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Age-Dependent Population Diffusion with External Constraint*

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Abstract. We present a simple model for age dependent population diffusion when the dynamics is submitted to external constraints. Existence, uniqueness and dependence on the parameters of the solution are discussed.

Key words: Degenerating elliptic operators – Variational inequalities – Unilateral constraints – Population diffusion – Renewal equation

Introduction

In this paper a mathematical model of an age-dependent population with diffusion in a bounded set of $\mathbb{R}^3$ and with an external constraint is treated.

In this model, the dynamics of the population is described by a function $u(t, a, x)$ such that for every open set Ω of $\mathbb{R}^3$ and every interval $[a_1, a_2]$, the integral

$$\int_{a_1}^{a_2} da \int_{\Omega} u(t, a, x)\, dx$$

gives the number of individuals of age between a_1 and a_2 living at the time t in the region Ω. Thus $u(t, a, x)$ represents the density of the individuals of age a at the time t and at position x.

We assume that the population develops with a constraint depending on the environment as follows: the density remains less than or equal to a given function $\psi(t, a, x)$ and moreover, when it is strictly less than ψ, it is ruled by the usual partial differential equation (see e.g. [3], [4], [5] and references there):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - \Delta u + \mu u = f. \qquad (*)$$

1) Here $\mu(t, a, x)$ is the rate of mortality, characteristic of the species, that is considered as divergent to $+\infty$ as $a \to A$, where A is the maximal age for the species;

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2) $f(t, a, x)$ is a factor, possibly zero, that takes into account possible external increase of population.

We further assume that:

3) birth is described by the "*renewal equation*" (see e.g. [3], [4], and [5])

$$u(t, 0, x) = \int_0^A \beta(t, a, x) u(t, a, x)\, da,$$

where β represents the rate of fertility;

4) the initial density of population in known;

5) the population does not leave the region Ω, i.e.

$$\frac{\partial u}{\partial \eta} = 0, \qquad \text{on } \partial\Omega.$$

This problem can be solved in terms of variational inequalities and can be set into equations as follows:

Find a function u such that

$$u \leqslant \psi,$$

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u - \Delta u - f \leqslant 0 \\ &\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u - \Delta u - f\right)(u - \psi) = 0 \end{aligned}\right\} \quad t > 0, \quad 0 < a < A, \quad x \in \Omega$$

$$\frac{\partial u}{\partial \eta} = 0, \qquad t > 0, \quad 0 < a < A, \quad x \in \partial\Omega,$$

$$u(0, a, x) = u_0(a, x), \qquad 0 < a < A, \quad x \in \Omega,$$

$$u(t, 0, x) = \int_0^A \beta(t, a, x) u(t, a, x)\, da, \qquad t > 0, \quad x \in \Omega.$$

A problem of this kind with rigid control of birth, i.e. $u(t, 0, u) = b(t, x)$, has been studied by M. G. Garroni and L. Lamberti [2].

In this paper, using the results of existence and regularity of [2] and the methods of M. Langlais [6], [7] we obtain the existence and the uniqueness of the solution under weaker hypotheses than those of [2].

We also rediscover all the biologically intuitive properties connecting the density of the population to the other parameters of the problem.

The plan of the paper is the following: In §I we introduce notations, hypotheses and preliminary results. In §II we show existence, uniqueness and investigate the properties of solutions. §III contains the proofs.

§I. Notations, Hypotheses, and Preliminary Results

1. Notations

We denote by Ω a bounded open subset of $\mathbb{R}^N$, with regular boundary $\partial\Omega$ and generic element $x = (x_1, \ldots, x_N)$. Δ is the Laplacian and ∇ is the gradient

$$\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}; \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}\right).$$

$H^1(\Omega)$ and $H^2(\Omega)$ are the usual Sobolev spaces of order 1 and 2 (see [9], for instance). By $\langle\ ,\ \rangle$ we denote the duality between $H^1(\Omega)$ and its dual.

Let η be the unit exterior normal; the normal exterior derivative is defined by

$$\frac{\partial}{\partial \eta} = \sum_{i=1}^{N} \eta_i \frac{\partial}{\partial x_i}.$$

T and A are two strictly positive and finite real numbers; $t \in]0, T[$ and $a \in]0, A[$. $\mathcal{O}$ is the open set $]0, T[\times]0, A[$.

If H is a Hilbert space and U an open set of $\mathbb{R}^p$, $L^2(U; H)$ is the (Hilbert) space of measurable functions of U with values in H s.t. $\int_U \|v(y)\|_H^2\, dy < +\infty$.

If u is a real function, we denote by u^+ its positive and by u^- its negative parts (so that $u = u^+ - u^-$).

∂_t and ∂_a indicate partial differentiation in $\mathcal{D}'(\mathcal{O}; [H^1(\Omega)]')$.

2. *Hypotheses*

We consider a real-valued function μ on $\mathcal{O} \times \Omega$ such that:

$(\mu)_1$ $\qquad \mu \in C^0([0, T] \times [0, A[\times \bar{\Omega}), \qquad \mu(t, a, x) \geqslant 0 \quad \text{in} \quad \mathcal{O} \times \Omega;$

the behaviour of μ at $a = A$ is given by the divergency condition (see [5]):

$$(\mu)_2 \qquad \begin{cases} 0 < t < A, \quad x \in \Omega, \quad \lim_{a \to A} \int_0^t \mu(\tau, a - t + \tau, x)\, d\tau = +\infty, \\ A < t < T, \quad x \in \Omega, \quad \lim_{a \to A} \int_0^a \mu(t - a + \alpha, \alpha, x)\, d\alpha = +\infty; \end{cases}$$

we also assume that

$$(\mu)_3 \qquad \nabla\mu \in [L^\infty(\mathcal{O} \times \Omega)]^N.$$

Given a real valued function β on $\mathcal{O} \times \Omega$ such that

$$(\beta)_1 \qquad \beta \in L^\infty(\mathcal{O} \times \Omega),$$

$$(\beta)_2 \qquad \beta(t, a, x) \geqslant 0, \qquad \text{a.e. in } \mathcal{O} \times \Omega,$$

$$(\beta)_3 \qquad \sup_{(t,x) \in]0,T[\times \Omega} \int_{]0,A[} [\beta^2(t, a, x) + |\nabla\beta|^2(t, a, x)]\, da \leqslant c_1 < +\infty.$$

Remark 1. The main hypothesis is $(\mu)_2$. It ensures that the solution of the problem vanishes at $a = A$ (see Theorem 3). If μ is independent of t (and of x), it can be written more simply

$$\int_0^A \mu(a)\, da = +\infty;$$

$(\mu)_2$ means that the integral of μ is infinite on all line segments parallel to the first besectrix in the plane (t, a) whose end points are $a = 0$ and $a = A$, and $t = 0$ and $a = A$.

