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Construction of Basic Functions for Numerical Utilisation of Ritz's Method*

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1. Introduction

We consider an elliptic partial differential equation, the operator being symmetric definite positive. It exists an equivalent problem of variation (see [1]) which is specially suitable for numerical treatment (see [2]). For this purpose, it is necessary to change the initial problem into a problem with a finite number of unknowns. We adopt the Ritz's discretisation procedure in which we restrict the space of the functions to be a space of finite dimension. For this, we divide the given domain into simple geometrical elements (segments for the one-dimensional problems, triangles for the two-dimensional problems). To each lattice point P, we associate one or more functions with value zero except on the elements adjacent to this point (the interval formed of the two segments admitting P for endpoint for the one-dimensional problems, the polygon formed of triangles admitting P for vertex for the two-dimensional problems). These functions (called hereafter basic functions) form the basis of the mentioned finite dimensional space. This method presents several avantages: the construction of the basic functions is relatively easy; the linear system has a band structure. It has been proposed by Courant [3] for problems of variation where the maximum order of derivatives is one. Recently, specialists in structural analysis, in particular Clough (Berkeley, see [4]) and Zienkiewicz (Wales, Swansea, see [5]) have obtained, under the name of finite element method, basic functions for two-dimensional problems of variation where the maximum order of derivatives is two.

In this paper, we give a mathematical form of Clough's and Zienkiewicz's results and generalize them. We introduce a notion of completion and we prove the sufficiency and very often the necessity of the conditions to obtain this completion. The practical value of the results has been verified experimentally by a general and entirely automatic program of calculus of plate in bending (see [9, 10]).

Definitions. 1. A *piecewise continuous* function on a closed interval I is continuous everywhere on I except at a finite number of points. A piecewise continuous function is not necessarily bounded and may be undefined at the points of discontinuity.

* This article is drawn from a thesis presented at the Ecole Polytechnique de l'Université de Lausanne [9]. I want to express my gratitude to Prof. J. Descloux for his suggestions and helpful assistance.

2. A *domain* S is a subspace of the two-dimensional closed plane, bounded and limited by a finite number of segments of straightline (triangle, rectangle, ...); the simple connexity is not required.

3. A function is *piecewise continuous* on a domain if it is continuous everywhere with the exception of a finite number of segments of straightline.

4. The *support* of a function is the closure of the set where it is different from zero.

5. The derivative $\partial f/\partial x = f_x$ of a function is called continuous on a domain S if 1) it is continuous in the interior of S, 2) it can be extended to a continuous function defined on S. This definition can clearly be extended for f_y, for the partial derivatives of greater order and for the notion of "function of class C^k on S".

Convention. All sets of definition of functions which we consider are closed.

Problem 1. Given: I the interval with endpoints 0 and 1, Θ the set of functions of class C^1 on I, of second derivative piecewise continuous and square integrable, Φ the set of functions of class C^3 on I; construct a sequence $E_1, E_2, \ldots$, not necessarily embedded, of linear subspaces of Θ, of finite dimension $n_1, n_2, \ldots$ such that for each $f \in \Phi$, there exists a sequence $w_1 \in E_1$, $w_2 \in E_2, \ldots$ with:

$$\lim_{k\to\infty} \int_0^1 [(f - w_k)^2 + (f_x - w_{kx})^2 + (f_{xx} - w_{kxx})^2]\, dx = 0. \tag{1}$$

Problem 2. Given: S a domain of boundary Z, Θ the set of functions of class C^1 on S possessing partial derivatives of second order piecewise continuous and square integrable. Φ the set of functions of class C^3 on S; construct a sequence $E_1, E_2, \ldots$, not necessarily embedded, of linear subspaces of Θ, of finite dimension $n_1, n_2, \ldots$ such that for each $f \in \Phi$, there exists a sequence $w_1 \in E_1$, $w_2 \in E_2, \ldots$ with:

$$\begin{aligned}\lim_{k\to\infty} \iint_S [(f - w_k)^2 + (f_x - w_{kx})^2 + (f_y - w_{ky})^2 \\ + (f_{xx} - w_{kxx})^2 + (f_{yy} - w_{kyy})^2 + (f_{xy} - w_{kxy})^2]\, dx\, dy = 0.\end{aligned} \tag{2}$$

Remarks. 1. No boundary conditions are introduced in Problems 1 and 2. In fact, for direct application of the Ritz's method, the elements of the set Θ should satisfy the boundary conditions. However very often the methods of construction for the spaces E_k can be modified in order to verify the boundary conditions arising frequently in the problems of mathematical physics.

2. Let:

$$Q(f) = a_1 f^2 + a_2 f f_x + a_3 f f_y + a_4 f f_{xx} + \cdots + b_1 f_x^2 + b_2 f_x f_y + \cdots \tag{3}$$

be a quadratic form in the variables $f, f_x, f_y, f_{xx}, f_{yy}, f_{xy}$ whose coefficients $a_1, a_2, \ldots, b_1, b_2, \ldots$ are supposed to be integrable and bounded. Setting $g_k = f - w_k$, one obtains by Schwarz's inequality:

$$\begin{aligned}\iint_S Q(f - w_k)\, dx\, dy \leq \max |a_1| \iint_S g_k^2\, dx\, dy \\ + \max |a_2| \left(\iint_S g_k^2\, dx\, dy\right)^{\frac{1}{2}} \left(\iint_S g_{kx}^2\, dx\, dy\right)^{\frac{1}{2}} + \cdots.\end{aligned} \tag{4}$$

If the relation (2) is verified, then:

$$\lim_{k\to\infty} \iint_S Q(f - w_k)\, dx\, dy = 0. \tag{5}$$

If the formulation of Problem 1 and 2 does not take into consideration the boundary conditions, it is relatively independent of the differential form of the energy Q.

