ON THE INSTABILITY OF PARAMETRICALLY EXCITED TWO DEGREES OF FREEDOM VIBRATING SYSTEMS WITH VISCOUS DAMPING

Emilio Massa*

SOMMARIO: *Nel presente lavoro si esamina con un particolare metodo di perturbazione la stabilità dei sistemi vibranti a due gradi di libertà eccitati parametricamente e con resistenza viscosa.*

Si ottengono espressioni analitiche in forma esplicita in seconda approssimazione dei confini fra stabilità ed instabilità. Si dà anche un criterio di massima per giudicare entro quali limiti l'approssimazione risulta accettabile.

Si pone in evidenza come anche nelle vibrazioni eccitate parametricamente lo smorzamento possa avere un effetto instabilizzante analogo a quello già noto nei sistemi elastici non conservativi.

SUMMARY: *The stability of vibrating systems having two degrees of freedom subjected to parametric excitation and with viscous damping are studied with a special perturbation method.*

Analytical expressions in explicit form in second approximation are obtained for the transition from stability to instability. A general criterion for judging the limits within which the approximation is acceptable is indicated.

It is shown that even in parametrically excited vibrating systems damping can have a destabilising effect similar to the destabilising effect known to exist in nonconservative elastic systems.

1. Introduction.

In a previous paper [1] the author considered the regions of instability of vibrating systems with one degree of freedom subjected to parametric excitation and viscous damping by applying a special method of perturbation that gives excellent approximation in cases relevant to technical applications and in such a way that the regions of instability are expressed simply, in terms of parameters suitable for the applications.

However, in technical problems it is not always admissible to formulate a phenomenon with only one degree of freedom, even if in the phenomenon in question different degrees of freedom in different frequency intervals are considered. Further, as is known [2], [3], [4], when more than one degree of freedom is considered new types of instability appear and the effects of damping may be different from the effects expected, as will be seen in this paper. For, in the case of one degree of freedom, stable and unstable solutions are separated by periodic solutions whereas with multiple degrees of freedom the transition from stable to unstable solutions may occur even in correspondence with nonperiodic solutions. Furthermore, the connexions, including those due to damping, between the various degrees of freedom, introduce new possibilities of instability.

Many studies in which parametric excitation plays a part have appeared over the past few years, including some on parametric excitation in general (see [3] Chap. IV, [4] [5] and annexed references). These last studies mainly establish general methods for seeking the regions of instability and stability, and these methods make it possible to identify some of the properties of these regions and to determine them in first approximation.

Tondl [6] deals with interesting applications to the problems of the stability of rotating shafts and he goes on to develop, with reference to studies of Russian workers, a special method [6] App. III, suitable for determining fields of stability in general. But this method too is virtually applicable only in first approximation and not even in this case are explicit formulas supplied for the transition from stability to instability, nor is the approximation of the results obtained discussed.

This paper deals with a peculiar method of perturbation, which differs from those proposed by other workers and which permits the transition to successive approximations and enables one to obtain explicit formulas for the transition from stability to instability. To apply the method to concrete cases we only examine the cases of systems with two degrees of freedom and with constant coefficients of inertia and viscous damping and coefficients of elasticity varying according to a harmonic law with equal frequency and phase (on the other hand the method expounded can be extended also to systems with several degrees of freedom). It is thus possible to appraise the effects of the successive approximations on the solutions in first approximation and, as the formulas are in explicit form, some highly interesting peculiarities of behavior due to damping can be highlighted. In fact, the method shows that in parametric instability too, in the so-called mixed cases, damping can have a destabilising effect similar to that found by Ziegler [7], Bolotin [8], and Hermann [9], for nonconservative elastic systems with two degrees of freedom.

* Professore di Meccanica Applicata alle Macchine, Politecnico di Milano

2. List of symbols.

m_1, m_2	=	coefficients of inertia of the system (> 0)
x_1, x_2	=	free coordinates starting from the resting position
k_{11}, $k_{12} = k_{21}$, k_{22}	=	mean coefficients of the elastic rigidity of the system
Δk_{11}, $\Delta k_{12} = \Delta k_{21}$, Δk_{22}	=	amplitude of the variations of the coefficients of rigidity
r_{11}, $r_{12} = r_{21}$, r_{22}	=	coefficients of viscous damping
ω	=	angular frequency of the variation in rigidity
t	=	time
w_1, w_2	=	principal coordinates (see Sec. 3)
ω_1, ω_2	=	principal angular frequencies (see Sec. 3)
ΔK_{11}, $\Delta K_{12} = \Delta K_{21}$, ΔK_{22}	=	amplitude of the rigidity variations in the principal coordinates
R_{11}, $R_{12} = R_{21}$, R_{22}	=	coefficients of viscous damping in the principal coordinates
$R_m = (R_{11} + R_{22})/2$	=	mean viscous damping coefficient (of the direct coefficients in the principal coordinates).

Non dimensional magnitudes:

$\varphi \;= 1/2\ \omega t$
$\Omega \;= \omega/\omega_1$
$a \;= \omega_2/\omega_1$
$\varkappa \;= \Delta K_{11}/(m_1\omega_1^2)$
$\gamma_{11} \;= \Delta K_{11}/\Delta K_{11} = 1$ (1)
$\gamma_{12} \;= \Delta K_{12}/\Delta K_{11}$
$\gamma_{22} \;= \Delta K_{22}/\Delta K_{11}$
$h \;= R_m/(2m\omega_1)$
$\varrho_{11} \;= R_{11}/R_m$
$\varrho_{12} \;= R_{12}/R_m$
$\varrho_{22} \;= R_{22}/R_m$.