This amounts to the main modification of the hypotheses of [2], where μ was assumed to belong to L^p.

Hypotheses $(\mu)_3$ and $(\beta)_3$ are technical. The others appear natural.

As to the data (u_0, ψ, f), we assume at the outset that

$$u_0 \in L^2(]0, A[; H^1(\Omega)),$$
$$u_0(a,x) \geqslant 0, \qquad \text{a.e. in }]0, A[\times \Omega,$$

The "obstacle" ψ is a regular function. More precisely ψ satisfies:

$$(\psi)_1 \qquad \begin{cases} \psi \in L^2(\mathcal{O}; H^2(\Omega)), \\ \dfrac{\partial\psi}{\partial\eta} \geqslant 0, \qquad \text{on } \mathcal{O} \times \partial\Omega, \\ \dfrac{\partial\psi}{\partial t} + \dfrac{\partial\psi}{\partial a} \in L^2(\mathcal{O} \times \Omega), \end{cases}$$

$$(\psi)_2 \qquad \begin{cases} \psi(t,a,x) \geqslant 0, \qquad \text{a.e. in } \mathcal{O} \times \Omega, \\ \psi(0,a,x) \geqslant u_0(a,x), \qquad \text{a.e. in }]0, A[\times \Omega, \\ \psi(t,0,x) \geqslant \displaystyle\int_0^A \beta(t,a,x)\psi(t,a,x)\,da, \qquad \text{a.e. in }]0, T[\times \Omega, \end{cases}$$

and note that, thanks to $(\psi)_1$, conditions $(\psi)_2$ make sense (see next section).

Finally, the right-hand side f satisfies

$$f \in L^2(\mathcal{O} \times \Omega),$$
$$f(t,a,x) \geqslant 0, \qquad \text{a.e. in } \mathcal{O} \times \Omega.$$

Remark 2. (i) Conditions $(\psi)_2$ are as natural as the positivity of the data.

(ii) Conditions $(\psi)_1$ and $(\psi)_2$ are usual when regular solutions are desired.

(iii) Another important modification to [2] is that no hypothesis is made on the term $\mu\psi$.

3. *Preliminary Results*

We begin with a trace result that will be essential in what follows. This result is known in case $H^1(\Omega)$ is replaced by $L^2(\Omega)$ (see [1], [9]) or by $H_0^1(\Omega)$ (see [6]).

Lemma 0. *A_0 is a strictly positive real number and $\mathcal{O}_0 =]0, T[\times]0, A_0[$. Let $u \in L^2(\mathcal{O}_0; H^1(\Omega))$ s.t. $(\partial_t + \partial_a)u$ belongs to $L^2(\mathcal{O}_0; [H^1(\Omega)]')$. Then:*

i) *for all t_0 in $]0, T[$ and all a_0 in $]0, A_0[$, u has a trace at $t = t_0$ belonging to $L^2(]0, A_0[\times \Omega)$ and at $a = a_0$ belonging to $L^2(]0, T[\times \Omega)$. The "trace applications" are continuous in the strong and weak topology;*

ii) *the following equality (Ostrogradski formula) holds:*

$$\int_{\mathcal{O}_0} \langle (\partial_t + \partial_a)u, u \rangle \,dt\,da = \frac{1}{2}\left\{ \int_{]0,A_0[\times\Omega} u^2(T,a,x)\,da\,dx \right.$$
$$\left. + \int_{]0,T[\times\Omega} u^2(t,A_0,x)\,dt\,dx \right\} -$$

$$-\frac{1}{2}\left\{\int_{]0,A_0[\times\Omega} u^2(0,a,x)\,da\,dx + \int_{]0,T[\times\Omega} u^2(t,0,x)\,dt\,dx\right\}. \qquad (0)$$

For the proof we can proceed as in [9] or adapt the proofs of [1], [6].

Remark 3. This result will be exploited both as it stands with $A_0 = A$ and in the following form: A_0 is such that $a < A_0 < A$ and u is a function defined on $\mathcal{O} \times \Omega$ s.t.

$$u \in L^2(\mathcal{O}; H^1(\Omega)),$$

$$(\partial_t + \partial_a)u + \mu u \in L^2(\mathcal{O}\colon [H^1(\Omega)]')$$

which implies that u satisfies the hypotheses of Lemma 0.

The second result concerns the regularity of the linear problems.

Theorem 0. *Under the hypotheses of Section I.2, for all given*

$$(u_0, b, f) \in L^2([0,A[\times\Omega) \times L^2(]0,T[\times\Omega) \times L^2(\mathcal{O}; [H^1(\Omega)]'),$$

there exists a unique

$$u \in L^2(\mathcal{O}; H^1(\Omega)) \text{ s.t. } (\partial_t + \partial_a)u + \mu u$$

belongs to $L^2(\mathcal{O}; H^1(\Omega)]')$, *which is solution of*

$$\forall v \in L^2(\mathcal{O}; H^1(\Omega)),$$

$$\int_{\mathcal{O}} \langle(\partial_t + \partial_a)u + \mu u, v\rangle\,dt\,da + \int_{\mathcal{O}\times\Omega} \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dt\,da\,dx = \int_{\mathcal{O}} \langle f, v\rangle\,dt\,da;$$

$$u(0,a,x) = u_0(a,x), \qquad a.e.\ in\ ]0,A[\times\Omega;$$

$$u(t,0,x) = b(t,x), \qquad a.e.\ in\ ]0,T[\times\Omega.$$

Moreover, if

$$(u_0, b, f) \in L^2(]0,A[; H^1(\Omega)) \times L^2(]0,T[\times H^1(\Omega)) \times L^2(\mathcal{O}\times\Omega),$$

then the solution u is in $L^2(\mathcal{O}; H^2(\Omega))$ *and satisfies:*

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u - \Delta u = f, \qquad a.e.\ in\ \mathcal{O}\times\Omega,$$

$$\frac{\partial u}{\partial \eta} = 0, \qquad \text{on } \mathcal{O}\times\partial\Omega,$$

$$\|u\|_{L^2(\mathcal{O}; H^2(\Omega))} \leqslant c\,\{\|f\|_{L^2(\mathcal{O}\times\Omega)}, \|u_0\|_{L^2(0,T; H^1(\Omega))}, \|b\|_{L^2(0,T; H^1(\Omega))},$$
$$|\nabla\mu|_{[L^\infty(\mathcal{O}\times\Omega)]^N}\}.$$

These results can be proved by passing to the limit on those contained in [2]. One can also proceed as follows: first, prove the result for $\mu = 0$, by using for instance Galerkin's method; then for bounded μ by using a fixed-point method, and finally for any μ by a passage to the limit (see also [6] for bounded μ and limit conditions of Dirichlet type).

