty and finally let R_{2h}^* be those points of R_h^* where neither $J_1(P)$ nor $J_2(P)$ is empty. We have

$$
\begin{aligned}
& h^2 \sum_{S_h} (\Delta_h V)^2 + h^{-2} \sum_{P \in R_h^*} \Big[\gamma(P)\, V(P) - \sum_{S \in J_2(P)} V(S)\Big] V(P) \\
& \geq h^2 \sum_{S_h} (\Delta_h V)^2 + h^{-2} \sum_{P \in R_{1h}^*} \gamma(P)\, V^2(P) + \\
& \quad + h^{-2} \sum_{P \in R_{2h}^*} \Big[\gamma(P)\, V(P) - \sum_{S \in J_2(P)} V(S)\Big] V(P).
\end{aligned} \tag{5.13}
$$

Now since $V(P)=0,\ P\notin R_h$

$$h^2\sum_{S_h}(\Delta_h V)^2 = h^2\sum_{S_h}[V_{x\bar{x}}^2 + 2V_{xy}^2 + V_{y\bar{y}}^2] \tag{5.14}$$

and hence because of the assumption on $\dot{R}_h$

$$h^2\sum_{S_h}(\Delta_h V)^2 \geqq h^2 \sum_{S_h-[R_{1h}^*\cup R_{2h}^*]} (\Delta_h V)^2 + h^{-2} \sum_{P\in R_{1h}^*\cup R_{2h}^*} j(P)V^2(P), \tag{5.15}$$

where $j(P)$ is the number of points in $J_1(P)$. Thus combining (5.13) and (5.15) we have

$$\begin{aligned} h^2\sum_{S_h}(\Delta_h V)^2 + h^{-2}\sum_{P\in R_h^*}\Big[\gamma(P)\,V(P) - \sum_{S\in J_2(P)} V(S)\Big]V(P) \\ \geqq h^2 \sum_{S_h-[R_{1h}^*\cup R_{2h}^*]}(\Delta_h V)^2 + h^{-2}\sum_{P\in R_{1h}^*}\big(\gamma(P)+j(P)\big)\,V^2(P) + \\ + h^{-2}\sum_{P\in R_{2h}^*}\Big[\big(\gamma(P)+j(P)\big)\,v(P) - \sum_{S\in J_2(P)} V(S)\Big]V(P). \end{aligned} \tag{5.16}$$

Now for any $P\in R_{1h}^*\cup R_{2h}^*$ it is easy to see from the definition of $\dot{R}_h$ and $\gamma(P)$ that

$$\gamma(P)+j(P)\geqq \tfrac{7}{8}\,j(P)\geqq\tfrac{7}{8}. \tag{5.17}$$

This estimate will suffice for R_{1h}^* however we must examine R_{2h}^* more closely. From the definition of $\gamma(P)$ we have

$$\gamma(P)+j(P)\geqq \sum_{Q\in J_1(P)}\gamma(P,Q) \tag{5.18}$$

where

$$\gamma(P,Q)=2\left[\frac{1}{\alpha^2(P)} - 2\left(\frac{\alpha(Q)}{\alpha(P)}\right)^2\right].$$

By the definition of $J_2(P)$ we have that $S\in J_2(P)$ if and only if $P\in J_2(S)$ and with each such pair P, S there is a unique $Q=J_1(P)\cap J_1(S)$. Thus we have

$$\begin{aligned} h^{-2}\sum_{P\in R_{2h}^*}\Big[\gamma(P)+j(P)\,V(P) - \sum_{S\in J_2(P)}V(S)\Big]V(P) \\ \geqq h^{-2}\sum_{(P,Q,S)\in T}[\gamma(P,Q)\,V^2(P) - 2V(P)\,V(S) + \gamma(S,Q)\,V^2(S)] \end{aligned} \tag{5.19}$$

where T is the set of triples satisfying $P,S\in R_{1h}^*$, $S\in J_2(P)$, $Q=J_1(P)\cap J_1(S)$. Now if $Q\in\dot{R}_{1h}$ then it is easily verified that $\gamma(P,Q)\geqq 2$. In case $Q\in\dot{R}_{2h}$ we could have at worst

$$\gamma(P,Q)\geqq 2\left[\frac{1-2\alpha^2}{(1+\alpha)^2}\right] \quad \text{for} \quad \alpha<\frac{1}{3}.$$