As obvious $(\varrho_{11} + \varrho_{22})/2 = 1$.

3. Transformation into principal coordinates.

Let us consider a system with two degrees of freedom, with constant inertia and viscous damping parameters and with harmonically variable elastic parameters with equal frequency and phase (2). This set of conditions is not only of great interest in the case of linear vibrations with variable elastic parameters but also occurs normally in the study of the stability, according to the small perturbation method, of periodic solutions of systems with two degrees of freedom with cubical nonlinear elasticity viscous damping, and harmonic excitation.

The system of differential equations representing the vibrations of the system under consideration (Fig. 1) may always be expressed easily in the form:

$$\begin{aligned} m_1\ddot{x}_1 + (k_{11} - \Delta k_{11}\cos\omega t)x_1 + (k_{12} - \Delta k_{12}\cos\omega t)x_2 +& \\ + r_{11}\dot{x}_1 + r_{12}\dot{x}_2 = 0& \\ m_2\ddot{x}_2 + (k_{21} - \Delta k_{21}\cos\omega t)x_1 + (k_{22} - \Delta k_{22}\cos\omega t)x_2 +& \\ + r_{21}\dot{x}_1 + r_{22}\dot{x}_2 = 0& \end{aligned} \tag{3.1}$$

(1) In the present work γ_{11} has been defined equal to 1. In other cases however it may be useful to define the γ's in other ways: for this reason we have left the symbol γ_{11} (> 0).

(2) Note that the amplitude may be positive or negative (the variations in the parameters may be either in phase or in counterphase).

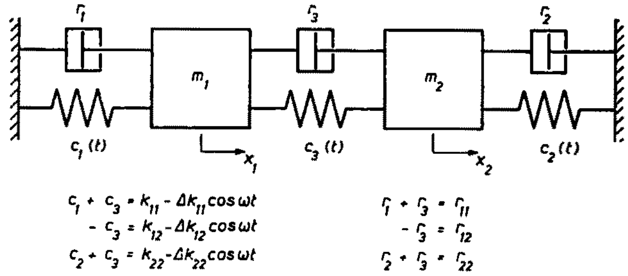

Fig. 1. Scheme of the vibrating system.

with

$$r_{11} \geqslant 0\,, \quad r_{22} \geqslant 0\,, \quad r_{11}r_{22} - r_{12}^2 \geqslant 0\,.$$

The meaning of the symbols is indicated in Sec. 2.

As in applying the method of perturbation the viscous terms and the variations in the elastic terms are regarded as small it is useful to carry out a transformation of coordinates by means of which the two equations are uncoupled in the other terms. Let us call these the *principal coordinates* of the system and denote them as w_1, w_2 and call the corresponding vibrations *principal vibrations*. Of all the possible transformations let us choose as the simplest and most significant the one for which the coupling coefficients, both elastic and viscous, remain equal.

Let us consider the differential system obtained from (3.1) omitting the terms that are to be regarded as small according to what has just been said:

$$\begin{aligned} m_1\ddot{x}_1 + k_{11}x_1 + k_{12}x_2 &= 0 \\ m_2\ddot{x}_2 + k_{12}x_1 + k_{22}x_2 &= 0. \end{aligned} \tag{3.2}$$

As is well known, it has a general solution with the form:

$$\begin{aligned} x_1 &= X_1^{(1)}\sin(\omega_1 t + \varphi_1) + X_1^{(2)}\sin(\omega_2 t + \varphi_2) \\ x_2 &= X_2^{(1)}\sin(\omega_1 t + \varphi_1) + X_2^{(2)}\sin(\omega_2 t + \varphi_2)\,, \end{aligned} \tag{3.3}$$

where:

$$\left.\begin{matrix}\omega_1^2 \\ \omega_2^2\end{matrix}\right\rangle \frac{1}{2}\left(\frac{k_{11}}{m_1} + \frac{k_{22}}{m_2}\right) \pm \sqrt{\frac{1}{4}\left(\frac{k_{11}}{m_1} - \frac{k_{22}}{m_2}\right)^2 + \frac{k_{12}^2}{m_1 m_2}} \tag{3.4}$$

$$\frac{X_2^{(1)}}{X_1^{(1)}} = a_1 = -\frac{k_{11} - m_1\omega_1^2}{k_{12}} \tag{3.5}$$

$$-\frac{X_2^{(2)}}{X_1^{(2)}} = a_2 = \frac{k_{11} - m_1\omega_2^2}{k_{12}} \tag{3.6}$$

with

$$m_1 - a_1 a_2 m_2 = 0\,. \tag{3.7}$$

$X_1^{(1)}$, $X_1^{(2)}$, φ_1, φ_2 are constants dependent on the initial conditions.

Let us first assume that ω_1^2 and ω_2^2 are different. Let ω_1 denote the smaller of the two (that is $a = \omega_2/\omega_1 > 1$).

Let the reference system be chosen in such a way that a_1 and a_2 are both positive. As the following always holds $k_{11} - m_1\omega_1^2 > 0$, $k_{11} - m_1\omega_2^2 < 0$, it follows that $k_{12} = k_{21}$ is always negative.