§II. Results

1. Existence and Uniqueness

We consider the following problem:

Find

$$u \in L^2(\mathcal{O}; H^2(\Omega)) \qquad \text{s.t.,}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u \in L^2(\mathcal{O} \times \Omega),$$

$$u(0,a,x) = u_0(a,x), \qquad \text{a.e. in }]0,A[\times \Omega,$$

$$u(t,0,x) = \int_0^A \beta(t,a,x) u(t,a,x)\, da, \qquad \text{a.e. in }]0,T[\times \Omega,$$

$$\frac{\partial u}{\partial \eta} = 0, \qquad \text{on } \mathcal{O} \times \partial\Omega, \tag{1}$$

and is a solution of

$$u(t,a,x) \leqslant \psi(t,a,x), \qquad \text{a.e. in } \mathcal{O} \times \Omega,$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u - \Delta u \leqslant f, \qquad \text{a.e. in } \mathcal{O} \times \Omega,$$

$$\int_{\mathcal{O} \times \Omega} \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u - \Delta u - f \right) \cdot (u - \psi)\, dt\, da\, dx = 0. \tag{2}$$

We prove:

Theorem 1. *Under the hypotheses of I.2, the problem* (1), (2) *admits a unique solution.*

The proof will be given at the end of the paper.

Remark 4. Under the previous hypotheses, problem (1), (2) is equivalent to the variational inequality:

Find u satisfying (1) and solution of

$$\forall v \in L^2(\mathcal{O} \times \Omega), \qquad \forall v \leqslant \psi, \quad \text{a.e. in } \mathcal{O} \times \Omega^{1},$$

$$\int_{\mathcal{O} \times \Omega} \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + \mu u - \Delta u - f \right) \cdot (v - u)\, dt\, da\, dx \geqslant 0. \tag{3}$$

We shall solve problem (1), (3).

2. Properties

A few side results can be deduced from the existence result.

Theorem 2. *The solution of* (1), (2) *is positive.*

[1] In what follows we shall omit writing a.e., if there is no danger of confusion

Let, for $i = 1, 2$, $(\mu^i, \beta^i, u_0^i, \psi^i, f^i)$ satisfy the hypotheses of Section I.2; and let u^i be the solution of the variational inequality (1), (3). *If we assume that*

$$\mu^2 \leqslant \mu^1, \qquad \beta^1 \leqslant \beta^2, \qquad u_0^1 \leqslant u_0^2, \qquad \psi^1 \leqslant \psi^2, \qquad f^1 \leqslant f^2, \qquad \text{in } \mathcal{O} \times \Omega,$$

then $u^1 \leqslant u^2$, in $\mathcal{O} \times \Omega$.

Thus, we have the following properties, which are intuitive from a biological viewpoint: the solution is positive, it decreases as a function of μ, and increases as function of the other parameters.

Hypothesis $(\mu)_2$ has not yet been exploited.

Theorem 3. *Under the hypotheses of I.2, the solution of problem* (1), (2) *or* (1), (3), *is such that*

$$u(t, A, x) = 0, \qquad \text{in }]0, T[\times \Omega. \tag{4}$$

The proof of this result is immediate, and is independent of the remaining part of the paper.

Let g be given in $L^2(\mathcal{O} \times \Omega)$ and w solution in $L^2(\mathcal{O} \times \Omega)$ of

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} + \mu w = g, \qquad & \text{in } \mathcal{O} \times \Omega, \\ w(0, a, x) = w_0(a, x), \qquad & \text{in }]0, A[\times \Omega, \\ w(t, 0, x) = w_1(t, x), \qquad & \text{in }]0, T[\times \Omega, \end{aligned}$$

where w_0 and w_1 are elements of $L^2(]0, A[\times \Omega)$ and $L^2(]0, T[\times \Omega)$ respectively.

By computing w by the method of characteristics, it is clear that condition $(\mu)_2$ ensures that $w(t, A, x) = 0$, in $]0, T[\times \Omega$. [1]

This result can thus be applied to u, which satisfies the same hypotheses as w (see [1]).

Remark 5. A few hypotheses may be weakened. For instance, the data do not have to be positive, and ψ minimized by a regular function.

One can also weaken the formulations (1), (2), look for weak solutions and study the regularity of the latter by dual estimates; this will allow to weaken the hypotheses of ψ.

The previous results still hold if one considers the interval $]0, T[\times]0, \infty[\times \Omega$.

§III. Proofs

In order to obtain the existence in the V.I. (1), (3) a penalization method is used. More precisely, for $\varepsilon > 0$, we seek $u_\varepsilon \in L^2(\mathcal{O}: H^2(\Omega))$ solution of

[1] We have indeed for $a > t$, $x \in \Omega$

$$w(t, a, x) = \exp\left(- \int_0^t \mu(\tau, a - t + \tau, x)\, d\tau \right) \Bigg[w_0(a - t, x) + \int_0^t \exp\left(\int_0^\tau \mu(\vartheta, a - t + \vartheta, x)\, d\vartheta \right) g(\tau, a - t + \tau, x)\, d\tau \Bigg];$$

analogously for $a < t$, $x \in \Omega$

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+\mu u-\Delta u+\frac{1}{\varepsilon}(u-\psi)^{+}=f,\qquad \text{in } \mathcal{O}\times\Omega;$$

$$u(0,a,x)=u_0(a,x),\qquad \text{in }]0,A[\times\Omega;$$

$$u(t,0,x)=\int_0^A \beta(t,a,x)u(t,a,x)\,da,\qquad \text{in }]0,T[\times\Omega;$$

$$\frac{\partial u}{\partial \eta}=0,\qquad \text{on } \mathcal{O}\times\partial\Omega, \tag{5}$$

and then, letting $\varepsilon\to 0$, we obtain a solution of (1), (3).