3. The Problems 1 and 2 refer to expressions of energy with first and second derivatives. The same questions can be considered for expressions of energy with first derivatives only or with derivatives of higher order than two. The problem with first derivatives is very easy to solve for functions of one or two variables. If the expression of energy contains derivatives of higher order than two, the one-dimensional case is relatively simple but the two-dimensional case becomes extremely intricate. Further generalizations are possible, e.g. those concerning the functions with more than two variables.

For the particular functions considered in this paper, our main result can be roughly expressed in the following way: the sequence $E_1, E_2, \ldots$ will satisfy the condition of Problem 1 (Problem 2) if and only if the functions $1, x, x^2$ ($1, x, y, x^2, y^2, xy$) belong to E_k for all k.

A list of functions for two-dimensional second order variational problems is given in [10].

2. Basic Functions for One-Dimensional Problems of Variation of Second Order

Let I be the interval $0 \leq x \leq 1$, $x_i = ih$, $i = 1, 2, \ldots, N$ be the coordinates of the points P_i of the mesh, $h = 1/N$ be the stepsize of the mesh; the interval $[x_i, x_{i+1}]$ is an element of the mesh (see Fig. 1).

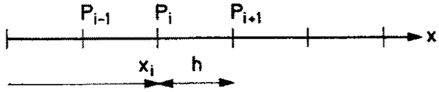

Fig. 1

To each point P_i, we associate the functions $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ defined on I, possessing the following properties: 1) they admit the interval $[x_{i-1}, x_{i+1}]$ for support, 2) they are of class C^1 with second derivative piecewise continuous and square integrable, 3) they verify the relations illustrated in the Fig. 2:

$$\tilde{\alpha}_i(x_i) = 1, \quad \tilde{\alpha}_{ix}(x_i) = 0, \quad \tilde{\beta}_i(x_i) = 0, \quad \tilde{\beta}_{ix}(x_i) = 1. \tag{6}$$

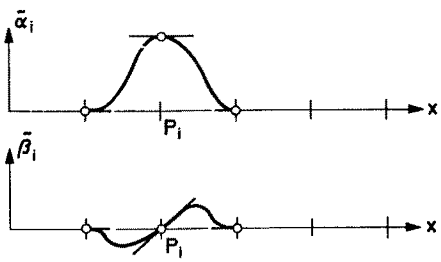

Fig. 2

One considers the function:

$$w(x)=\sum_{i=0}^{N}\left(w_i\,\tilde{\alpha}_i(x)+w_{xi}\tilde{\beta}_i(x)\right), \tag{7}$$

where w_i, w_{xi} are parameters representing respectively the value of the function w and of its derivative with respect to x at the point P_i. The set E_N of the functions w is a subspace of Θ (see Section 1) of dimension $2(N+1)$; the $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ form a basis.

The functions $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ are constructed from the two functions α and β which have the following properties: defined for $-\infty<x'<+\infty$, α, β admit the interval $[-1, +1]$ for support; they are of class C^1 with second piecewise continuous derivative square integrable; they verify the relations:

$$\alpha(0)=1, \quad \alpha_{x'}(0)=0, \quad \beta(0)=0, \quad \beta_{x'}(0)=1. \tag{8}$$

Now, we set:

$$\tilde{\alpha}_i(x)=\alpha(x'), \quad \tilde{\beta}_i(x)=h\,\beta(x'), \quad x'=\frac{x-x_i}{h}. \tag{9}$$

Let (see Fig. 3):

$$\alpha_1(x')=\begin{cases}\alpha(x') & \text{pour } x'\geqq 0,\\ 0 & \text{pour } x'<0;\end{cases} \quad \beta_1(x')=\begin{cases}\beta(x') & \text{pour } x'\geqq 0,\\ 0 & \text{pour } x'<0;\end{cases}$$
$$\alpha_2(x')=\begin{cases}\alpha(x'-1) & \text{pour } x'\leqq 1,\\ 0 & \text{pour } x'>1;\end{cases} \quad \beta_2(x')=\begin{cases}\beta(x'-1) & \text{pour } x'\leqq 1,\\ 0 & \text{pour } x'>1.\end{cases} \tag{10}$$

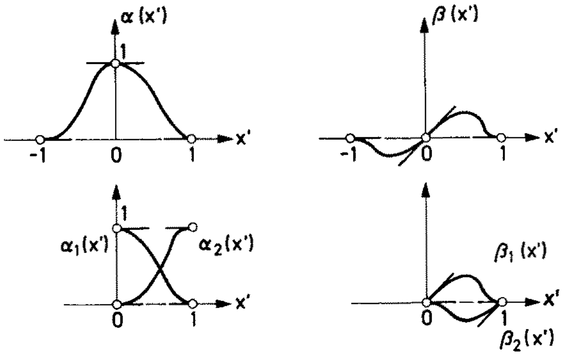

Fig. 3

Property 1. For each function $f\in\Phi$, of class C^3, defined on I, there exists a sequence $w_1\in E_1$, $w_2\in E_2, \ldots, w_N\in E_N, \ldots$ such that:

$$\lim_{N\to\infty}\int_0^1 \left[(f-w_N)^2+(f_x-w_{Nx})^2+(f_{xx}-w_{Nxx})^2\right]dx=0. \tag{11}$$