Supposing that:

$$x_1 = \sqrt{a_2}\, w_1 + \sqrt{a_1}\, w_2$$
$$x_2 = a_1 \sqrt{a_2}\, w_1 - a_2 \sqrt{a_1}\, w_2 \tag{3.8}$$

we have:

$$w_1 = \frac{x_1 \sqrt{a_2} + x_2/\sqrt{a_2}}{a_1 + a_2}$$
$$w_2 = \frac{x_1 \sqrt{a_1} - x_2/\sqrt{a_1}}{a_1 + a_2}. \tag{3.9}$$

These w_1 and w_2 are principal coordinates and with equal coupling coefficients. Indeed by multiplying the second (3.1) equation by a_1 and adding it to the first and multiplying the second (3.1) by a_2 and subtracting it to the first, bearing in mind the preceding positions, with some transitions we reach the differential system:

$$\begin{aligned} m_1\ddot{w}_1 + m_1\omega_1^2 w_1 - \Delta K_{11}\cos\omega t \cdot w_1 - \Delta K_{12}\cos\omega t \cdot w_2 + \\ + R_{11}\dot{w}_1 + R_{12}\dot{w}_2 = 0 \end{aligned}$$
$$(3.10)$$
$$\begin{aligned} m_1\ddot{w}_2 + m_1\omega_2^2 w_2 - \Delta K_{21}\cos\omega t \cdot w_1 - \Delta K_{22}\cos\omega t \cdot w_2 + \\ + R_{21}\dot{w}_1 + R_{22}\dot{w}_2 = 0 \end{aligned}$$

with

$$\Delta K_{11} = \frac{a_2}{a_1 + a_2}(\Delta k_{11} + 2\Delta k_{12} a_1 + \Delta k_{22} a_1^2)$$

$$\Delta K_{12} = \Delta K_{21} = \frac{\sqrt{a_1 a_2}}{a_1 + a_2}(\Delta k_{11} - \Delta k_{12} a_2 + \Delta k_{12} a_1 - \Delta k_{22} a_1 a_2) \tag{3.11}$$

$$\Delta K_{22} = \frac{a_1}{a_1 + a_2}(\Delta k_{11} - 2\Delta k_{12} a_2 + \Delta k_{22} a_2^2)$$

$$R_{11} = \frac{a_2}{a_1 + a_2}(r_{11} + 2 r_{12} a_1 + r_{22} a_1^2)$$

$$R_{12} = R_{21} = \frac{\sqrt{a_1 a_2}}{a_1 + a_2}(r_{11} - r_{12} a_2 + r_{12} a_1 - r_{22} a_1 a_2) \tag{3.12}$$

$$R_{22} = \frac{a_1}{a_1 + a_2}(r_{11} - 2 r_{12} a_2 + r_{22} a_2^2).$$

It is $R_{11} \geqslant 0$, $R_{22} \geqslant 0$, $R_{11}R_{22} - R_{12}^2 \geqslant 0$.

Let us now go over to the nondimensional form. By introducing the nondimensional symbols listed in Sec. 2 into (3.10) and denoting with ′ the derivatives with respect to φ, by simple substitutions and transformations we obtain:

$$\begin{aligned} (\Omega/2)^2 w_1'' + w_1(1 - \varkappa\gamma_{11}\cos 2\varphi) - w_2\varkappa\gamma_{12}\cos 2\varphi + \\ + w_1' b\,\Omega\varrho_{11} + w_2' b\,\Omega\varrho_{12} = 0 \end{aligned}$$
$$(3.13)$$
$$\begin{aligned} (\Omega/2)^2 w_2'' - w_1\varkappa\gamma_{12}\cos 2\varphi + w_2(a^2 - \varkappa\gamma_{22}\cos 2\varphi) + \\ + w_1' b\,\Omega\varrho_{12} + w_2' b\,\Omega\varrho_{22} = 0\,. \end{aligned}$$

4. Premises on stability.

We would recall some results taken from the general literature on linear systems with periodic coefficients.

System (3.13) admits of particular solutions of the form

$$w_1 = e^{\bar{\nu}\varphi}\bar{f}(\varphi)\,, \qquad w_2 = e^{\bar{\nu}\varphi}\bar{g}(\varphi)\,, \tag{4.1}$$

with constant $\bar{\nu}$, more generally complex (determined up to $2ni$, n any integer, $i = \sqrt{-1}$), and with $\bar{f}(\varphi)$ and $\bar{g}(\varphi)$ periodic with period π. With a simpler expression of the exponent we also obtain:

$$w_1 = e^{\nu\varphi}f(\varphi)\,, \qquad w_2 = e^{\nu\varphi}g(\varphi)\,, \tag{4.2}$$

with constant ν more generally complex (determined up to ni) and with $f(\varphi)$ and $g(\varphi)$ periodic with period 2π.

In the case in point, as this is a system with two second order differential equations, we obtain, as known, four exponents ν and hence if ν are all distinct four pairs of functions $f(\varphi)$ and $g(\varphi)$, or four linearly independent solutions out of which the general solution is constructed. If the ν values all possess a real part less than zero the general solution vanishes as φ increases and in that case there is asymptotic stability; if some of the ν values have a real part zero whilst the others remain less than zero the general solution is bounded (it does not vanish as φ increases) and there is stability; if finally some of the ν values have a real part greater than zero general solution is not bounded as φ increases and there is instability. These conclusions as to stability or instability are valid even in the case of multiple values of exponents ν except in the case of a multiple value zero, in which case there is in general instability likewise.

In the transitions from stability to instability one of the ν values thus has a real part zero, but the reverse is not always true, for there may be another ν with a real part greater than zero (3); further, in the absence of viscous damping the ν values have a real part zero throughout the region of stability and the transition from stability to instability occurs at values of ν whose imaginary parts are also equal, that is there are two coincident values of ν [2].