From a technical viewpoint it is more convenient to work on a weaker formulation of (5): we seek $u_\varepsilon\in L^2(\mathcal{O}: H^1(\Omega))$ solution of

$$\forall v\in L^2(\mathcal{O}:H^1(\Omega)),$$

$$\int_{\mathcal{O}}\langle(\partial_t+\partial_a)u+\mu u, v\rangle\,dt\,da+\int_{\mathcal{O}\times\Omega}\nabla \mathbf{u}\times\nabla \mathbf{v}\,dt\,da\,dx$$

$$+\frac{1}{\varepsilon}\int_{\mathcal{O}\times\Omega}(u-\psi)^{+}v\,dt\,da\,dx=\int_{\mathcal{O}\times\Omega}fv\,dt\,da\,dx;$$

$$u(0,a,x)=u_0(a,x),\qquad \text{in }]0,A[\times\Omega,$$

$$u(t,0,x)=\int_0^A \beta(t,a,x)u(t,a,x)\,da,\qquad \text{in }]0,T[\times\Omega.$$

In order to pass from (6) to (5) it is sufficient to apply the regularity results to the equations.

Remark 6. If we perform a change of variables $u=e^{\lambda t}\tilde{u}, f=e^{\lambda t}\tilde{f}, \psi=e^{\lambda t}\tilde{\psi}$, then $\tilde{u}$ is solution of (5) or (6) with μ replaced by $\tilde{\mu}=\mu+\lambda$, f by $\tilde{f}$ and ψ by $\tilde{\psi}$. This will be tacitly done in the sequel, with λ sufficiently large (i.e. $\lambda>\frac{1}{2}c_1$, c_1 defined in $(\beta)_3$).

1. Penalized Equations: The Case μ Bounded

In this section μ is assumed to be bounded in $\mathcal{O}\times\Omega$ i.e. hypotheses $(\mu)_2$ is replaced by

$(\mu)_2^*$ $$\mu\in L^\infty(\mathcal{O}\times\Omega).$$

The following lemma is basic.

Lemma 1. *Under the hypotheses of I.2 with $(\mu)_2$ replaced by $(\mu)_2^*$ and $\varepsilon>0$, for all $b\in L^2(]0,T[\times\Omega)$, there exists a unique $u\in L^2(\mathcal{O}:H^1(\Omega))$ such that*

$$(\partial_t+\partial_a)u\in L^2(\mathcal{O}:H^1(\Omega)]')$$

which is solution of

$$\forall v\in L^2(\mathcal{O};H^1(\Omega)),$$

$$\int_{\mathcal{O}} \langle (\partial_t + \partial_a)u, v\rangle \, dt\, da + \int_{\mathcal{O}\times\Omega} \left[(\lambda + \mu)uv + \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \frac{1}{\varepsilon}(u - \psi)^+ v\right] dt\, da\, dx$$
$$= \int_{\mathcal{O}\times\Omega} fv \, dt\, da\, dx;$$
$$u(0, a, x) = u_0(a, x), \qquad]0, A[\times \Omega;$$
$$u(t, 0, x) = b(t, x), \qquad]0, T[\times \Omega. \tag{7}$$

Proof. The proof follows either by using the results of [2] or the results on the linear problems and the technique of the maximal-monotone operators (see [7] for the case of the Dirichlet problem).

Remark 7. Since

$$\frac{1}{\varepsilon}(u - \psi)^+ \in L^2(\mathcal{O} \times \Omega), \qquad \text{if} \qquad b \in L^2(0, T; H^1(\Omega))$$

then u satisfies

$$u \in L^2(\mathcal{O}; H^2(\Omega)),$$
$$\frac{\partial u}{\partial \eta} = 0, \qquad \text{on } \mathcal{O} \times \partial\Omega,$$
$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (\lambda + \mu)u - \Delta u + \frac{1}{\varepsilon}(u - \psi)^+ = f, \qquad \text{in } \mathcal{O} \times \Omega;$$

(By Theorem 0, because of $(\mu)_2^*$ the term μu is in $L^2(\mathcal{O} \times \Omega)$).

For the proof of Theorem 2 we need the following lemma.

Lemma 2. *Under the hypotheses of Lemma 1,*

i) *if b is positive (in $]0, T[\times \Omega$), the same holds (in $\mathcal{O} \times \Omega$) for the solution of problem (7).*

More generally:

ii) *for $i = 1, 2$, let $(b^i, u_0^i, \psi^i, f^i)$ satisfy the hypotheses of Lemma 1, and let u^i be the solution of (7). Then if*

$$b^1 \leqslant b^2, \qquad u_0^1 \leqslant u_0^2, \qquad \psi^1 \leqslant \psi^2, \qquad f^1 \leqslant f^2,$$

we have $u^1 \leqslant u^2$.

iii) *Let ε_1 and ε_2 s.t. $0 < \varepsilon_1 < \varepsilon_2$ and let u^i the solution of (7) for $\varepsilon = \varepsilon_i$, $i = 1, 2$. Then $u^1 \leqslant u^2$.*

iv) *Let μ^1 and μ^2 satisfy $(\mu)_2^*$ and let u^i the solution of (7) for $\mu = \mu^i$, $i = 1, 2$. Then, if $b \geqslant 0$, and $\mu^1 \leqslant \mu^2$ we have $u^2 \leqslant u^1$.*

Proof. These results are a consequence of the weak maximum principle, satisfied by the degenerate elliptic operators. Since, by hypothesis, f and u_0 are positive, it is clear that if b is positive the same holds for u.

The second property (comparison result) is also classic.

The third property is a standard property of the penalized equations (see [10], [11], [7]).

