Numer. Math. 12, 384—393 (1968)

Asymptotic Properties of Minimum Norm and Optimal Quadratures*

ROBERT E. BARNHILL**

Received April 17, 1967

Summary. This paper contains three types of asymptotic results for certain quadratures applied to a Hilbert space of analytic functions. These results concern the following: bounds on the norm of a certain error functional; the convergence of the weights and nodes of a minimum norm quadrature to the weights and nodes of the corresponding Gaussian quadrature; and the convergence of optimal quadratures.

1. Introduction

The purpose of this paper is to discuss error bounds for certain quadratures defined on spaces of analytic functions. The function space is $L^2(E_\varrho)$, the space of functions analytic inside the ellipse E_ϱ such that $\iint_{E_\varrho} |f(z)|^2 dx\, dy$ exists, the integral being taken over the region enclosed by the ellipse. The ellipse E_ϱ has foci at ± 1, semimajor axis a, semiminor axis $b=(a^2-1)^{\frac{1}{2}}$ and $\varrho=(a+b)^2$. $L^2(E_\varrho)$ is a Hilbert space and the sequence $\{P_m^*(z)\}_{m=0}^{\infty}$, where $P_m^*(z)= 2[(m+1)/\pi]^{\frac{1}{2}}(\varrho^{m+1}-\varrho^{-m-1})^{-\frac{1}{2}} U_m(z)$, where $U_m(z)$ is the m-th Chebyshev polynomial of the second kind, is complete and orthonormal in $L^2(E_\varrho)$. For a discussion of $L^2(E_\varrho)$ the reader is referred to DAVIS [8]. An important property of the ellipses E_ϱ is that they approach the interval $[-1, 1]$ as $\varrho \to 1$. Thus a function analytic on $[-1, 1]$ can be continued to a function that is analytic in some E_ϱ.

Let $Q_n(f)=\sum_{k=1}^{n} A_k f(z_k)$ and $R_n(f)=\int_{-1}^{1} f - Q_n(f)$, where the A_k and the z_k are, in general, functions of n. The idea of using the Riesz Representation Theorem for Hilbert space to compute $\|R_n\|$ and then using the Schwarz inequality $|R_n(f)| \leqq \|R_n\| \cdot \|f\|$ is due to DAVIS [7]. DAVIS and RABINOWITZ [28], VALENTIN [25], YANAGIHARA [31], WILF [30] and the author [1—3, 5] have suggested the idea of minimizing $\|R_n\|$ with respect to the A_k and/or the z_k. Such rules are called minimum norm (MN) rules. A similar idea is to use the so-called hypercircle inequality or a modified form of it. The hypercircle inequality appeared first in SYNGE [24], and then in GOLOMB and WEINBERGER [10] and DAVIS [8], the differences in the formulas of the latter two references resulting from certain orthonormalizations. Modi-

---

* The research reported in this document was supported by the Office of Naval Research, Contract Nonr 562(36) to Brown University and by the National Science Foundation, Grant GP 5906 to the University of Utah.

** Department of Mathematics, University of Utah, Salt Lake City, Utah. On leave during the 1966—67 academic year at the Division of Applied Mathematics, Brown University, Providence, Rhode Island.

fications have been discussed by the author [5]. The quadrature rules derived from the hypercircle inequality and its variants are called optimal rules.

There are many bounds available for $|R_n(f)|$, most of which have the feature that $R_n(f)$ can be written as the product of a term involving f and of a term independent of f, the Peano-Sard Theorem being a classical example. For quadrature rules, this theorem has been generalized to involve derivatives (of f) of lesser order by SARD [19, 20]; a discussion is also given in STROUD and SECREST [23]. It has also been generalized to involve higher-order derivatives. This work amounts to finding certain Euler's expansions for the Peano-Sard kernel. The general idea and its application to the trapezoidal and Simpson's rules is given by KRYLOV [15]; it is extended to Romberg integration by MEINGUET [17]. This last article also has a good general discussion of estimating the remainders of linear approximation rules.