Hence, to determine the conditions in which the transition from stability to instability occurs with damping, we must first of all determine when the real part of a ν is equal to zero and then check that all the other ν values do not possess a real part greater than zero; in the absence of damping we have to seek the point when two ν values are coincident with a real part zero and then check that the other two ν values have not a real part greater than zero.

The conditions in which ν has a real part equal to zero are determined here by a perturbation method.

5. Description of the perturbation method.

The method of perturbation consists in obtaining solutions of (3.13) in the form of a series development of the parameter $\varkappa$, developing also $\Omega/2$ in series of $\varkappa$, start-

(3) Contrary to what happens in cases of systems with one degree of freedom where one of the two characteristic exponents has a real part zero and the other is certainly not positive.

ing with the generating solutions for $\varkappa = 0$, $\Omega/2 = a_0$, $b = 0$. It is assumed that $\varkappa$ is always sufficiently small for all the developments to be convergent. It is known [3] that the instabilities are in the neighborhood of $\Omega/2$ equal to $1/m$, a/m (called direct cases) or $(a \pm 1)/2m$ (called mixed cases) with m being any positive integer; a_0 correspondingly assumes different values and there are different developments. We shall suppose that in the cases under consideration the a_0 values will all be distinct, which first of all requires that $a \neq 1$, or as already assumed $\omega_2 \neq \omega_1$. The instabilities for the various values of m are said to be of order m. Further, the term b on which the damping depends is also supposed to be small; for the instabilities to be of order m, b must be of the same order as $\varkappa^m$, that is:

$$b = \mu \varkappa^m, \qquad \mu = \frac{b}{\varkappa^m}. \tag{5.1}$$

For equations (4.2) the solutions are put in the form:

$$w_1 = e^{\nu\varphi} f, \qquad w_2 = e^{\nu\varphi} g, \tag{5.2}$$

$$f = f_0 + \varkappa f_1 + \varkappa^2 f_2 + \dots \tag{5.3}$$

$$g = g_0 + \varkappa g_1 + \varkappa^2 g_2 + \dots \tag{5.3'}$$

$$\nu = \nu_0 + \varkappa \nu_1 + \varkappa^2 \nu_2 + \dots \tag{5.4}$$

$$\Omega/2 = a_0 + a_1 \varkappa + a_2 \varkappa^2 + \dots \tag{5.5}$$

where ν_0, ν_1, ν_2, ... are more generally complex like ν; a_0, a_1, a_2, ... real like $\Omega/2$ and $f_0, f_1, f_2, \dots, g_0, g_1, g_2, \dots$ periodic with period 2π.

By substituting the above expressions in (3.13) and separating the terms of the same order in $\varkappa$ we obtain the following equations for $\varkappa^0$, $\varkappa^1$, $\varkappa^2$ respectively:

$$\begin{cases} a_0^2(\nu_0^2 f_0 + 2\nu_0 f_0' + f_0'') + f_0 = 0 \\ a_0^2(\nu_0^2 g_0 + 2\nu_0 g_0' + g_0'') + a^2 g_0 = 0 \end{cases} \tag{5.6}$$

$$\begin{cases} a_0^2(\nu_0^2 f_1 + 2\nu_0 f_1' + f_1'') + f_1 = H_1 \\ a_0^2(\nu_0^2 g_1 + 2\nu_0 g_1' + g_1'') + a^2 g_1 = K_1 \end{cases} \tag{5.7}$$

$$\begin{cases} a_0^2(\nu_0^2 f_2 + 2\nu_0 f_2' + f_2'') + f_2 = H_2 \\ a_0^2(\nu_0^2 g_2 + 2\nu_0 g_2' + g_2'') + a^2 g_2 = K_2, \end{cases} \tag{5.8}$$

where H_1 and K_1 are expressions containing f_0 and g_0, their derivatives and complex exponential functions of φ, H_2 and K_2 containing f_0, g_0, f_1, g_1, their derivatives and likewise complex exponential functions of φ: these expressions may easily be obtained by carrying out the above substitutions.

Equations (5.6) enable us to determine the generating functions f_0 and g_0 for $\varkappa = 0$. For the various values of a_0 stated above and except for one multiply constant (assumed to be 1) all these functions have the form,

for $a_0 = 1/m$:

$$\nu_0 = 0 \qquad f_0 = e^{mi\varphi} + \lambda_0 e^{-mi\varphi}, \quad g_0 = 0 \tag{5.9}$$

$$\nu_0 = \pm (a+1) mi \qquad f_0 = 0, \quad g_0 = e^{\mp mi\varphi}, \tag{5.9'}$$

for $a_0 = a/m$:

$$\nu_0 = 0 \qquad f_0 = 0, \quad g_0 = e^{mi\varphi} + \lambda_0 e^{-mi\varphi} \tag{5.10}$$

$$\nu_0 = \pm \frac{a+1}{a} mi \qquad f_0 = e^{\mp mi\varphi}, \quad g_0 = 0, \tag{5.10'}$$

for $a_0 = (a+1)/2m$:

$$\nu_0 = \frac{1-a}{1+a} mi \qquad f_0 = e^{mi\varphi}, \quad g_0 = \lambda_0 e^{-mi\varphi} \tag{5.11}$$

$$\nu_0 = \frac{a-1}{a+1} mi \qquad f_0 = e^{-mi\varphi}, \quad g_0 = \lambda_0 e^{mi\varphi}, \tag{5.11'}$$

for $a_0 = (a-1)/2m$:

$$\nu_0 = \frac{1+a}{1-a} mi \qquad f_0 = e^{mi\varphi}, \quad g_0 = \lambda_0 e^{-mi\varphi} \tag{5.12}$$

$$\nu_0 = \frac{1+a}{a-1} mi \qquad f_0 = e^{-mi\varphi}, \quad g_0 = \lambda_0 e^{mi\varphi}. \tag{5.12'}$$

λ_0 is an unknown constant, more generally complex, which may assume two values, as will be shown later on. Thus starting from the four generating solutions just given, for each value of a_0 in the neighborhood of which there may be instability, we can obtain four particular solutions for constructing the general solution. We would point out right away, without going into further detail, that, as may easily be checked by the development of the calculations, there are unbounded solutions only as from the generating solutions containing the parameter λ_0: and so from now on we shall be concerned only with these latter solutions.