But in this case we have

$$\gamma(S,Q)\geqq 2\left[\frac{1-2\alpha^2}{1+\alpha^2}\right], \qquad \alpha<\frac{1}{3}$$

(see Fig. 1). In any case a simple calculation shows that

$$\begin{aligned} h^{-2}\sum_{(P,Q,S)\in T}[\gamma(P,Q)\,V^2(P) - 2V(P)\,V(S) + \gamma(S,Q)\,V^2(S)] \\ \geqq \tfrac{1}{10}\,h^{-2}\sum_{P\in R_{2h}^*}V^2(P). \end{aligned} \tag{5.20}$$

Combining (5.16), (5.17), (5.19) and (5.20) we have

$$h^2 \sum_{S_h} (\Delta_h V)^2 + h^{-2} \sum_{P \in \dot{R}_h^*} \Big[\gamma(P)\, V(P) - \sum_{S \in J_h(P)} V(S)\Big] V(P) \geqq \tfrac{1}{10}\, h^2 \sum_{S_h} (\Delta_h V)^2. \tag{5.21}$$

The case in which the assumption on $\dot{R}_h$ at the beginning of the proof is not satisfied can be dealt with again by examining terms of $h^2 \sum_{S_h} (\Delta_h V)^2$ and (5.21) can be shown to hold more generally.

Thus combining (5.11) and (5.21) we have

$$h^2 \sum_{S_h} (\Delta_h V)^2 \leqq 10 \Big\{ h^2 \sum_{R_h'} V \Delta_h^2 V + h^2 \sum_{R_h^*} V \bar{\Delta}_h^2 V \Big\}. \tag{5.22}$$

In view of uniqueness in the discrete Dirichlet problem (5.22) tells us immediately that the solution of (5.9) is unique. But for linear systems uniqueness implies existence for any given F.

In order to obtain the estimate (5.10) note that the number of points in $N_2(R_h^*)$ is $O(h^{-1})$ and by the definition of the norm $\|V\|_{N_2(R_h^*)}$ $\Big(\|V\|_{N_2(R_h^*)} = \Big(h \sum_{P \in N_2(R_h^*)} V^2(P)\Big)^{\frac{1}{2}}\Big)$ and the fact that $V(P)=0$, $P \notin R_h$, we have

$$h^{-\frac{3}{2}} \|V\|_{R_h^*} \leqq C \|\Delta_h V\| \tag{5.23}$$

where C is a constant which does not depend on h. In addition to this we need the well known inequality

$$\|V\| \leqq C \|\Delta_h V\|. \tag{5.24}$$

Now we have for e

$$|\Delta_h^2 e|_{R_h'} \leqq C h^2 \qquad |\bar{\Delta}_h^2 e|_{R_h^*} \leqq C h^{-1} \qquad e(P) = 0, \quad P \notin R_h. \tag{5.25}$$

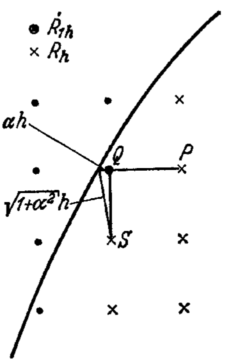

Fig. 1

If we set $V=e$ in (5.22)—(5.24) and apply the Schwarz inequality to (5.22) it follows, in view of (5.25), that

$$\|e\|_{R_h^*} \leqq C h^3. \tag{5.26}$$

But by Lemma 3.3 the estimate (5.10) follows. This completes the proof of the theorem.

Corollary 1. There exists a constant C independent of h such that

$$\|e\|_{(2p)} \leqq C h^2 \tag{5.27}$$

for any integer $p \geqq 1$ and h sufficiently small.

Proof. Set $V=e$ in Lemma 3.1 and apply Theorem 5.1.

Corollary 2. There exists a constant C independent of h such that

$$|e|_{R_h} \leqq C h^2 |\log 1/h|^{\frac{1}{2}} \tag{5.28}$$

for h sufficiently small.

Proof. Set $V=e$ in Lemma 3.2 and apply Theorem 5.1.

Appendix

Formulation of Difference Problem for Inhomogeneous Boundary Data

We suppose for simplicity that the boundary is smooth and that u is smooth in $\bar{R}$.