The results of this paper include estimates of $\|R_n^{MN}\|$ in terms of n, the number of evaluation points and similar estimates for a composite MN rule. These estimates are obtained by comparison with the corresponding results for the remainder of Gaussian quadrature with n points, R_n^G. The estimates of $\|R_n^G\|$ are obtained by a method which is in spirit similar to that used by HÄMMERLIN [12—14]. HÄMMERLIN obtained results for the norms of certain Newton-Cotes rules for two function spaces, H_2 and $L^2(E_\varrho)$. The first of these is the Hardy space of analytic functions square summable on C_r, C_r being the circle centered at the origin with radius $r>1$. The complete orthonormal sequence is simpler for H_2, being essentially the complex monomials. HÄMMERLIN achieved his results by the use of special properties of the Euler-Maclaurin expansions of the Newton-Cotes rules and these were generalized by Meinguet to Romberg integration. The proofs involved showing that the Euler-Maclaurin expansions of certain functions (the even members of the complete sequences) were alternating series, so that the error in truncating was bounded by the first neglected term. A similar result for Gaussian quadratures is apparently unknown, although related results are given in an article by STENGER [22].

Results are given concerning the behavior of the A_k and z_k in a MN or an optimal rule, as $\varrho\to\infty$. The proof of this theorem follows along the lines of one by VALENTIN [25] for a Hilbert space of analytic functions similar to $L^2(E_\varrho)$. The result is that the A_k and z_k, for a given n, converge to the corresponding A_k^G and z_k^G of the Gaussian quadrature with the same n. This theorem is capable of rather wide generalization, as Valentin points out. This result also implies that if all $\varrho>1$ are considered, then $\|R_n^{MN}\|\leqq\|R_n^G\|$ is a sharp result.

The last theoretical results of the paper concern the convergence of the bounds given by the hypercircle inequality. A theorem giving sufficient conditions that the limit of the radii of the hypercircles be zero is given and this theorem yields various corollaries pertaining to quadrature rules for analytic functions.

2. Asymptotic Properties of $\|R_n^{MN}\|$

The first theorem gives an upper bound on $\|R_n^G\|$, where R_n^G is the remainder of the n-point Gaussian quadrature.

Theorem 1. For the space $L^2(E_\varrho)$, $\|R_n^G\| < \gamma(n) \cdot \beta(n, \varrho) \cdot \Sigma(n, \varrho)$ where

$$\gamma(n) = \frac{2^{2n+1}}{(2n+1)\,(2n)!} \left\{\frac{(n!)^2}{(2n)!}\right\}^2 \frac{1}{1 \cdot 3 \ldots (4n+1)},$$

$$\beta(n, \varrho) = \{4/[\pi \varrho (1 - p^{-4n-2})]\}^{\frac{1}{2}},$$

$$\Sigma(n, \varrho) = \left\{\sum_{k=n}^{\infty} \frac{(2k+1)^3 [(2k+1)^2 - 1]^{4n}}{(\varrho^2)^k}\right\}^{\frac{1}{2}}.$$

Proof. Firstly,

$$\|R_n^G\|^2 = \sum_{m=0}^{\infty} |R_n^G(P_m^*)|^2,$$

and

$$|R_n^G(P_m^*)|^2 = \alpha(m, \varrho)\, [R_n^G(U_m)]^2,$$

where

$$\alpha(m, \varrho) = 4(m+1)/[\pi(\varrho^{m+1} - \varrho^{-m-1})].$$

By the traditional remainder formula,

$$R_n^G(U_m) = \frac{2^{2n+1}}{(2n+1)\,(2n)!} \left\{\frac{(n!)^2}{(2n)!}\right\}^2 U_m^{(2n)}(\eta), \qquad \eta \text{ in } (-1, 1).$$

Now

$$|U_m^{(k)}(x)| \leqq \frac{(m+1)\,[(m+1)^2 - 1] \ldots [(m+1)^2 - k^2]}{1 \cdot 3 \ldots (2k+1)} \quad \text{on } [-1, 1]$$

with equality at ± 1, $k \leqq m$ (see the reference in HÄMMERLIN [14] to TODD). Hence

$$|R_n^G(P_m^*)|^2 \leqq \alpha(m, \varrho) \left\{\frac{2^{2n+1}}{(2n+1)\,(2n)!} \left\{\frac{(n!)^2}{(2n)!}\right\}^2\right\}^2$$
$$\cdot \left\{\frac{(m+1)\,[(m+1)^2 - 1] \ldots [(m+1)^2 - (2n)^2]}{1 \cdot 3 \ldots (4n+1)}\right\}^2 = [\gamma(n)]^2\, \alpha(m, \varrho)\, \psi(m),$$

where $\psi(m) = \{(m+1)\,[(m+1)^2 - 1] \ldots [(m+1)^2 - (2n)^2]\}^2$ and

$$\gamma(n) = \frac{2^{2n+1}}{(2n+1)\,(2n)!} \left\{\frac{(n!)^2}{(2n)!}\right\}^2 \frac{1}{1 \cdot 3 \ldots (4n+1)}$$