Once f_0 and g_0 are obtained, H_1 and K_1 prove to be known functions of φ in the form of a linear combination of exponential functions of the type $e^{\pm p i\varphi}$ with $p = m$, $m \pm 2$.

To obtain f_1 and g_1 we must now integrate equations (5.7). For f_1 and g_1 to be periodic, as they must be, the terms $e^{\pm mi\varphi}$, known as secular terms, which would give rise to nonperiodic particular integrals, must be zero in H_1 and K_1. The condition in which the secular terms are cancelled supplies two complex equations and hence four real equations in the five unknown quantities $\mathrm{Re}\,\lambda_0$, $\mathrm{Re}\,\nu_1$, $\mathrm{Im}\,\lambda_0$, $\mathrm{Im}\,\nu_1$ (real and imaginary parts of λ_0 and ν_1) and a_1. As in the transition from stability to instability it is necessary that $\mathrm{Re}\,\nu_1 = 0$, the above four equations generally make it possible to determine the remaining four unknowns. Then f_1 and g_1 (periodic functions in 2π) are determined by integrating equation (5.7) with secular terms zero in the way stated earlier. These f_1 and g_1 values are sums of the particular integrals of the complete system (5.7) plus the periodic part of the general integral of the associated homogeneous system. This latter part is in the same form as the generating solution with λ_1 in place of λ_0 (4); the part in $e^{mi\varphi}$ may without loss of generality be incorporated in that of the generating solution, for the general solution is defined except for a multiply constant assumed to be 1.

Once f_1 and g_1 are obtained, H_2 and K_2 prove to be known functions of φ in the form of a linear combination of exponential functions of the type $e^{\pm qi\varphi}$ with $q = m$, $m \pm 2$, $m \pm 4$. So that f_2 and g_2 may be periodic the secular terms in H_2 and K_2 must be made equal to zero. The related cancelling condition supplies another four equations in the five unknowns $\mathrm{Re}\lambda_1$, $\mathrm{Re}\,\nu_2$, $\mathrm{Im}\lambda_1$, $\mathrm{Im}\,\nu_2$ and a_2, by means of which, since $\mathrm{Re}\,\nu_2 = 0$ at the boundary of instability, the remaining four unknowns are determined.

Later f_2 and g_2 may be determined, still taking into account a term in $\lambda_2 e^{-mi\varphi}$ with λ_2 as complex unknown constant, proceeding by successive approximations. The calculations rapidly become more complicated, however, and so we shall confine ourselves to determining the boundaries of stability as far as terms of the order of $\varkappa^2$, that is (oversigning the magnitudes for $\mathrm{Re}\,\nu_1 = \mathrm{Re}\,\nu_2 = 0$) in the form:

$$\overline{\Omega}/2 = a_0 + \overline{a}_1\varkappa + \overline{a}_2\varkappa^2.$$

As a_0, $\overline{a}_1$ and $\overline{a}_2$ depend through μ also on b, the above relation sets the boundaries of the possible regions of instability in the form $\overline{\Omega} = f(\varkappa, b)$, i. e. in a form that, even though it contains many terms, is always explicit and such that successive approximations add only terms of a higher order in $\varkappa$ and b without affecting the terms already obtained. This is the advantage of the new method we are proposing.

It has therefore enabled us to discuss the results obtained and to draw interesting conclusions.

Before turning to these, we would like to give an example of the procedure described above for $m = 1$ in the direct case for $\Omega/2$ in the neighborhood of $a_0 = 1$ and in the mixed case for $\Omega/2$ in the neighborhood of $a_0 = (\alpha + 1)/2$.

In the case of $a_0 = 1$, Eqs. (5.3), (5.3'), (5.4), (5.5), become:

$$\begin{aligned} f &= f_0 + \varkappa f_1 + \varkappa^2 f_2\,, & f_0 &= e^{i\varphi} + \lambda_0 e^{-i\varphi} \\ g &= g_0 + \varkappa g_1 + \varkappa^2 g_2\,, & g_0 &= 0 \\ \nu &= \nu_0 + \varkappa\nu_1 + \varkappa^2\nu_2\,, & \nu_0 &= 0 \\ \Omega/2 &= a_0 + \varkappa a_1 + \varkappa^2 a_2\,, & a_0 &= 1\,. \end{aligned} \tag{5.13}$$

Equations (5.6) become:

$$f_0'' + f_0 = 0\,, \qquad g_0'' + \alpha^2 g_0 = 0 \tag{5.14}$$

and are identically satisfied by the f_0 and g_0 assumed.