Theorem 1. The Property 1 is satisfied if and only if the functions $\alpha_1, \beta_1, \alpha_2, \beta_2$ satisfy the relations:

$$\alpha_1(x')+\alpha_2(x') \qquad\qquad =1, \tag{12}$$

$$\alpha_2(x')+\beta_1(x')+\beta_2(x')=x', \tag{13}$$

$$\tfrac{1}{2}\alpha_2(x') \qquad +\beta_2(x')=\tfrac{1}{2}x'^2. \tag{14}$$

Proof. The condition is necessary. Suppose that the Property 1 is satisfied, in particular for $f=1$; there exists a sequence $w_1\in E_1$, $w_2\in E_2, \ldots, w_N\in E_N, \ldots$ such that $\lim_{N\to\infty} I_N=0$ where:

$$I_N=\int_0^1 (w_N-1)^2 dx.$$

Let:

$$\lambda=\min_{p_1,p_2,q_1,q_2}\int_0^1 [p_1\alpha_1(x')+p_2\alpha_2(x')+q_1\beta_1(x')+q_2\beta_2(x')-1]^2 dx'.$$

On the interval $[x_i, x_{i+1}]$, w_N takes the form:

$$w_N(x)=w_i\alpha_1\left(\frac{x-x_i}{h}\right)+w_{x\,i}h\,\beta_1\left(\frac{x-x_i}{h}\right)$$
$$+w_{i+1}\alpha_2\left(\frac{x-x_i}{h}\right)+w_{x\,i+1}h\,\beta_2\left(\frac{x-x_i}{h}\right);$$

I_N becomes:

$$I_N=\sum_{i=0}^{N-1}\int_{ih}^{(i+1)h}\left[w_i\alpha_1\left(\frac{x-x_i}{h}\right)+w_{x\,i}h\,\beta_1\left(\frac{x-x_i}{h}\right)\right.$$
$$\left.+w_{i+1}\alpha_2\left(\frac{x-x_i}{h}\right)+w_{x\,i+1}h\,\beta_2\left(\frac{x-x_i}{h}\right)-1\right]^2 dx$$
$$\geqq\sum_{i=0}^{N-1}h\lambda=\lambda.$$

By hypothesis, $\lim_{N\to\infty} I_N=0$, thus $\lambda=0$. The relation (12) follows immediately. In the same way, the relations (13) and (14) can be obtained by using the Property 1 for the function $f(x)=x$ and $f(x)=\frac{1}{2}x^2$.

To show the sufficiency of the condition, one chooses for parameters w_i, $w_{x\,i}$ in (7): $w_i=f(x_i)$, $w_{x\,i}=f_x(x_i)$, $i=0, 1, \ldots, N$; therefore, the functions w_N are constructed by interpolation. Using Taylor's series, one shows that there exists a constant C, dependant from f, but independant from the function w_N such that:

$$N(f-w_N)=\left(\int_0^1 [(f-w_N)^2+(f_x-w_{N\,x})^2+(f_{xx}-w_{N\,xx})^2 dx]\right)^{\frac{1}{2}}\leqq C h. \tag{15}$$

Letting $N\to\infty$, one gets (11).

Remarks. 1. One verifies easily that the Theorem 1 can also take the form: the Property 1 is satisfied if and only if the functions 1, x, x^2 belong to the subspaces E_N for $N=1, 2, \ldots$.

2. If f is the solution of a variational problem, w_N its Ritz's approximation, then (15) and (4) prove that the energy norm of $f-w_N$ converges towards zero like h. Assuming f of class C^{2+m} it is possible to construct with analogous principles, w-functions allowing convergence like h^m.

3. The theorem remains valid if we divide the interval [0, 1] into variable stepsizes. On an element $[x_i, x_{i+1}]$ of length h_i, w takes the form:

$$w(x)=w_i\alpha_1\left(\frac{x-x_i}{h_i}\right)+w_{x\,i}h_i\beta_i\left(\frac{x-x_i}{h_i}\right)$$
$$+w_{i+1}\alpha_2\left(\frac{x-x_i}{h_i}\right)+w_{x\,i+1}h_i\beta_2\left(\frac{x-x_i}{h_i}\right). \tag{16}$$

For a decomposition $\mathscr{D}$, one defines $H_{\mathscr{D}} = \max_{i=0,1,\ldots,N-1} (h_i)$. Then, the Property 1 becomes: for all sequence of decomposition $\mathscr{D}_1, \mathscr{D}_2, \ldots$ with $\lim_{k\to\infty} H_{\mathscr{D}_k} = 0$ and for each function $f \in \Phi$, of class C^3, defined on I, there exists a sequence $w \in_1 E_{\mathscr{D}_1}$, $w_2 \in E_{\mathscr{D}_2}, \ldots$ such that:

$$\lim_{k\to\infty} \int_0^1 [(f - w_k)^2 + (f_x - w_{kx})^2 + (f_{xx} - w_{kxx})^2]\, dx = 0. \tag{17}$$

Theorem 1 is then still valid.