The last property is proved similarly; noting $(u^2 - u^1)^+ \in L^2(\mathcal{O}; H^1(\Omega))$, we obtain

$$\int_{\mathcal{O}} \langle [\partial_t + \partial_a](u^2 - u^1), (u^2 - u^1)^+ \rangle \, dt\, da$$
$$+ \int_{\mathcal{O}\times\Omega} (\lambda + \mu^1)(u^2 - u^1)(u^2 - u^1)^+ \, dt\, da\, dx$$
$$+ \int_{\mathcal{O}\times\Omega} \nabla(u^2 - u^1) \cdot \nabla(u^2 - u^1)^+ \, dt\, da\, dx$$
$$+ \frac{1}{\varepsilon}\int_{\mathcal{O}\times\Omega} [(u^2 - \psi)^+ - (u^1 - \psi)^+](u^2 - u^1)^+ \, dt\, da\, dx$$
$$+ \int_{\mathcal{O}\times\Omega} (\mu^2 - \mu^1)u^2(u^2 - u^1)^+ \, dt\, da\, dx = 0.$$

The last two terms of this equality are positive (as $b \geqslant 0$ ensures that $u^i \geqslant 0$, $i = 1, 2$). The first one is also positive since

$$(u^2 - u^1)(0, a, x) = (u^2 - u^1)(t, 0, x) = 0.$$

It follows that $(u^2 - u^1)^+ = 0$, i.e. $u^2 \leqslant u^1$ in $\mathcal{O} \times \Omega$.

Problem (6) is solved by a fixed point method.

Lemma 3. *Under the hypotheses I.2, with* $(\mu)_2$ *replaced by* $(\mu)_2^*$, *then problem* (6) *admits a unique solution in* $L^2(\mathcal{O}; H^1(\Omega))$.

Proof. For a given w in $L^2(\mathcal{O}; H^1(\Omega))$, Sw denotes the solution of (7) with

$$b(t, x) = \int_0^A \beta(t, a, x) w(t, a, x)\, da. \tag{8}$$

S is an application of $L^2(\mathcal{O}; H^1(\Omega))$ in itself. We are left with proving that S is strictly contracting.

Let w_1 and w_2 be two elements of $L^2(\mathcal{O}; H^1(\Omega))$. By elementary computations we obtain:

$$\int_{\mathcal{O}} \langle [\partial_t + \partial_a](Sw_1 - Sw_2), Sw_1 - Sw_2 \rangle \, dt\, da$$
$$+ \int_{\mathcal{O}\times\Omega} [(\lambda + \mu)(Sw_1 - Sw_2)^2 + |\nabla(Sw_1 - Sw_2)|^2] \, dt\, da\, dx$$
$$+ \frac{1}{\varepsilon}\int_{\mathcal{O}\times\Omega} [(Sw_1 - \psi)^+ - (Sw_2 - \psi)^+] \cdot [Sw_1 - Sw_2] \, dt\, da\, dx = 0. \tag{9}$$

The penalization term, i.e. the last term, is positive; the initial data at $t = 0$ and $a = 0$ are

$$[Sw_1 - Sw_2](0, a, x) = 0, \qquad [Sw_1 - Sw_2](t, 0, x) = \int_0^A \beta(w_1 - w_2)\, da.$$

From equality (0) (see Section I.3) we have:

$$\int_{\mathcal{O}} \langle [\partial_t + \partial_a](Sw_1 - Sw_2), Sw_1 - Sw_2 \rangle \, dt \, da$$

$$= \frac{1}{2} \int_{]0,A[\times \Omega} [Sw_1 - Sw_2]^2(T, a, x) \, da \, dx$$

$$+ \frac{1}{2} \int_{]0,T[\times \Omega} [Sw_1 - Sw_2]^2 (t, A, x) \, dt \, dx$$

$$- \frac{1}{2} \int_{]0,T[\times \Omega} \left[\int_0^A \beta(w_1 - w_2) \, da \right]^2 dt \, dx.$$

By substituting this in (9) and using $(\beta)_3$ we have in particular

$$\int_{\mathcal{O} \times \Omega} [(\lambda + \mu)(Sw_1 - Sw_2)^2 + |\nabla(Sw_1 - Sw_2)|^2] \, dt \, da \, dx$$

$$\leqslant \frac{1}{2} c_1 \int_{\mathcal{O} \times \Omega} (w_1 - w_2)^2 \, dt \, da \, dx$$

and we deduce (see the choice of λ in Remark 6) that S is strict contraction.

Before passing to the case of μ whatsoever we draw from Lemma 2 a few properties of the solution of (6).

Lemma 4. *Under the hypotheses of Lemma 3, the solution u of problem* (6) *is positive, it depends in an increasing manner on* (β, u_0, ψ, f) *and on* ε, *and it depends in a decreasing way on* μ.

Proof. It is sufficient to prove that the results of Lemma 2 still hold for the previous fixed point. We prove, for instance, the last property.

Let μ^1 and μ^2 with $\mu^1 \leqslant \mu^2$, and let S^i the application S corresponding to $\mu = \mu^i$, $i = 1, 2$. Set

$$u^{i,0} = 0, \qquad i = 1, 2,$$

$$u^{i,n+1} = S^i(u^{i,n}), \qquad n \geqslant 0, \quad i = 1, 2.$$

From the last part of Lemma 2 we have $0 \leqslant u^{2,1} \leqslant u^{1,1}$. Let now $v^{2,2}$ be the solution of (7) with $\mu = \mu^2$, $w = u^{1,1}$, in (8). From Lemma 2 if follows $0 \leqslant u^{2,2} \leqslant v^{2,2}$ (see (ii)), $v^{2,2} \leqslant u^{1,2}$ (see (iv)). Finally, let $0 \leqslant u^{2,2} \leqslant u^{1,2}$. An easy induction shows that $\forall n \geqslant 0$, $0 \leqslant u^{2,n} \leqslant u^{1,n}$, and since the sequence $(u^{i,n})_n$ converges to the solution u^i of (6) with $\mu = \mu^i$, we obtain the desired result. ∎

Remark 8. Note that (6), and thus also (5), has been solved under the hypothesis μ bounded.

2. *Penalized Equation: General Case*

In order to pass from the case of bounded μ to that of an arbitrary μ we introduce the sequence $(\mu_n)_{n \in \mathbb{N}}$ defined by

$$\mu^n = \mu \wedge n = \operatorname{Inf}(\mu, n), \qquad n \in \mathbb{N}.$$

As we know how to solve (5) and (6) with μ replaced by μ^n, we perform a passage to the limit.