Since $R_n^G(U_m) = 0$ for $m = 0, 1, \ldots, 2n-1$ and for $m = 2k+1$; $k = n, n+1, \ldots,$ we have

$$\|R_n^G\|^2 \leqq [\gamma(n)]^2 \sum_{m=2n}^{\infty} \alpha(m, \varrho)\, \psi(m);$$

that is,

$$\|R_n^G\|^2 \leqq [\gamma(n)]^2 \sum_{k=n}^{\infty} \alpha(2k, \varrho)\, \psi(2k).$$

Now

$$\sum_{k=n}^{\infty} \alpha(2k, \varrho)\, \psi(2k)$$
$$= \sum_{k=n}^{\infty} \frac{4}{\pi} \frac{(2k+1}{(\varrho^{2k+1} - \varrho^{-2k-1})} \{(2k+1)\,[(2k+1)^2 - 1] \ldots [(2k+1)^2 - (2n)^2]\}^2$$

and

$$\varrho^{2k+1} - \varrho^{-(2k+1)} = \varrho^{2k+1}(1 - \varrho^{-2(2k+1)}) > \varrho^{2k+1}(1 - \varrho^{-2(2n+1)})$$

for $k > n$, so that

$$\sum_{k=n}^{\infty} \alpha(2k,\varrho)\,\psi(2k) < \frac{4}{\pi}\,\frac{1}{1-\varrho^{-4n-2}} \sum_{k=n}^{\infty} \frac{(2k+1)^3}{\varrho^{2k+1}}\,[(2k+1)^2-1]^{4n}$$

$$= \frac{4}{\pi}\left(\frac{1}{1-\varrho^{-4n-2}}\right) \sum_{k=n}^{\infty} \frac{(2k+1)^3\,[(2k+1)^2-1]^{4n}}{\varrho^{2k+1}}$$

which implies the desired result. Q.E.D.

Corollary 1. Let R_n^{MN} denote the remainder of the MN quadrature with n points. Then $\|R_n^{MN}\| < \gamma(n)\,\beta(n,\varrho)\,\Sigma(n,\varrho)$. (This follows directly from the fact that $\|R_n^{MN}\| = \min\limits_{A,z} \|R_n\| \leqq \|R_n^G\|$.)

Corollary 2. Consider a composite MN quadrature that uses n points on s subintervals of $[-1, 1]$, each subinterval being of length $h = 2/s$. Then $\|R_{ns}^{MN}\| \leqq a \cdot \gamma^*(n)\,\beta(n,\varrho)\,\Sigma(n,\varrho)$, where $\gamma^*(n) = h^{2n+1}\,\gamma(n)$.

Proof. The n-point Gaussian quadrature remainder on the interval $[a, b]$, $\overline{R}_m^G$, is given by

$$\overline{R}_n^G(f) = (b-a)^{2n+1} \frac{(n!)^4}{(2n+1)\,[(2n)!]^3}\, f^{(2n)}(\varepsilon), \qquad a < \varepsilon < b.$$

Hence the only change in the bound on $\|R_n^G\|$ of Theorem 1 is that $\gamma(n)$ is replaced by

$$\gamma^*(n) \equiv (b-a)^{2n+1} \cdot \gamma(n),$$

where $b - a = h$. However, $\|\overline{R}_n^G\|$ is taken over a subinterval of length h. Therefore

$$\|R_{ns}^G\| \leqq s \cdot \|\overline{R}_n^G\| \quad \text{and} \quad \|R_{ns}^{MN}\| \leqq \|R_{ns}^G\|$$

implies the desired result. Q.E.D.

3. Numerical Results

For the function $f(z) = z \sin z \cos z$, $a = 1.5$, and three point rules, we have the following numerical results. The optimal quadrature based on the 3 Gaussian nodes has error 0.51677(−02)[1] with error bound 0.41135. The minimum norm and optimal (with respect to nodes) quadratures have error 0.47762(−02) with optimal error bound 0.33212(−01). Thus, the minimization with respect to the nodes yields a slightly better error, but an error bound that is better by an order of magnitude. A second example is the function $f(z) = 1/(1+z^2)$, with $a = 1.5$. The four point Gauss rule yields an error of 0.21689(−02), while the optimal rule has an error of 0.21323(−02) and an error bound of 0.32271(−02). Thus, the optimal error bound is close to the error in this example. For computational details and more numerical results, the reader is referred to [2—4, 33].