4. It is easy to take into consideration boundary conditions. Consider e.g. the condition $f = 0$ at a point R. Thus the decomposition of the interval I requires R to be a point P_j of the mesh. Therefore:

$$w(x) = \sum_{\substack{i=0\\ i\neq j}}^{N} w_i\, \tilde{\alpha}_i(x) + \sum_{i=0}^{N} w_{xi}\, \tilde{\beta}_i(x). \tag{18}$$

Other conditions can be considered ($f_x = 0$ at a point, $f = f_x = 0$ at a point) and combined. Correspondant spaces Θ and $E_{\mathscr{D}}$ are defined for each problem.

Example of Functions $\alpha_1, \beta_1, \alpha_2, \beta_2$. Any set of functions $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfying the conditions (8), (10), (12), (13), (14) can be obtained in the following way. One chooses arbitrarily one of the functions $\alpha_1, \beta_1, \alpha_2, \beta_2$ subjected to the conditions implied by (8). The other functions are then uniquelly determined. For example:

$$\begin{aligned} \alpha_1(x') &= 2x'^3 - 3x'^2 + 1, \\ \beta_1(x') & x'^3 - 2x'^2 + x', \\ \alpha_2(x') &= -2x'^3 + 3x'^2, \\ \beta_2(x') &= x'^3 - x'^2. \end{aligned} \tag{19}$$

3. Basic Functions for Two-Dimensional Problems of Variation of Second Order

Let us consider a domain S whose boundary Z is formed by segments of straightline (see Definition in Section 1); one sets on this domain a mesh of triangles (see Fig. 4). An element of the mesh is one of the triangles; N is the number of points of the mesh and NE is the number of elements of the mesh. A typical element $P_1 P_2 P_3$ is characterised by the geometrical datas: (x_1, y_1), (x_2, y_2), (x_3, y_3): coordinates of P_1, P_2, P_3 respectively; L_1, L_2, L_3: length of the sides $P_2 P_3$, $P_3 P_1$, $P_1 P_2$ respectively; $\varkappa_1, \varkappa_2, \varkappa_3$: angle at the vertex P_1, P_2, P_3 respectively; $C_1 = L_2 L_3 \cos\varkappa_1$, $C_2 = L_3 L_1 \cos\varkappa_2$, $C_3 = L_1 L_2 \cos\varkappa_3$; $h = \max(L_1, L_2, L_3)$; $\Delta = \frac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]$: surface of the triangle.

For a decomposition $\mathscr{D}$ of the domain S into triangles, one defines:

$H_{\mathscr{D}} = \max_{j=1,2,\ldots,NE} (hj)$: parameter of fineness,

$R_{\mathscr{D}} = \min_{j=1,2,\ldots,NE} \min(\varkappa_{1j}, \varkappa_{2j}, \varkappa_{3j}, \pi - \varkappa_{1j}, \pi - \varkappa_{2j}, \pi - \varkappa_{3j})$: parameter of regularity.

A sequence of decomposition $\mathscr{D}_1, \mathscr{D}_2, \ldots$ is said regular if: 1) $\lim_{k\to\infty} H_{\mathscr{D}_k} = 0$, 2) there exists a number $\varepsilon > 0$, independant of $\mathscr{D}_k$ and such that: $R_{\mathscr{D}_k} > \varepsilon$, $k = 1, 2, \ldots$.

One associates to each point P_i of the mesh three basic functions $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i$. Let us consider a point P_1 and the polygon formed of triangles admitting P_1 for vertex (hachured polygon of the Fig. 4). The basic functions $\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1$ associated to the point P_1 verify the following properties:

1) They admit the polygon for support; 2) they are of class C^1 and of piecewise continuous partial derivatives of second order square integrable; 3) they verify the relations:

$$\begin{aligned} &\tilde{\alpha}_1(P_1) = 1, \quad \tilde{\alpha}_{1x}(P_1) = 0, \quad \tilde{\alpha}_{1y}(P_1) = 0, \\ &\tilde{\beta}_1(P_1) = 0, \quad \tilde{\beta}_{1x}(P_1) = 1, \quad \tilde{\beta}_{1y}(P_1) = 0, \\ &\tilde{\gamma}_1(P_1) = 0, \quad \tilde{\gamma}_{1x}(P_1) = 0, \quad \tilde{\gamma}_{1y}(P_1) = 1. \end{aligned} \tag{20}$$

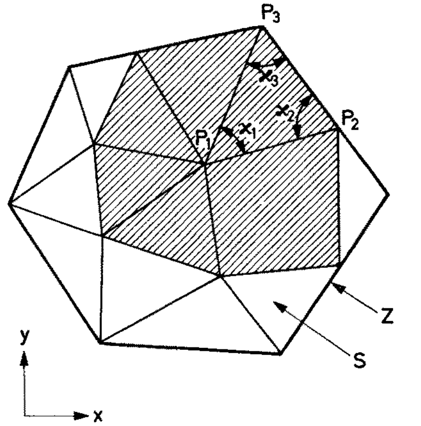

Fig. 4

For a decomposition $\mathscr{D}$ of the mesh, one considers the functions:

$$w(x) = \sum_{i=1}^{N} [w_i \tilde{\alpha}_i(x, y) + w_{xi} \tilde{\beta}_i(x, y) + w_{yi} \tilde{\gamma}_i(x, y)], \tag{21}$$

where w_i, w_{xi}, w_{yi} are parameters representing respectively the value of the function w, of its derivative with respect to x and of its derivative with respect to y at a point P_i. The set $E_{\mathscr{D}}$ of the functions w defined on S and relative to the decomposition $\mathscr{D}$ is a subset of Θ (see Section 1) of finite dimension $3N$, a basis being $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i$.