Equations (5.7) become:

$$f_1'' + f_1 = H_1\,, \qquad g_1'' + \alpha^2 g_1 = K_1\,, \tag{5.15}$$

where, taking account of f_0 and g_0,

$$\begin{aligned} H_1 = \gamma_{11}(e^{i\varphi} + \lambda_0 e^{-i\varphi})\,\frac{e^{2i\varphi} + e^{-2i\varphi}}{2} + 2a_1(e^{i\varphi} + \lambda_0 e^{-i\varphi}) \\ - 2(\nu_1 + \mu_1\varrho_{11})\,(ie^{i\varphi} - i\lambda_0 e^{-i\varphi}) \end{aligned} \tag{5.16}$$

$$K_1 = \gamma_{12}(e^{i\varphi} + \lambda_0 e^{-i\varphi})\,\frac{e^{2i\varphi} + e^{-2i\varphi}}{2} - 2\mu\varrho_{12}(ie^{i\varphi} - i\lambda_0 e^{-i\varphi})\,.$$

(4) Indeed the homogeneous differential system in the systems (5.6) (5.7) (5.8) has always the same form.

The conditions in which the secular terms, which are those in $e^{i\varphi}$ and $e^{-i\varphi}$ in H_1, vanish are:

$$\begin{cases} \gamma_{11}\lambda_0/2 + 2a_1 - 2i(\nu_1 + \mu\varrho_{11}) = 0 \\ \gamma_{11}/(2\lambda_0) + 2a_1 + 2i(\nu_1 + \mu\varrho_{11}) = 0\,. \end{cases} \tag{5.17}$$

Assuming $\lambda_0 = |\lambda_0|(\cos\theta_0 + i\,\mathrm{sen}\,\theta_0)$, $\nu_1 = \nu_{1R} + i\nu_{1I}$ we have:

$$\begin{cases} \gamma_{11}|\lambda_0|\cos\theta_0 \; + 4a_1 + 4\nu_{1I} = 0 \\ \gamma_{11}\cos\theta_0/|\lambda_0| + 4a_1 - 4\nu_{1I} = 0 \\ \gamma_{11}|\lambda_0|\sin\theta_0 \; - 4\mu\varrho_{11} - 4\nu_{1R} = 0 \\ \gamma_{11}\sin\theta_0/|\lambda_0| - 4\mu\varrho_{11} - 4\nu_{1R} = 0 \end{cases} \tag{5.18}$$

and hence

$$|\lambda_0| = 1\,, \qquad \nu_{1I} = 0\,, \qquad \sin\theta_0 = 4(\mu\varrho_{11} + \nu_{1R})/\gamma_{11} \tag{5.19}$$

$$a_1 = \mp\sqrt{\gamma^2_{11}/16 - (\mu\varrho_{11} + \nu_{1R})^2}\,. \tag{5.19'}$$

Taking account of equation (5.5) in first approximation, we have:

$$\Omega/2 = 1 \mp \sqrt{\varkappa^2\gamma^2_{11}/16 - (b\varrho_{11} + \varkappa\nu_{1R})^2}\,. \tag{5.20}$$

Then, assuming $\nu_{1R} = 0$, we obtain the expressions of Ω on the boundary of instability in first approximation, which we shall denote as $\overline{\Omega}_1'$, $\overline{\Omega}_1''$ in terms of the parameters of the system:

$$\left.\begin{matrix}\overline{\Omega}'_1 \\ \overline{\Omega}_1''\end{matrix}\right\rangle = 2 \mp 2\sqrt{\varkappa^2\gamma^2_{11}/16 - b^2\varrho^2_{11}}\,. \tag{5.21}$$

As for $\overline{\Omega}_1' < \Omega < \overline{\Omega}_1''$ the value of ν_{1R} is positive, when Ω lies between the above two values there is instability.

We may now proceed to the next approximation. By integrating equations (5.15) (taking into account Eqs. (5.16) where the secular terms have been cancelled) we obtain:

$$\begin{aligned} f_1 &= -\gamma_{11}\,\frac{e^{3i\varphi} + \lambda_0 e^{-3i\varphi}}{16} + \lambda_1 e^{-i\varphi} \\ g_1 &= \frac{\gamma_{12}}{2}\,\frac{e^{3i\varphi} + \lambda_0 e^{-3i\varphi}}{\alpha^2 - 9} + \left(\frac{\gamma_{12}\lambda_0}{2} - 2\mu\varrho_{12}i\right)\frac{e^{i\varphi}}{\alpha^2 - 1} + \\ &\qquad + \left(\frac{\gamma_{12}}{2} + 2\mu\varrho_{22}\lambda_0 i\right)\frac{e^{-i\varphi}}{\alpha^2 - 1}. \end{aligned} \tag{5.22}$$

In the same way as in the first approximation, the H_2 and K_2 values are then determined; the secular terms, that is those in $e^{i\varphi}$ and $e^{-i\varphi}$ in H_2, are cancelled; the four consequent secular equations are solved and we obtain for $\mathrm{Re}\,\nu_2 = 0$ the values of

$$\nu_{2I} = 0\,, \qquad \lambda_1 = -i\mu\varrho_{11} \tag{5.23}$$

and

$$\overline{a}_2 = -\frac{\gamma^2_{11}}{64} - \frac{\gamma^2_{12}}{8}\,\frac{1}{\alpha^2 - 9} - \frac{\gamma^2_{12}}{8}\,\frac{1}{\alpha^2 - 1} - \frac{\mu^2\varrho^2_{11}}{2} + \frac{2\mu^2\varrho^2_{12}}{\alpha^2 - 1} \tag{5.23'}$$

We thus arrive at the corrective term $\overline{a}_2\varkappa^2$ which, added to the values (5.21), yields the expressions of Ω on the boundary of instability in second approximation, denoted as $\overline{\Omega}_2'$ and $\overline{\Omega}_2''$, as shown in the final table.