Lemma 5. *Under the hypotheses of Section I.2, for all $\varepsilon > 0$ there exists a unique u_ε in $L^2(\mathcal{O}; H^1(\Omega))$ s.t. $(\partial_t + \partial_a)u_\varepsilon + \mu u_\varepsilon \in L^2(\mathcal{O}; [H^1(\Omega)]')$ and is a solution of (6).*

Proof. Let us start by proving the existence.

Let u_n be the solution of (6) with $\mu = \mu^n$ (see Lemma 3). Let $v = u_n$ in (6); integrating by parts (i.e., using equality (0)) we have:

$$\begin{aligned}
&\frac{1}{2}\int_{]0A[\times\Omega} u_n^2(T,a,x)\,da\,dx + \frac{1}{2}\int_{]0,T[\times\Omega} u_n^2(t,A,x)\,dt\,dx \\
&\quad + \int_{\mathcal{O}\times\Omega} \Big[(\lambda + \mu^n)u_n^2 + |\nabla u_n|^2 + \frac{1}{\varepsilon}(u_n - \psi)^+ u_n\Big]\,dt\,da\,dx \\
&= \int_{\mathcal{O}\times\Omega} f\cdot u_n\,dt\,da\,dx + \frac{1}{2}\int_{]0,A[\times\Omega} u_0^2\,da\,dx \\
&\quad + \int_{]0,T[\times\Omega}\left[\int_0^A \beta(t,a,x)u_n(t,a,x)\right]^2 dt\,dx. \qquad (10)
\end{aligned}$$

Since for all n we have $u_n \geqslant 0$ in $\mathcal{O} \times \Omega$, we have

$$\int_{\mathcal{O}\times\Omega} (\mu^n u_n^2 + u_n^2 + |\nabla u_n|^2)\,dt\,da\,dx \leqslant k, \qquad (11)$$

where k is a constant independent of n and ε.

From equation (6) we can now deduce $([\partial_t + \partial_a]u_n + \mu^n u_n)_n$ bounded in $L^2(\mathcal{O}; [H^1(\Omega)]')$. On the other hand (see Lemma 4), the sequence $(u_n)_n$ is decreasing. Thus from the sequence $(u_n)_n$ we can extract a sequence $(u_k)_k$ such that for $k \to \infty$

$$\begin{aligned}
u_k &\to u_\varepsilon && \text{in } L^2(\mathcal{O}; H^1(\Omega)) \text{ weakly},\\
\sqrt{\mu^k u_k} &\to w && \text{in } L^2(\mathcal{O}\times\Omega) \text{ weakly},\\
(\partial_t + \partial_a)u_k + \mu^k u_k &\underset{k\to\infty}{\to} h && \text{in } L^2(\mathcal{O}; [H^1(\Omega)]') \text{ weakly}.
\end{aligned}$$

The monotonicity of u_n ensures that

$$\begin{aligned}
u_k &\to u_\varepsilon && \text{in } L^2(\mathcal{O}\times\Omega) \text{ strongly},\\
(u_k - \psi)^+ &\to (u_\varepsilon - \psi)^+ && \text{in } L^2(\mathcal{O}\times\Omega) \text{ strongly}.
\end{aligned}$$

Hypothesis $(\mu)_1$ ensures that for φ in $\mathcal{D}(\mathcal{O}\times\Omega)$ and k sufficiently large

$$\int_{\mathcal{O}\times\Omega} \sqrt{\mu^k}\, u_k\varphi\,dt\,da\,dx = \int_{\mathcal{O}\times\Omega} \sqrt{\mu}\, u_k\varphi\,dt\,da\,dx \underset{k\to\infty}{\to} \int_{\mathcal{O}\times\Omega} \sqrt{\mu}\, u_\varepsilon\varphi\,dt\,da\,dx$$

and thus $(\sqrt{\mu^k}\, u_k)$ converges to $\sqrt{\mu}\, u_\varepsilon$ in $\mathcal{D}'(\mathcal{O}\times\Omega)$ and

$$w = \sqrt{\mu}\, u_\varepsilon.$$

Similarly, $([\partial_t + \partial_a]u_k + \mu^k]u_k)$ converges in $\mathscr{D}'(\mathcal{O}; H^1(\Omega)]')$ to $(\partial_t + \partial_a)u_\varepsilon + \mu u_\varepsilon$ and $h = (\partial_t + \partial_a)u_\varepsilon + \mu u_\varepsilon$.

In order to show that u_ε is solution of the penalized equation (6) we have yet to verify the initial conditions. We do this using the results of Section I.3.

Let A_0 be such that $0 \leqslant A_0 < A$. From the above we deduce (setting $\mathcal{O}_0 =]0, T[\times]0, A_0[$) for $k \to \infty$

$$u_k \to u_\varepsilon \qquad \text{in } L^2(\mathcal{O}_0; H^1(\Omega)) \text{ weakly,}$$

$$(\partial_t + \partial_a)u_k \to (\partial_t + \partial_a)u_\varepsilon \qquad \text{in } L^2(\mathcal{O}_0; [H^1(\Omega)]') \text{ weakly,}$$

consequently (see Lemma 0)

$$u_k(0, a, x) \to u_\varepsilon(0, a, x) \qquad \text{in } L^2(]0, A_0[\times \Omega) \text{ weakly,}$$

$$u_k(t, 0, x) \to u_\varepsilon(t, 0, x) \qquad \text{in } L^2(]0, T[\times \Omega) \text{ weakly.}$$

As

$$u_k(0, a, x) = u_0(a, x), \qquad \forall n \geqslant 0, \qquad \text{in }]0, A[\times \Omega,$$

$$u_k(t, a, x) = \int_0^A \beta u_k \, da \to \int_0^A \beta u_\varepsilon \, da \qquad \text{in } L^2(]0, T[\times \Omega) \text{ strongly,}$$

u_ε satisfies

$$u_\varepsilon(0, a, x) = u_0(a, x), \qquad \text{in }]0, A[\times \Omega,$$

$$u_\varepsilon(t, 0, x) = \int_0^A \beta u_\varepsilon \, da, \qquad \text{in }]0, T[\times \Omega$$

and is a solution of (6).