At an arbitrary point $P \in R_h^*$ we define

$$W_P(Q; u) = h^{-2} \sum_{S \in J_1(Q)} \left\{ \left(\frac{\alpha(S)}{\alpha(P)} \right)^2 u(\bar{P}) - u(\bar{S}) + \alpha(S) h \left[\frac{\alpha(S)}{\alpha(P)} u_n(\bar{P}) - u_n(\bar{S}) \right] \right\} \quad \text{if } Q \in J_0(P) \cap P$$

and

$$W_P(Q; u) = \frac{2}{\alpha^2(P) h^2} \left\{ u(\bar{P}) + \alpha(P) h u_n(\bar{P}) - \frac{\alpha^2(P) h^2}{2} \left[u_{ss}(\bar{P}) + k(\bar{P}) u_n(\bar{P}) \right] \right\} \quad \text{if } Q \in J_1(P).$$

We have used the notation:

a) $\bar{P}$ is the point of $\dot{R}$ closest to P (same for S).

b) $k(\bar{P})$ is the curvature of $\dot{R}$ at P.

c) $u_n(\bar{P})$ is the outward normal derivative of u at $\bar{P}$ on $\dot{R}$.

d) $u_{ss}(\bar{P})$ is the second derivative of u with respect to arc length on $\dot{R}$ at $\bar{P}$.

If, instead of (1.1), we have u and u_n as given functions of arc length on R then the difference problem (5.9) is replaced by

$$\begin{aligned} \Delta_h^2 U(P) &= F(P), && P \in R_h' \\ \bar{\Delta}_h^2 U(P) &= F(P) + \Delta_h W_P(P; U), && P \in R_h^* \\ U(P) &= 0, && P \notin R_h. \end{aligned}$$

Now for $e(P)$, as defined in Theorem 1, it then follows that we have

$$\begin{aligned} \Delta_h^2 e(P) &= O(h^2), && P \in R_h' \\ \bar{\Delta}_h^2 e(P) &= O(h^{-1}), && P \in R_h^* \\ e(P) &= 0, && P \notin R_h. \end{aligned}$$

All the previous results now follow for the inhomogeneous problem.

References

[1] Bramble, J. H.: Error estimates for difference methods in forced vibration problems. SIAM Series B 3, No. 1, 1—12 (1966).

[2] —, and B. E. Hubbard: On the formulation of finite difference analogues of the Dirichlet problem for Poisson's equation. Num. Math. 4, 313—327 (1962).

[3] — — A priori bounds on the discretization error in the numerical solution of the Dirichlet problem. Contributions to Diff. Eq. V 2, 229—252 (1963).

[4] — — New monotone type approximations for elliptic problems. Math. Comp. V 18, 349—367 (1964).

[5] — — Approximations of derivatives by finite difference methods in elliptic boundary value problems. Contributions to Diff. Eq. V 3, No. 4, 399—410 (1964).

[6] — — Approximation of solutions of mixed boundary value problems for Poisson's equation by finite differences. J. ACM, V 12, No. 1, 114—123 (1965).

[7] Bramble, J. H., and B. E. Hubbard: A finite difference analog of the Neumann problem for Poisson's equation. SIAM Series B 2, No. 1, 1—14 (1965).
[8] — — Eigenvalue and eigenvector estimates for the fixed membrane problem by difference methods. (To appear.)
[9] Courant, R., K. O. Friedrichs u. H. Lewy: Über die partiellen Differenzengleichungen der mathematischen Physik. Math. Ann. **100**, 32—74 (1928).
[10] Fichera, G.: On a unified theory of boundary value problems for elliptic-parabolic equations of second order. Boundary problems in differential equations, pp. 97—120. U. Wis. Press 1960.
[11] Miranda, C.: Formule di magrorazione e teorema di esistenza per le funzioni biarmoniche di due variabili. Giornale de Mat. di Battaglini **78**, 1—22 (1948/49).
[12] Thomée, V.: Elliptic difference operators and Dirichlet's problem. Contributions to Diff. Eq. V **3**, 301—324 (1964).
[13] Zlámal, M.: Asymptotic error estimates in solving elliptic equations of the fourth order by the method of finite differences. SIAM Series B **2**, 337—344 (1965).

University of Maryland
College Park, Maryland, USA