The bound on $\|R_n^G\|$ provided by Theorem 1 has been calculated for $n = 3$ and 4 and $a = 1.5$.[2] From a previous paper [Table 3 in reference 2], we have that, for $a = 1.5$, $\|R_3^G\| = 0.10364(-01)$ and $\|R_4^G\| = 0.17416(-02)$. ϱ is 6.8541 for

[1] Floating point notation.

[2] The author is indebted to Mr. Gregory M. Nielson for the new computations in this paragraph.

$a = 1.5$, and, for this ϱ, $\gamma(3)\,\beta(3,\varrho)\,\Sigma(3,\varrho) = 3.9547$, and $\gamma(4)\,\beta(4\,\varrho)\,\Sigma(4,\varrho) = 5.0249$. Since these bounds are so much larger (essentially, two orders of magnitude) than $\|R_n^G\|$ itself, it was conjectured that the upper bound

$$[(2k+1)^2-1]\ldots[(2k+1)^2-(2n)^2] < [(2k+1)^2-1]^{2n}$$

was too large and so the above right-hand bound was replaced by the left-hand side. This decreased the bound for $n=3$ to 2.5306 and the bound for $n=4$ to 2.7386. These bounds are still too conservative and so we must recommend to users of these quadrature rules that they use $\|R_n\|$ itself, rather than bounds on it.

4. Asymptotic Behavior of the Quadratures as $\rho \to \infty$

We shall prove the result stated in the introduction concerning the behavior of the quadrature weights and points of MN and optimal rules as $\varrho \to \infty$, i.e., as the functions considered approach entire functions.

Theorem 2. Assume that the n-point quadratures Q_n that follow are defined on $L^2(E_\varrho)$.

(1) If Q_n is a MN quadrature with fixed z_k, then the weights of Q_n converge, as $\varrho \to \infty$, to the weights of the interpolatory quadrature based on the z_k.

(2) If Q_n is a MN quadrature with the z_k variable, then the weights and the base points converge, as $\varrho \to \infty$, to the weights and base points of the n-point Gaussian quadrature.

(3) If Q_n is an optimal quadrature with fixed z_k, then the conclusion of statement (1) holds.

(4) If Q_n is an optimal quadrature of minimal norm with respect to the z_k, then the conclusion of statement (2) holds.

Proof. We prove statement (2) first. The first step is to show that

$$\lim_{\varrho\to\infty} |R_n^{MN}(U_m)| = 0 \quad \text{for} \quad m = 0, 1, \ldots, 2n-1.$$

By definition, $\|R_n^{MN}\|^2 \leqq \|R_n^G\|^2$, i.e.

$$\sum_{m=0}^{\infty} \frac{4(m+1)}{\pi(\varrho^{m+1}-\varrho^{-m-1})}\,[R_n^{MN}(U_m)]^2 \leqq \sum_{m=2n}^{\infty} \frac{4(m+1)}{\pi(\varrho^{m+1}-\varrho^{-m-1})}\,[R_n^G(U_m)]^2. \tag{8}$$

Multiplying this inequality by $\varrho^{2n}-\varrho^{-2n}$ and deleting all but the first term of the series on the left-hand side, we get

$$(\varrho^{2n}-\varrho^{-2n})\,\frac{4}{\pi(\varrho-\varrho^{-1})}\,[R_n^{MN}(U_0)]^2 \leqq \sum_{m=2n}^{\infty} \frac{4(m+1)}{\pi}\,\frac{(\varrho^{2n}-\varrho^{-2n})\,[R_n^G(U_m)]^2}{(\varrho^{m+1}-\varrho^{-m-1})}.$$

Taking limits of both sides as $\varrho \to \infty$, we find that $\lim_{\varrho\to\infty} \varrho^{2n-1}[R_n^{MN}(U_0)]^2 = 0$, which implies that $\lim_{\varrho\to\infty} [R_n^{MN}(U_0)]^2 = 0$. We include successive terms on the left-hand side of (8) to get $\lim_{\varrho\to\infty} \varrho^{2n-m-1}[R_n^{MN}(U_m)]^2 = 0$, $m = 0, 1, \ldots, 2n-1$, so that $\lim_{\varrho\to\infty} [R_n^{MN}(U_m)]^2 = 0$, $m = 0, 1, \ldots, 2n-1$.