The basic functions associated to the point P_1: $\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1$ are constructed from the functions $\alpha_k, \beta_k, \gamma_k, \varrho_k$, $k = 1, 2, 3$. Let us determine at first the properties of the functions $\alpha_1, \beta_1, \gamma_1, \varrho_1$: 1) they are defined on the rectangular triangle (0, 0), (1, 0), (0, 1); 2) they are of class C^1 and of partial derivatives of second order piecewise continuous square integrable; 3) they verify the relations:

$$\begin{aligned} &\alpha_1(0, 0) = 1, \quad \alpha_{1x'}(0, 0) = 0, \quad \alpha_{1y'}(0, 0) = 0, \\ &\beta_1(0, 0) = 0, \quad \beta_{1x'}(0, 0) = 1, \quad \beta_{1y'}(0, 0) = 0, \\ &\gamma_1(0, 0) = 0, \quad \gamma_{1x'}(0, 0) = 0, \quad \gamma_{1y'}(0, 0) = 1; \end{aligned} \tag{22}$$

4) the form of the functions along the three sides of the triangle satisfy the conditions:

$$\begin{aligned}
&\alpha_1(s,0) = u_1(s), && \alpha_1(0,s) = u_1(s), && \alpha_1(s,1-s) = 0,\\
&\alpha_{1y'}(s,0) = 0, && \alpha_{1x'}(0,s) = 0, && \alpha_{1n}(s,1-s) = 0;\\
&\beta_1(s,0) = u_2(s), && \beta_1(0,s) = 0, && \beta_1(s,1-s) = 0,\\
&\beta_{1y'}(s,0) = 0, && \beta_{1x'}(0,s) = u_{2s}(s), && \beta_{1n}(s,1-s) = 0;\\
&\gamma_1(s,0) = 0, && \gamma_1(0,s) = u_2(s), && \gamma_1(s,1-s) = 0,\\
&\gamma_{1y'}(s,0) = u_{2s}(s), && \gamma_{1x'}(0,s) = 0, && \gamma_{1n}(s,1-s) = 0;\\
&\varrho_1(s,0) = 0, && \varrho_1(0,s) = 0, && \varrho_1(s,1-s) = 0,\\
&\varrho_{1y'}(s,0) = 0, && \varrho_{1x'}(0,s) = 0, && \varrho_{1n}(s,1-s) = \sqrt{2}\,u_{1s}(s).
\end{aligned} \tag{23}$$

In Fig. 5, we have drafted the value of the function along the three sides of the triangle; the value of the normal derivative has been indicated by an arrow. The functions u_1 and u_2 have the following properties: defined on the interval $[0, 1]$, they are of class C^1 with second piecewise continuous derivative square integrable; they verify the relations (see Fig. 5):

$$\begin{aligned}
&u_1(0) = 1, \quad u_{1s}(0) = 0, \quad u_1(1) = 0, \quad u_{1s}(1) = 0;\\
&u_1(s) + u_1(1-s) = 1, \quad 0 \leqq s \leqq 1;\\
&u_2(0) = 0, \quad u_{2s}(0) = 1, \quad u_2(1) = 0, \quad u_{2s}(1) = 0.
\end{aligned} \tag{24}$$

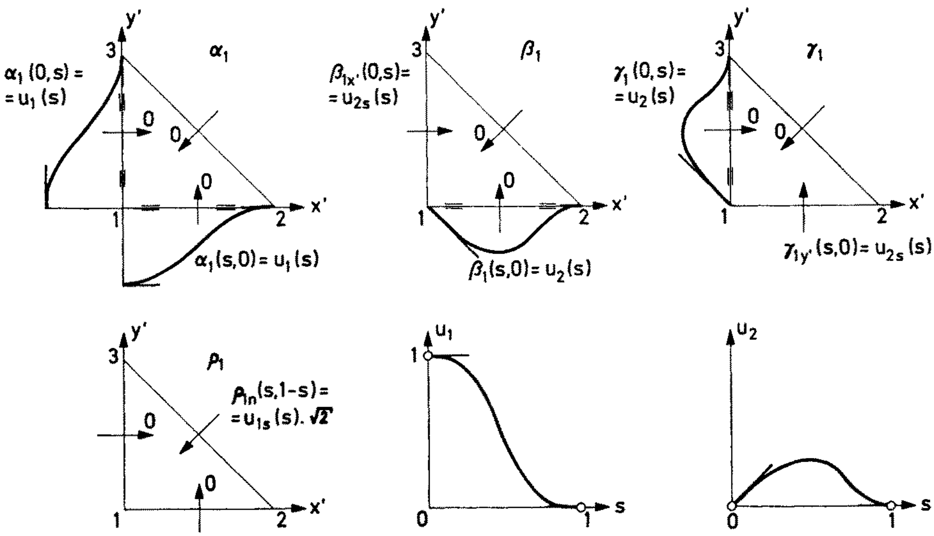

Fig. 5

Set:

$$\begin{aligned}
&\alpha_2(x',y') = \alpha_1(x'',y''), \quad \beta_2(x',y') = \beta_1(x'',y''),\\
&\gamma_2(x',y') = \gamma_1(x'',y''), \quad \varrho_2(x',y') = \varrho_1(x'',y''),
\end{aligned}$$

with

$$x'' = y', \quad y'' = 1 - x' - y';$$

$$\begin{aligned}
&\alpha_3(x',y') = \alpha_1(x''',y'''), \quad \beta_3(x',y') = \beta_1(x''',y'''),\\
&\gamma_3(x',y') = \gamma_1(x''',y'''), \quad \varrho_3(x',y') = \varrho_1(x''',y'''),
\end{aligned} \tag{25}$$

with

$$x''' = 1 - x' - y', \quad y''' = x'.$$

$\tilde{\alpha}_1^k, \tilde{\beta}_1^k, \tilde{\gamma}_1^k$ are the restrictions of functions $\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1$ on the k-th triangle of the polygon formed of triangles admitting P_1 for vertex:

$$\tilde{\alpha}_1(x, y) = \tilde{\alpha}_1^k(x, y), \quad \tilde{\beta}_1(x, y) = \tilde{\beta}_1^k(x, y), \quad \tilde{\gamma}_1(x, y) = \tilde{\gamma}_1^k(x, y), \tag{26}$$

on the k-th triangle of the polygon, $k = 1, 2, \ldots, NT$,
where NT is the number of triangular elements surrounding the point P_1. $\tilde{\alpha}_1^k, \tilde{\beta}_1^k, \tilde{\gamma}_1^k$ are constructed from the functions $\alpha_1, \beta_1, \gamma_1, \varrho_2, \varrho_3$. We choose the typical triangle $P_1 P_2 P_3$ for this construction.

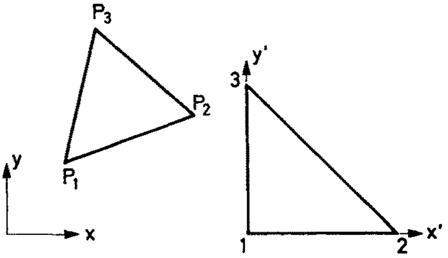

Fig. 6

Let:

$$\boldsymbol{x} = A\boldsymbol{x}' + \boldsymbol{z},$$

with:

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \boldsymbol{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \boldsymbol{z} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad A = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \tag{27}$$

be a linear application; the points 1, 2, 3 of the plane (x', y') (see Fig. 6) is mapped into the points P_1, P_2, P_3 of the plane (x, y). $\tilde{\alpha}_1^k, \tilde{\beta}_1^k, \tilde{\gamma}_1^k$ take the form:

$$\begin{aligned}
\tilde{\alpha}_1^k(x, y) &= \alpha_1(x', y') + \frac{C_1}{L_2^2}\varrho_2(x', y') + \frac{C_1}{L_3^2}\varrho_3(x', y'), \\
\tilde{\beta}_1^k(x, y) &= (x_2 - x_1)\beta_1(x', y') + (x_3 - x_1)\gamma_1(x', y') \\
&\quad - \frac{(y_3 - y_1)\Delta}{L_2^2}\varrho_2(x', y') + \frac{(y_2 - y_1)\Delta}{L_3^2}\varrho_3(x', y'), \\
\tilde{\gamma}_1^k(x, y) &= (y_2 - y_1)\beta_1(x', y') + (y_3 - y_1)\gamma_1(x', y') \\
&\quad + \frac{(x_3 - x_1)\Delta}{L_2^2}\varrho_2(x', y') - \frac{(x_2 - x_1)\Delta}{L_3^2}\varrho_3(x', y'), \\
\boldsymbol{x} &= A\boldsymbol{x}' + \boldsymbol{z}.
\end{aligned} \tag{28}$$

The procedure for the other triangles admitting P_1 for vertex is identical; a special linear application exists for each other triangle. So, we obtain the definitions of the basic functions associated to the point P_1. One verifies that the continuity of the basic functions and their partial derivatives of first order is realized. Moreover, the normal derivative of $\tilde{\alpha}_1$ along the sides of the triangles is zero. Leaving out the two last terms of the expressions $\tilde{\beta}_1^k, \tilde{\gamma}_1^k$, the continuity of the functions $\tilde{\beta}_1, \tilde{\gamma}_1$ and their partial derivatives of first order is still verified; then, the normal derivative along the sides of the triangles is proportional to the tangential derivative; however with this limitation, one cannot get the convergence.

The expression of w on the typical triangle $P_1 P_2 P_3$ is given by the expression:

$$w(x, y) = \xi^T(x', y')\, T^T \boldsymbol{w},$$

with

$$\begin{aligned} \xi^T &= (\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3, \varrho_1, \varrho_2, \varrho_3), \\ \boldsymbol{w}^T &= (w_1, w_{x1}, w_{y1}, w_2, w_{x2}, w_{y2}, w_3, w_{x3}, w_{y3}), \\ \boldsymbol{x} &= A\boldsymbol{x}' + \boldsymbol{z}. \end{aligned} \tag{29}$$

The matrix T is given in the Table. Thus, the expression of w includes three distinct terms: 1) $\xi(x', y')$ depends only from the functions defined on the rectangular triangle of the plane (x', y'), 2) T characterizes geometrically the element $P_1 P_2 P_3$, 3) $\boldsymbol{w}$ specifies the parameters associated to the three points of the element. The form of w proves to be specially advantageous for the numerical calculations.