Let us now prove *uniqueness*. Let u_1 and u_2 be two solutions of (6) and let $u = u_1 - u_2$. We take $v \doteq u \cdot \chi_{]0,A_0[}$[1] as test function in the equation admitting u_1 and u_2 as solutions. Taking the difference and using (0) we have

$$\frac{1}{2}\int_{]0,A_0[\times\Omega} u^2(T, a, x)\, da\, dx + \frac{1}{2}\int_{]0,T[\times\Omega} u^2(t, A_0, x)\, dt\, dx$$

$$+ \int_{\mathcal{O}_0\times\Omega} \left[(\lambda+\mu)u^2 + |\nabla u|^2 + \frac{1}{\varepsilon} u\{(u_1-\psi)^+ - (u_2-\psi)^+\}\chi_{]0,A_0[}\right] dt\, da\, dx$$

$$= \frac{1}{2}\int_{]0,T[\times\Omega} \left[\int_0^A \beta u\, da\right]^2 dt\, dx.$$

Since the penalization term is positive we have in particular:

$$\int_{\mathcal{O}_0\times\Omega} [\lambda u^2 + |\nabla u|^2]\, dt\, da\, dx \leqslant \frac{1}{2} c_1 \int_{\mathcal{O}\times\Omega} u^2\, dt\, da\, dx.$$

Letting A_0 tend to A we obtain the result: $u = 0$.

Remark 9. The sequence $(u_n)_n$ converges strongly in $L^2(\mathcal{O} \times \Omega)$ to the solution u_ε of (6).

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[1] χ_F denotes the characteristic function of the set F

We can now solve problem (5).

Lemma 6. *Under the hypotheses of Section I.2, there exists a unique* u_ε *in* $L^2(\mathcal{O}; H^2(\Omega))$ *s.t.* $(\partial u_\varepsilon/\partial t) + (\partial u_\varepsilon/\partial a) + \mu u_\varepsilon$ *belongs to* $L^2(\mathcal{O} \times \Omega)$ *and is solution of the penalized problem* (5).

Proof. We make use of the regularity of the linear problems. If we set

$$g = f - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+,$$

$$b = \int_0^A \beta u_\varepsilon \, da,$$

where u_ε is solution of (6), then u_ε is solution in $L^2(\mathcal{O}; H^1(\Omega))$ of

$$\forall v \in L^2(\mathcal{O}; H^1(\Omega)),$$

$$\int_{\mathcal{O}} \langle [\partial_t + \partial_a]u + \mu u, v\rangle \, dt\, da + \int_{\mathcal{O}\times\Omega} \nabla u \cdot \nabla v \, dt\, da\, dx = \int_{\mathcal{O}\times\Omega} g \cdot v \, dt\, da\, dx,$$

$$u(0, a, x) = u_0(a, x), \qquad \text{in }]0, A[\times \Omega,$$

$$u(t, 0, x) = b(t, x), \qquad \text{in }]0, T[\times \Omega.$$

The regularity result can now be applied. ■

Collecting the results of Lemmas 4, 5 and 6 we obtain:

Lemma 7. *Under the hypotheses of Lemma 6, the solution of problem* (5) *depends in an increasing way on* (β, u_0, ψ, f) *and* $\varepsilon > 0$, *and in a decreasing way on* μ.

We are now in a position to prove the results announced at the beginning.

3. Proofs of the Results

Consider the sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions of the penalized problem (5). We already know (see (11) and Lemma 7) that

$$\int_{\mathcal{O}\times\Omega} [\mu u_\varepsilon^2 + \lambda u_\varepsilon^2 + |\nabla u_\varepsilon|^2] \, dt\, da\, dx \leqslant k, \qquad k \text{ independent of } \varepsilon > 0,$$

and that the sequence u_ε is decreasing, as ε decreases to 0.

We show that

$$\left(\frac{1}{\varepsilon}(u_\varepsilon - \psi)^+\right)_{\varepsilon>0} \qquad \text{is bounded in } L^2(\mathcal{O} \times \Omega). \tag{12}$$

The properties of ψ imply that $(u_\varepsilon - \psi)^+$ is an element of $L^2(\mathcal{O}: H^1(\Omega))$. For all A_0 such that $0 < A_0 < A$ we have (setting $\mathcal{O}_0 =]0, T[\times]0, A_0[$)

$$\int_{\mathcal{O}_0} \left\langle \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial a} + \mu\right](u_\varepsilon - \psi), (u_\varepsilon - \psi)^+ \right\rangle dt\, da +$$

$$+\int_{\mathcal{O}_0\times\Omega}\left[\lambda(u_\varepsilon-\psi)(u_\varepsilon-\psi)^+ + \nabla(u_\varepsilon-\psi)\cdot\nabla(u_\varepsilon-\psi)^+\right.$$

$$\left.+\frac{1}{\varepsilon}(u_\varepsilon-\psi)(u_\varepsilon-\psi)^+\right]dt\,da\,dx$$

$$=\int_{\mathcal{O}_0}\left\langle f-\frac{\partial\psi}{\partial t}-\frac{\partial\psi}{\partial a}-(\lambda+\mu)\psi,(u_\varepsilon-\psi)^+\right\rangle dt\,da$$

$$+\int_{\mathcal{O}_0\times\Omega}\nabla\psi\cdot\nabla(u_\varepsilon-\psi)^+\,dt\,da\,dx. \tag{13}$$

The condition $\partial\psi/\partial\eta \geqslant 0$, on $\mathcal{O}\times\partial\Omega$ ensures that

$$\int_{\mathcal{O}_0\times\Omega}\Delta\psi\cdot(u_\varepsilon-\psi)^+\,dt\,da\,dx \geqslant -\int_{\mathcal{O}_0\times\Omega}\nabla\psi\times\nabla(u_\varepsilon-\psi)^+\,dt\,da\,dx;$$

since λ, μ and ψ are positive, the right member of equality (13) is bounded from above by

$$\int_{\mathcal{O}_0\times\Omega}\left(f-\frac{\partial\psi}{\partial t}-\frac{\partial\psi}{\partial a}+\Delta\psi\right)(u_\varepsilon-\psi)^+\,dt\,da\,dx.$$

The regularity of ψ and that of u_ε imply

$$\int_{\mathcal{O}_0}\left\langle\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\mu\right](u_\varepsilon-\psi)(u_\varepsilon-\psi)^+\right\rangle dt\,da$$

$$=\int_{\mathcal{O}_0\times\Omega}\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right](u_\varepsilon-\psi)^+\cdot(u_\varepsilon-\psi)^+\,dt\,da\,dx$$

$$+\int_{\mathcal{O}_0\times\Omega}\mu[(u_\varepsilon-\psi)^+]^2\,dt\,da\,dx. \tag{14}$$