We now recall the algebraic derivation of Gaussian quadratures given in KOPAL [32]. That is, we write the $2n \times 2n$ nonlinear system of equations to be satisfied if the Gaussian quadrature is to be exact for polynomials of degree not greater than $2n-1$. We define $p_n(x)=\prod_{i=1}^{n}(x-x_i)=x^n+\sum_{i=0}^{n-1} c_i x^i$, where the x_i are the Gauss nodes and, by appropriate combinations of equations in the $2n\times 2n$ system, we obtain a system linear in the coefficients c_i. We can solve for the c_i, since the determinant involved is a Vandermonde and we are assuming that the x_i are distinct. The roots of a polynomial are continuous functions of its coefficients and so we can get the Gaussian x_i. With the x_i in hand, the quadrature weights A_i are found by solving the appropriate linear system.

We recall now that $R_n^{MN}(U_m)\to 0$ as $\varrho\to\infty$, $m=0,\ldots,2n-1$ and that U_m is a polynomial of degree m, so that these conditions are equivalent to $R_n^{MN}(z^m)\to 0$ as $\varrho\to\infty$, $m=0,\ldots,2n-1$. The latter conditions are equivalent to the original nonlinear system for Gaussian quadrature, if perturbation terms that go to zero as $\varrho\to\infty$ are added to the constant vector. Then the argument for the algebraic derivation of the Gaussian quadratures can be followed to obtain statement (2). For the calculations involved in the above discussion, the reader is referred to a similar case in VALENTIN [25].

The argument for statement (1) is analogous. We show that, for fixed z_k, $R_n^{MN}(z^m)\to 0$ as $\varrho\to\infty$ for $m=0,\ldots,n-1$ and the rest of the argument is the same as the last part of the preceding one.

If Q_n is an optimal quadrature with the z_k given, then the hypercircle inequality can be written as follows:

$$|R_n(f)|\leqq\|L\|_U(\|f\|^2-\|u\|^2)^{\frac{1}{2}},$$

where u is the center of the hypercircle and U is a subspace parallel to the hypercircle. However, VALENTIN has shown that the A_k calculated for the hypercircle inequality are the same as the A_k that minimize $\|R_n\|$ with fixed z_k. (He proved this by showing that the minimum property of the appropriate Fourier expansion in terms of an orthonormal sequence was equivalent to setting up the corresponding normal equations.) This means that statement (3) holds, by application of statement (1). Since the optimal quadrature in (4) is defined to have the property that $\|L\|_U$ is a minimum with respect to the z_k, statement (4) follows from statement (2). Q.E.D.

Remark 1. Since $\|R_n\|$ is a continuous function of the vectors A and z, Corollary 1 to Theorem 1 is the strongest statement that can be made about $\|R_n^{MN}\|$ for all ϱ. In fact, $\|R_n^{MN}\|$ is the same as $\|R_n^G\|$ to several digits for ϱ that are not large. This can be seen by comparing, for example, Table 3 in reference [2] with Table 3 in reference [3], from which we note that, for $a=1.5$ ($\varrho=6.9$), $\|R_4^{MN}\|$ differs from $\|R_4^G\|$ by about 0.5×10^{-6}.

Remark 2. Similar results can be proved for the space $L^2(C_r)$, C_r the unit circle. The formulas are considerably neater, because the monomials form a complete orthogonal system for $L^2(C_r)$. However, the ellipses E_ϱ collapse to the interval of integration $[-1,1]$ as $\varrho\to 1$, whereas the circles C_r do not, as $r\to 1$. Therefore, if we have a function analytic on $[-1,1]$ but with a nearby singularity, such as $f(z)=1/(0.01+z^2)$, then f is in no $L^2(C_r)$, $r>1$, but f is in $L^2(E_\varrho)$ for sufficiently small $\varrho>1$.

5. The Convergence of Optimal Quadratures

This section contains conditions under which certain upper bounds on the remainder of the optimal quadratures tend to zero as $n \to \infty$. We begin with a discussion of the hypercircle inequality which leads to a theorem applicable to spaces more general than $L^2(E_\varrho)$.