Let $E_{\mathscr{D}}$ be the set of functions w whose expression on a triangle has the form (29).

Property 2. For any regular sequence of decomposition $\mathscr{D}_1, \mathscr{D}_2, \ldots$ and for each function $f \in \Phi$, of class C^3, defined on I, there exists a sequence $w_1 \in E_1$, $w_2 \in E_2, \ldots$ such that:

$$\begin{aligned} \lim_{k\to\infty} \iint_S [(f - w_k)^2 + (f_x - w_{kx})^2 + (f_y - w_{ky})^2 + (f_{xx} - w_{kxx})^2 \\ + (f_{yy} - w_{kyy})^2 + (f_{xy} - w_{kxy})^2]\, dx\, dy = 0. \end{aligned} \tag{30}$$

Theorem 2. The Property 2 is verified if and only if the functions $\alpha_k, \beta_k, \gamma_k, \varrho_k$ satisfy the six relations:

$$\begin{aligned} &\sum_{k=1}^{3} (\alpha_k + \varrho_k) && = 1, \\ &\alpha_2 + \beta_1 - \beta_2 - \gamma_2 + \gamma_3 + \varrho_1 - \varrho_2 + \varrho_3 && = x', \\ &\alpha_3 + \beta_2 - \beta_3 + \gamma_1 - \gamma_3 + \varrho_1 + \varrho_2 - \varrho_3 && = y', \\ &\tfrac{1}{2}\alpha_2 - \beta_2 - \gamma_2 + \tfrac{1}{2}\varrho_1 + \varrho_3 && = \tfrac{1}{2}x'^2, \\ &\tfrac{1}{2}\alpha_3 - \beta_3 - \gamma_3 + \tfrac{1}{2}\varrho_1 + \varrho_2 && = \tfrac{1}{2}y'^2, \\ &\beta_2 + \gamma_3 + \tfrac{1}{2}\varrho_1 - \tfrac{1}{2}\varrho_2 - \tfrac{1}{2}\varrho_3 && = x' y'. \end{aligned} \tag{31}$$

In order to prove the necessity, one can choose the particular domain S: $0 \leq x$, $y \leq 1$ divided into isoceles triangles (see Fig. 7); relations (31) are obtained from

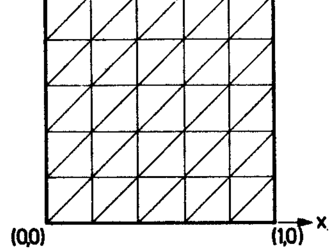

Fig. 7

Table

$T =$

1										$\frac{C_1}{L_2^2}$	$\frac{C_1}{L_3^2}$
	$x_2 - x_1$	$x_3 - x_1$								$-\frac{(y_3 - y_1)\Delta}{L_2^2}$	$\frac{(y_2 - y_1)\Delta}{L_3^2}$
	$y_2 - y_1$	$y_3 - y_1$								$\frac{(x_3 - x_1)\Delta}{L_2^2}$	$-\frac{(x_2 - x_1)\Delta}{L_3^2}$
			1						$\frac{C_2}{L_1^2}$		$\frac{C_2}{L_3^2}$
				$x_3 - x_2$	$x_1 - x_2$				$\frac{(y_3 - y_2)\Delta}{L_1^2}$		$-\frac{(y_1 - y_2)\Delta}{L_3^2}$
				$y_3 - y_2$	$y_1 - y_2$				$-\frac{(x_3 - x_2)\Delta}{L_1^2}$		$\frac{(x_1 - x_2)\Delta}{L_3^2}$
						1			$\frac{C_3}{L_1^2}$	$\frac{C_3}{L_2^2}$	
							$x_1 - x_3$	$x_2 - x_3$	$-\frac{(y_2 - y_3)\Delta}{L_1^2}$	$\frac{(y_1 - y_3)\Delta}{L_2^2}$	
							$y_1 - y_3$	$y_2 - y_3$	$\frac{(x_2 - x_3)\Delta}{L_1^2}$	$-\frac{(x_1 - x_3)\Delta}{L_2^2}$	

the condition (30) for $f=1$, x, y, $\frac{1}{2}x^2$, $\frac{1}{2}y^2$, xy. In order to prove the sufficiency of the condition, one constructs by an interpolation method a particular sequence w_k satisfying (30).

Remarks. 1. The Theorem 2 can also take the form: the Property 2 is satisfied if and only if the functions 1, x, y, x^2, y^2, xy belong to the subspace $E_{\mathscr{D}}$ for each decomposition of the domain S.

2. The relations (31) imply that the normal derivative of the three basic functions along the sides of the triangles is a linear function.