Further approximations may be obtained by proceeding in the same way.

In the mixed case of $a_0 = (\alpha+1)/2$, Eqs. (5.3), (5.3'), (5.4), (5.5), become:

$$\begin{aligned} f &= f_0 + \varkappa f_1 + \varkappa^2 f_2 & f_0 &= e^{i\varphi} \\ g &= g_0 + \varkappa g_1 + \varkappa^2 g_2 & g_0 &= \lambda_0 e^{-i\varphi} \\ \nu &= \nu_0 + \varkappa\nu_1 + \varkappa^2\nu_2 & \nu_0 &= i(1-\alpha)/(1+\alpha) \\ \Omega/2 &= a_0 + \varkappa a_1 + \varkappa^2 a_2 & a_0 &= (\alpha+1)/2\,. \end{aligned} \tag{5.24}$$

The values of f_0 and g_0 given satisfy (5.6) identically. In Eqs. (5.7)

$$H_1 = (\gamma_{11}e^{i\varphi} + \gamma_{12}\lambda_0 e^{-i\varphi})\frac{e^{2i\varphi}+e^{-2i\varphi}}{2} + \frac{2a_1}{a_0}e^{i\varphi} - $$

$$-2(a_0^2\nu_1 + 2\varrho_{11}\mu a_0)(\nu_0+i)e^{i\varphi} - 2\varrho_{12}\mu a_0(\nu_0-i)\lambda_0 e^{-i\varphi} \tag{5.25}$$

$$K_1 = (\gamma_{22}\lambda_0 e^{-i\varphi} + \gamma_{12}e^{i\varphi})\frac{e^{2i\varphi}+e^{-2i\varphi}}{2} + \frac{2a_1\alpha^2}{a_0}\lambda_0 e^{-i\varphi} - $$

$$-2(a_0^2\nu_1 + 2\varrho_{22}\mu a_0)(\nu_0-i)\lambda_0 e^{-i\varphi} - 2\varrho_{12}\mu a_0(\nu_0+i)e^{i\varphi}\,.$$

From the conditions in which the secular terms, that is those in $e^{i\varphi}$ of H_1 and in $e^{-i\varphi}$ of K_1, are cancelled, we obtain:

$$\left\{\begin{aligned} &\gamma_{12}\lambda_0 + 4a_1/a_0 - 4(a_0^2\nu_1 + \varrho_{11}\mu a_0)(\nu_0+i) = 0 \\ &\gamma_{12}/\lambda_0 + 4\alpha^2 a_1/a_0 - 4(a_0^2\nu_1 + \varrho_{22}\mu a_0)(\nu_0-i) = 0\,. \end{aligned}\right. \tag{5.26}$$

Assuming $\lambda_0 = |\lambda_0|(\cos\theta_0 + i\sin\theta_0)$, $\nu_1 = \nu_{1R} + i\nu_{1I}$ we have:

$$\left\{\begin{aligned} &\frac{2a_1}{\alpha+1} + \frac{\alpha+1}{2}\,\nu_{1I} = -\frac{\gamma_{12}}{4}|\lambda_0|\cos\theta_0 \\ &\frac{2\alpha a_1}{\alpha+1} - \frac{\alpha+1}{2}\,\nu_{1I} = -\frac{\gamma_{12}}{4\alpha}\,\frac{1}{|\lambda_0|}\cos\theta_0 \\ &\mu\varrho_{11} + \frac{\alpha+1}{2}\,\nu_{1R} = \frac{\gamma_{12}}{4}|\lambda_0|\sin\theta_0 \\ &\mu\varrho_{22} + \frac{\alpha+1}{2}\,\nu_{1R} = \frac{\gamma_{12}}{4\alpha}\,\frac{1}{|\lambda_0|}\sin\theta_0\,, \end{aligned}\right. \tag{5.27}$$

from which it follows that:

$$a_1 = \pm\frac{\left(\mu\varrho_{11} + \frac{\alpha+1}{2}\nu_{1R}\right) + \left(\mu\varrho_{22} + \frac{\alpha+1}{2}\nu_{1R}\right)}{2\sqrt{\left(\mu\varrho_{11} + \frac{\alpha+1}{2}\nu_{1R}\right)\left(\mu\varrho_{22} + \frac{\alpha+1}{2}\nu_{1R}\right)}}\,\cdot$$

$$\cdot\sqrt{\frac{\gamma^2_{12}}{16\alpha} - \left(\mu\varrho_{11} + \frac{\alpha+1}{2}\nu_{1R}\right)\left(\mu\varrho_{22} + \frac{\alpha+1}{2}\nu_{1R}\right)}\,. \tag{5.28}$$

Taking (5.5) into account and assuming $\nu_{1R} = 0$, we obtain the expressions of Ω on the boundary of instability in first approximation (recalling that $(\varrho_{11}+\varrho_{22})/2 = 1$):

$$\left.\begin{matrix}\overline{\Omega}_1' \\ \overline{\Omega}_1''\end{matrix}\right\rangle = \alpha + 1 \mp \frac{2}{\sqrt{\varrho_{11}\varrho_{22}}}\sqrt{\frac{\gamma^2_{12}\varkappa^2}{16\alpha} - b^2\varrho_{11}\varrho_{22}} \tag{5.29}$$

and it is obvious that there is instability for $\overline{\Omega}_1' < \Omega < \overline{\Omega}_1''$. Continuing on, we obtain the successive approximations.