A further consequence is (see condition $(\psi)_2$)

$$(u_\varepsilon-\psi)^+(0,a,x)=[(u_\varepsilon-\psi)(0,a,x)]^+=0$$

and (see again $(\psi)_2$)

$$(u_\varepsilon-\psi)^+(t,0,x)=[(u_\varepsilon-\psi)\cdot(t,0,x)]^+\leqslant\left[\int_0^A\beta(u_\varepsilon-\psi)\,da\right]^+$$

$$\leqslant\int_0^A\beta(u_\varepsilon-\psi)^+\,da.$$

Therefore, the first term of the left side of (13) is minorized by (integrating by parts (14) and taking into account $(\beta)_3$)

$$-\frac{1}{2}c_1\int_{\mathcal{O}\times\Omega}[(u_\varepsilon-\psi)^+]^2\,dt\,da\,dx.$$

Dividing both sides of (13) by ε we obtain

$$\frac{\lambda}{\varepsilon}\int_{\mathcal{O}_0\times\Omega}[(u_\varepsilon-\psi)^+]^2\,dt\,da\,dx+\frac{1}{\varepsilon}\int_{\mathcal{O}_0\times\Omega}|\nabla(u_\varepsilon-\psi)^+|^2\,dt\,da\,dx$$
$$+\frac{1}{\varepsilon^2}\int_{\mathcal{O}_0\times\Omega}[(u_\varepsilon-\psi)^+]^2\,dt\,da\,dx$$
$$\leqslant\int_{\mathcal{O}_0\times\Omega}\left[f-\frac{\partial\psi}{\partial t}-\frac{\partial\psi}{\partial a}-\Delta\psi\right]\frac{(u_\varepsilon-\psi)^+}{\varepsilon}\,dt\,da\,dx$$
$$+\frac{1}{2\varepsilon}c_1\int_{\mathcal{O}\times\Omega}[(u_3-\psi)^+]^2\,dt\,da\,dx. \tag{15}$$

If we make A_0 tend to A we can replace $\mathcal{O}_0$ by $\mathcal{O}$ in (15) and deduce (12); thus from (5) it follows that

$$\left(\frac{\partial u_\varepsilon}{\partial t}+\frac{\partial u_\varepsilon}{\partial a}+\mu u_\varepsilon-\Delta u_\varepsilon\right)_{\varepsilon>0}\qquad\text{is bounded in } L^2(\mathcal{O}\times\Omega),$$

and also (see the regularity of the linear problem)

$$(u_\varepsilon)_{\varepsilon>0}\text{ bounded in } L^2(\mathcal{O};H^2(\Omega)).$$

The monotonicity of the sequence (u_ε) ensures that it converges strongly in $L^2(\mathcal{O}\times\Omega)$, i.e. $u_\varepsilon\to u$ in $L^2(\mathcal{O}\times\Omega)$ for $\varepsilon\to 0$. On the other hand, we can extract a subsequence $(u_{\varepsilon'})_{\varepsilon'}$ such that when $\varepsilon'\to 0$

$$u_{\varepsilon'}\to u\qquad\text{in } L^2(\mathcal{O};H^2(\Omega))\text{ weakly,}$$
$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\mu\right]u_{\varepsilon'}-\Delta u_{\varepsilon'}\to l\qquad\text{in } L^2(\mathcal{O}\times\Omega)\text{ weakly.}$$

It is clear that

$$\frac{\partial u}{\partial\eta}=0,\qquad\text{on }\mathcal{O}\times\partial\Omega$$

and (see proof of Lemma 5)

$$l=\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\mu\right]u-\Delta u.$$

It follows (see also Lemma 5)

$$u(0,a,x)=u_0(a,x),\qquad\text{in }]0,A[\times\Omega;$$
$$u(t,0,x)=\int_0^A\beta u\,da,\qquad\text{in }]0,T[\times\Omega.$$

Finally, [as $(1/\varepsilon)(u_\varepsilon-\psi)^+$ is bounded in $L^2(\mathcal{O}\times\Omega)$], we deduce that $(u-\psi)^+=0$, i.e.

$$u\leqslant\psi,\qquad\text{in }\mathcal{O}\times\Omega.$$

Let now v be an element of $L^2(\mathcal{O} \times \Omega)$, s.t. $v \leqslant \psi$ in $\mathcal{O} \times \Omega$. Note that for all v

$$\int_{\mathcal{O}\times\Omega}\left(\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\mu\right]u_{\varepsilon'}-\Delta u_{\varepsilon'}-f\right)(v-u_{\varepsilon'})\,dt\,da\,dx$$

$$=\frac{1}{\varepsilon'}\int_{\mathcal{O}\times\Omega}[(v-\psi)^+-(u_{\varepsilon'}-\psi)^+](v-u_{\varepsilon'})\,dt\,da\,dx\geqslant 0. \tag{16}$$

The strong convergence of $u_{\varepsilon'}$ in $L^2(\mathcal{O} \times \Omega)$ insures that the limit u satisfies inequality (3): it is sufficient to make ε' tend to 0 in (16). ■

As to uniqueness note that if u^1 and u^2 are two solutions of (1), (3), then the difference $u = u^1 - u^2$ satisfies

$$\int_{\mathcal{O}\times\Omega}\left[\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+\mu u-\Delta u\right]u\,dt\,da\,dx\leqslant 0,$$

$$u(0,a,x)=0, \qquad \text{in }]0,A[\times\Omega,$$

$$u(t,0,x)=\int_0^A \beta u\,da, \qquad \text{in }]0,T[\times\Omega,$$

$$\frac{\partial u}{\partial \eta}=0, \qquad \text{on } \mathcal{O}\times\partial\Omega.$$

Since

$$\int_{\mathcal{O}\times\Omega}\left[\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+\mu u\right]u\,dt\,da\,dx=\lim_{A_0\to A}\int_{\mathcal{O}_0\times\Omega}\left[\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}+\mu u\right]u\,dt\,da\,dx$$

the proof that $u = 0$ is along the lines of that of Lemma 5. ■

The proof of Theorem 1 is thus completed. That of Theorem 2 is immediate from Lemma 7.

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