3. It is easy to introduce boundary conditions. Consider for example the particular condition $f=0$ along a segment $\overline{Z}$ of slope tg α. For any decomposition, $\overline{Z}$ is composed of sides of the triangles and its endpoints are meshpoints. P_j being a meshpoint on $\overline{Z}$, we set:

$$w_j=0, \qquad w_{tj}=w_{xj}\cos\alpha+w_{yj}\sin\alpha=0, \tag{32}$$

where w_{tj} represents the derivative with respect to the direction $\overline{Z}$ at the point P_j. w can be written (we assume that $|\text{tg}\,\alpha|<1$):

$$\begin{aligned} w(x,y)=&\sum_{\substack{i\\ p_i\notin\overline{Z}}}[w_i\tilde{\alpha}_i(x,y)+w_{xi}\tilde{\beta}_i(x,y)+w_{yi}\tilde{\gamma}_i(x,y)]\\ &+\sum_{\substack{j\\ p_j\in\overline{Z}}}w_{yj}(-\text{tg}\,\alpha\,\tilde{\beta}_j+\tilde{\gamma}_j).\end{aligned} \tag{33}$$

Then $w(P)=0$ for $P\in\overline{Z}$. Other conditions ($f=0$ at a point, $f=f_x=f_y=0$ along a segment of straightline) and combinations of conditions can be considered. For all "reasonable" conditions, Theorem 2 is valid.

Examples of Functions $\alpha_k, \beta_k, \gamma_k, \varrho_k$, $k=1, 2, 3$. Solutions of Eqs. (31) are given in [4, 5]. However our presentation is more synthetical and better adaptated to the numerical calculation. For these solutions:

$$\begin{aligned} \alpha_1&=1+2x'^3+2y'^3-3x'^2-3y'^2-4x'y'(1-x'-y')\\ &\quad+\tfrac{1}{3}\varrho_1-\tfrac{2}{3}\varrho_2-\tfrac{2}{3}\varrho_3,\\ \beta_1&=x'(1-x'-y')^2+\tfrac{1}{2}x'y'(1-x'-y')+\tfrac{1}{12}\varrho_1+\tfrac{5}{12}\varrho_2-\tfrac{3}{12}\varrho_3,\\ \gamma_1&=y'(1-x'-y')^2+\tfrac{1}{2}x'y'(1-x'-y')+\tfrac{1}{12}\varrho_1-\tfrac{3}{12}\varrho_2+\tfrac{5}{12}\varrho_3.\end{aligned} \tag{34}$$

For the solution of reference [5]:

$$\varrho_1=\frac{-6x'^2y'^2(1-x'-y')}{(1-x')(1-y')}\,; \tag{35}$$

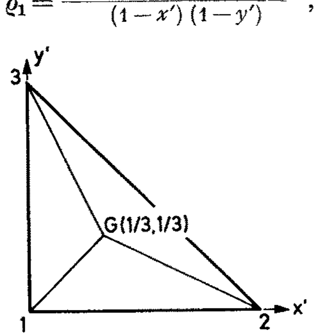

Fig. 8

for the solution of reference [4], one divides the triangle 1 2 3 into three small triangles 2 3 G, 3 1 G, 1 2 G (see Fig. 8):

$$\varrho_1 = \begin{cases} -(1-x'-y')\,[6x'y' + 5(1-x'-y')^2 - 3(1-x'-y')]: \text{ triangle } 2\,3\,G, \\ -x'^2(-x'+3y'): \text{ triangle } 3\,2\,G, \\ -y'^2(-y'+3x'): \text{ triangle } 1\,2\,G. \end{cases} \tag{36}$$

The formula (25) allows us to calculate the functions $\alpha_k, \beta_k, \gamma_k, \varrho_k$, $k=2, 3$.

Decomposition of the Domain S in Rectangles and Triangles. Basic functions for rectangular elements are much simpler than for triangles; their use is also easier. However triangular elements allow much more general domains. By associating basic functions at each point of the mesh, one can mix triangular and rectangular elements (see [9, 10]).

References

1. Mikhlin, S. G.: Variational methods in mathematical physics. New York: Macmillan 1964.
2. Engeli, M., Th. Ginsburg, H. Rutishauser, and E. Stiefel: Refined iterative methods for computation of the solution and the eigenvalues of self-adjoint boundary value problems. Mitteilungen aus dem Institut für angewandte Mathematik, Nr. 8, Zürich, 1959.
3. Courant, R.: Variational methods for the solution of problems of equilibrium and vibrations. Bull. Amer. Math. Soc. **49**, 1—23 (1943).
4. Clough, R. W., and J. L. Tocher: Finite element stiffness matrices for analysis of plates in bending. Proc. Conf. Matrix Methods in Struct. Mech., Air Force Inst. of Tech., Wright Patterson A.F. Base, Ohio, 1965.
5. Zienkiewicz, O. C.: The finite element method in structural and continuum mechanics. New York: McGraw Hill 1967.
6. Ciarlet, P. G., M. H. Schultz, and R. S. Varga: Numerical methods of high-order accuracy for nonlinear boundary value problems. I. One dimensional problem. Numerische Mathematik **9**, 394—430 (1967).
7. Schultz, M. H., and R. S. Varga: *L*-splines. Numerische Mathematik **10**, 345—369 (1967).
8. Birkhoff, G., M. H. Schultz, and R. S. Varga: Piecewise Hermite interpolation in one and two variables with applications to partial differential equations. Numerische Mathematik **11**, 232—256 (1968).
9. Goël, J.-J.: Utilisation numérique de la méthode de Ritz, application au calcul de plaque, thèse de doctorat. Ecole polytechnique de l'Université de Lausanne, 1968.
10. — List of basic functions for numerical utilisation of Ritz's method, application to the problem of the plate. Ecole polytechnique de l'Université de Lausanne, 1968.

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