Assume that we are given n linearly independent bounded linear functionals $L_1, \ldots, L_n$ which are to approximate a given bounded linear functional L and also assume a uniform bound on the functions to be considered, $\|f\| \leqq r < \infty$. The hypercircle inequality yields a linear approximation of the form

$$L(f) \simeq \sum_{k=1}^{n} A_k L_k(f)$$

where $\sum A_k L_k(f)$ is $L(u)$, u being the approximation to f which is optimal in the sense that $L_k(f) = L_k(u)$, $k = 1, \ldots, n$, $\|u\| \leqq r$, and the upper bound given by the hypercircle inequality on $|L(f) - L(u)|$ is minimal among all such functions u. The hypercircle inequality is the following:

$$|L(f) - L(u_n)| \leqq \|L\|_{U(n)} \cdot [r^2 - \|u_n\|^2]^{\frac{1}{2}}$$

where we have indicated the dependence of u and the subspace U on $L_1, \ldots, L_n$. We define the radius of the hypercircle to be $r_n \equiv \|L\|_{U(n)} \cdot [r^2 - \|u_n\|^2]^{\frac{1}{2}}$. The question to be considered is the following: under what conditions does $r_n \to 0$ as $n \to \infty$? If we take the view that f is some given function, then this question amounts to asking if f is unique under the conditions that $L_k(f) = \alpha_k$, $k = 1, 2, \ldots$ and $\|f\| \leqq r$.

The statement of the next theorem is simplified if we recall that X^* is the space of bounded linear functionals defined on the normed linear space X. We define S_r to be the set of functions f such that $\|f\| \leqq r < \infty$.

Theorem 3. Let X be a Hilbert space with $L_1, L_2, \ldots$ linearly independent elements in X^* and let f be a given function in S_r. If the L_k are complete in X^*, then $\lim r_n = 0$.

Proof. It suffices to show that $\lim_n \|L\|_{U(n)} = 0$. Now $\{L_k\}$ complete implies $\{L_k\}$ closed and so $\|E_n\| \to 0$ as $n \to \infty$, where E_n is the difference between L and its best linear approximation by $L_1, \ldots, L_n$. But $\|L\|_{U(n)} = \|E_n\|_{U(n)} \leqq \|E_n\|$, so that $\|L\|_{U(n)} \to 0$. Q.E.D.

Let B be a region (an open, connected set) in the complex plane. Then $L^2(B)$ is defined analogously to $L^2(E_\varrho)$ [8]. WALSH and DAVIS [27] considered the following questions: given the linearly independent L_k in $[L^2(B)]^*$ and constants $\beta_1, \beta_2, \ldots$, does there exist a function f in $L^2(E_\varrho)$ such that $L_k(f) = \beta_k$, $k = 1, \ldots$ and is this f unique? Their theorem that answers these questions is the following:

Theorem 4 (WALSH and DAVIS). Given the sequence $\{L_n\}$ linearly independent and in $[L^2(B)]^*$ and the constants $\beta_1, \beta_2, \ldots$, a necessary and sufficient condition that there exists a function f in $L^2(B)$ such that $L_n(f) = \beta_n$, $n = 1, 2, \ldots$ is that

$$\sum_{n=0}^{\infty} \left| \sum_{i=0}^{n} \Delta_{ni} \beta_i \right|^2 < \infty$$

where the Δ_{ni} come from a biorthogonalization of the L_n (see below). This solution, when it exists, is unique iff the L_n are complete.

The biorthogonalization is of the form

$$L_n^*(f) = \sum_{i=0}^{n} \Delta_{ni} L_i(f), \qquad L_n^*(\phi_m^*) = \delta_{mn},$$

where the ϕ_m^* are in $L^2(B)$ and the L_n^* in $[L^2(B)]^*$.

The above condition is then seen to be

$$\sum_{n=0}^{\infty} |L_n^*(f)|^2 < \infty.$$

Corollary. Let $L_k(g) = g(z_k)$, $k = 1, 2, \ldots$ be distinct point functionals defined on $L^2(B)$, B a region and let f be in S_r. If the sequence $\{z_k\}$ has a limit point in B, then $\lim r_n = 0$.

Proof. We remark that the z_k and $f(z_k)$ must satisfy the condition of Theorem 4 in order that f be in $L^2(B)$. Assuming that this condition is fulfilled, Theorem 3 implies that $r_n \to 0$, i.e., f is unique if the L_k are complete. But, by a uniqueness theorem for analytic functions, the z_k having an interior limit point implies that the L_k are complete [18]. Q.E.D.

We note that these theorems do not apply to cases such as the optimal quadratures based on the Gaussian base points, where a triangular scheme of the form

$$\begin{matrix} L_1^{(1)} & & \\ \vdots & & \\ L_n^{(1)} & \cdots & L_n^{(n)} \\ \vdots & & \end{matrix}$$

is involved. However, results can be formulated for the convergence of optimal quadratures based on points that occur in a cyclic order, such as the Newton-Cotes quadratures.

Another question of theoretical interest for $L^2(B)$ and point functionals is the following: assuming that the points have no interior limit point (of course, this implies at least one limit point on the boundary), how slowly must the points approach the boundary so that $g(z_k) = 0$, $k = 1, 2, \ldots$ implies $g \equiv 0$ on B? The answer to this question depends on the geometry of B and has apparently been satisfactorily resolved only for $B = U$, the open unit disc. For $L^2(U)$ the uniqueness question has been answered, the result being that if $\sum_{k=1}^{\infty} (1 - |z_k|)$ diverges, then the point functionals determine a unique f in $L^2(U)$. This result appears in LOKKI [16], who gives reference to F. and R. NEVANLINNA. The uniqueness question has been answered for the Hardy space H_2 [26] as follows: If g is in H_2 and vanishes in the points z_k of U, where $\prod_{k=1}^{\infty} |z_k|$ diverges, then $g \equiv 0$ on U. The divergence of $\pi|z_k|$ is equivalent to the divergence of $\sum (1 - |z_k|)$, so that the answer is the same for H_2 as for $L^2(U)$. The answer is also the same for the space H^∞ of functions analytic on U and bounded, in the sup norm, on U. For the spaces H_p, $0<p\leqq\infty$, the result can also be phrased in terms of a limiting boundary function as follows: let $g^*(e^{i\theta})=\lim_{r\to 1} g(r\,e^{i\theta})$, g in H_p. If $g^*=0$ on a set of the unit circle ($|z|=1$) that has positive measure, then $g\equiv 0$ on U [18].

6. Conclusions

The results contained in this paper are stated in Section 1 and thus need not be repeated here. At the beginning of the work that led to the results in Section 2, it was hoped that a bound on $\|R_n^{MN}\|$ in terms of n could be found directly. When this hope was not realized, it was decided to find a bound on $\|R_n^{MN}\|$ by finding one on $\|R_n^G\|$. The bound presented in Section 2 is found in a straightforward fashion and we might hope for a slightly better bound using an Euler-Maclaurin expansion of Gaussian quadrature. This would be analogous to the discussions given for the trapezoidal and Simpson's rules in KRYLOV [15] and HÄMMERLIN [12—14] and for Romberg integration by MEINGUET [17]. Unfortunately, for Gaussian quadrature the kernels of interest are not of one sign on the interval $[-1, 1]$. It appears that STENGER [22] has made an equivalent observation.

The results of Section 4 can be generalized to cubatures of analytic functions. If the space X is $L^2(E_\varrho\times E_\varrho)$ (defined below), then $\lim R_n=0$ if the functionals L_k are complete. $L^2(E_\varrho\times E_\varrho)$ is the space of functions $f(z, w)$ analytic inside $E_\varrho\times E_\varrho$ such that $\int|f|^2 dV$ exists, where the volume integral is taken over the four-dimensional real region enclosed by $E_\varrho\times E_\varrho$. Thus, for example, if the L_k are point functionals, $L_k(f)=f(z_k, w_k)$, and the points (z_k, w_k) have a limit point inside $E_\varrho\times E_\varrho$, and are not on an analytic hypersurface, then $\lim r_n=0$. Further applications of these results to cubatures will appear in a future paper [34].

Acknowledgements. The author is grateful to PHILIP DAVIS for setting up the year at Brown University during which most of this research was done, and for helpful discussions related to it. This research was supported in part by the Office of Naval Research under Contract Nonr 562(36) to Brown University and by the National Science Foundation under Grant GP 5906 to the University of Utah.

References

1. BARNHILL, R. E.: Complex quadratures with remainders of minimum norm. Numer. Math. **7**, 384—390 (1965).
2. —, and J. A. WIXOM: Quadratures with remainders of minimum norm I. Math. Comp. **21**, 66—75 (1967).
3. — — Quadratures with remainders of minimum norm II. Math. Comp. **21**, 382—387 (1967).
4. — Optimal quadratures in $L^2(E_\varrho)$ I. SIAM J. Num. Anal. **4**, 390—397 (1967).
5. — Numerical contour integration. U.S. Army Mathematics Research Center Report No. 519 (1964).
6. DAVIS, P. J.: Errors of numerical approximation for analytic functions. J. Rat. Mech. Anal. **2**, 303—313 (1953).
7. — Errors of numerical approximation for analytic functions. In: Survey of Numerical Analysis (ed. J. TODD). New York: McGraw-Hill Book Co. 1962.
8. — Interpolation and Approximation. New York: Blaisdell 1963.
9. —, and P. RABINOWITZ: On the estimation of quadrature errors for analytic functions. Math. Tables Aids Comp. **88**, 193—203 (1954).

10. GOLOMB, M., and H. F. WEINBERGER: Optimal approximation and error bounds. In: On numerical approximation (ed. R. E. LANGER). Madison: University of Wisconsin Press 1959.
11. — Lectures on theory of approximation. Argonne National Laboratory 1962.
12. HÄMMERLIN, G.: Über ableitungsfreie Schranken für Quadraturfehler. Numer. Math. **5**, 226—233 (1963).
13. — Über ableitungsfreie Schranken für Quadraturfehler. II. Ergänzungen und Möglichkeiten zur Verbesserung. Numer. Math. **7**, 232—237 (1965).
14. — Zur Abschätzung von Quadraturfehlern für analytische Funktionen. Numer. Math. **8**, 334—344 (1966).
15. KRYLOV, V. I.: Approximate calculation of integrals. New York: Macmillan 1966.
16. LOKKI, O.: Über eine Klasse von analytischen Funktionen. Skandinaviske matematikerkongres 213—224 (1946).
17. MEINGUET, J.: Methods for estimating the remainder in linear rules of approximation. Numer. Math. **8**, 345—366 (1966).
18. RUDIN, W.: Real and complex analysis. New York: McGraw-Hill Book Co. 1966.
19. SARD, A.: Best approximate integration formulas; best approximation formulas. Amer. J. Math. **71**, 80—91 (1949).
20. — Linear approximation. Math. Surveys 9, Amer. Math. Soc. 1963.
21. STALLEY, R.: A generalization of the geometric series. Amer. Math. Monthly **56**, 325—327 (1949).
22. STENGER, F.: Bounds on the error of Gauss-type quadratures. Numer. Math. **8**, 150—160 (1966).
23. STROUD, A. H., and D. SECREST: Gaussian quadrature formulas. Englewood Cliffs, N.J.: Prentice-Hall 1966.
24. SYNGE, J. L.: The hypercircle in mathematical physics. Cambridge 1957.
25. VALENTIN, R. A.: Applications of functional analysis to optimal numerical approximation for analytic functions. Ph. D. thesis Brown University 1965.
26. WALSH, J. L.: Interpolation and approximation by rational functions in the complex domain. Amer. Math. Soc. Colloq. Publ., vol. XX (1960).
27. —, and P. J. DAVIS: Interpolation and orthonormal systems. J. d'Anal. Math. **2**, 1—28 (1952).
28. DAVIS, P. J., and P. RABINOWITZ: Numerical integration. New York: Blaisdell Book Co. 1967.
29. NEHARI, Z.: Conformal mapping. New York: McGraw-Hill Book Co. 1952.
30. WILF, H. S.: Exactness conditions in numerical quadrature. Numer. Math. **6**, 315—319 (1964).
31. YANAGIHARA, H.: A new method of numerical integration of Gauss type. Bull. Fukuoka Gakugei Univ., vol. 6, part III, 19—25 (1956) [in Japanese].
32. KOPAL, Z.: Numerical analysis. London: Chapman & Hall 1961.
33. BARNHILL, R. E.: Optimal quadratures in $L^2(E_\varrho)$. II. SIAM J. Num. Anal. **4**, 534—541 (1967).
34. — An error analysis for numerical multiple integration. I. Math. Comp. **22**, 98—109, II. 286—292 (1968).

Prof. R. E. BARNHILL
Department of Mathematics
University of Utah
206 Mathematics Building
Salt Lake City, Utah 84112, USA