The same results are obtained starting from the other two generating solutions (see (5.11')) $f_0 = e^{-i\varphi}$ $g_0 = \lambda_0 e^{+i\varphi}$, $\nu_0 = i(\alpha+1)/(\alpha-1)$.

By the same procedure starting from other values of a_0 we obtain the pair of values at which the real part of a characteristic exponent is zero. There is always instability when, ceteris paribus, Ω lies between at least one of these pairs and there is stability when it lies in none of them.

The expressions of $\overline{\Omega}'$ and $\overline{\Omega}''$ obtained in second approximation, i. e. including second order terms, are shown hereunder,

for $m = 1$:

$\Omega_0 = 2a_0 = 2$

$$\left.\begin{matrix}\overline{\Omega}' \\ \overline{\Omega}''\end{matrix}\right\rangle = 2\left[1 \mp \sqrt{\frac{\gamma^2_{11}\varkappa^2}{16} - b^2\varrho_{11}^2} - \left(\frac{\gamma_{11}^2}{64} + \frac{\gamma_{12}^2}{8(\alpha^2-1)} + \frac{\gamma_{12}^2}{8(\alpha^2-9)}\right)\varkappa^2 + b^2\left(\frac{2\varrho_{12}^2}{\alpha^2-1} - \frac{\varrho_{11}^2}{2}\right)\right] \tag{5.30.I}$$

$\Omega_0 = 2a_0 = 2\alpha$

$$\left.\begin{matrix}\overline{\Omega}' \\ \overline{\Omega}''\end{matrix}\right\rangle = 2\left[\alpha \mp \sqrt{\frac{\gamma^2_{22}\varkappa^2}{16\alpha^2} - b^2\varrho_{22}^2} - \left(\frac{\gamma_{22}^2}{64\alpha^3} + \frac{\gamma_{12}^2}{8\alpha(1-9\alpha^2)} + \frac{\gamma_{12}^2}{8\alpha(1-\alpha^2)}\right)\varkappa^2 + b^2\left(\frac{2\varrho_{12}^2}{1-\alpha^2} - \frac{\varrho_{22}^2}{2\alpha}\right)\right] \tag{5.30.II}$$

$\Omega_0 = 2a_0 = \alpha + 1$

$$\left.\begin{matrix}\overline{\Omega}' \\ \overline{\Omega}''\end{matrix}\right\rangle = 2\left[\frac{\alpha+1}{2} \mp \frac{1}{\sqrt{\varrho_{11}\varrho_{22}}}\sqrt{\frac{\gamma^2_{12}\varkappa^2}{16\alpha} - b^2\varrho_{11}\varrho_{22}} + \right.$$

$$+\left(\frac{\gamma_{11}^2}{16(1+\alpha)(3+\alpha)} - \frac{\gamma_{11}^2}{16(1-\alpha^2)} - \frac{\gamma_{22}^2}{16\alpha(\alpha^2-1)} + \right.$$

$$+\frac{\gamma_{22}^2}{16\alpha(1+\alpha)(1+3\alpha)} - \frac{\gamma_{12}^2}{32\alpha(\alpha+1)} -$$

$$\left.\left. - \frac{\gamma_{12}\varrho_{12}(\alpha\gamma_{11}+\gamma_{22})(\varrho_{22}-\varrho_{11})}{16\alpha(1-\alpha^2)\varrho_{11}\varrho_{22}}\right)\varkappa^2 - b^2\left(\frac{\varrho_{12}^2}{1+\alpha} + \frac{\alpha\varrho_{11}^2+\varrho_{22}^2}{4\alpha}\right)\right] \tag{5.30.III}$$

$\Omega_0 = 2a_0 = \alpha - 1$ no value of $\overline{\Omega}'$ and $\overline{\Omega}''$;

for $m = 2$:

$\Omega_0 = 2a_0 = 1$

$$\left.\begin{matrix}\overline{\Omega}' \\ \overline{\Omega}''\end{matrix}\right\rangle = 2\left[\frac{1}{2} - \left(\frac{\gamma_{11}^2}{24} + \frac{\gamma_{12}^2}{16}\,\frac{1}{\alpha^2-4} + \frac{\gamma_{12}^2}{16\alpha^2}\right)\varkappa^2 \mp \right.$$

$$\left. \mp\sqrt{\left(\frac{\gamma^2_{11}}{16} + \frac{\gamma^2_{12}}{16\alpha^2}\right)^2\varkappa^4 - \frac{b^2\varrho^2_{11}}{4}}\right] \tag{5.30.IV}$$

$$\Omega_0 = 2a_0 = a$$

$$\left.\begin{matrix}\overline{\Omega}' \\ \overline{\Omega}''\end{matrix}\right\rangle = 2\left[\frac{a}{2} - \left(\frac{\gamma_{22}^2}{24a^3} + \frac{\gamma_{12}^2}{16a}\,\frac{1}{1-4a^2} + \frac{\gamma_{12}^2}{16a}\right)\varkappa^2 \mp \right.$$

$$\left. \mp \sqrt{\left(\frac{\gamma^2{}_{22}}{16a^3} + \frac{\gamma^2{}_{12}}{16a}\right)^2 \varkappa^4 - \frac{b^2\varrho^2{}_{22}}{4}}\,\right] \qquad (5.30.\mathrm{V})$$

$$\Omega_0 = 2a_0 = (a+1)/2$$

$$\left.\begin{matrix}\overline{\Omega}' \\ \overline{\Omega}''\end{matrix}\right\rangle = 2\left[\frac{a+1}{4} - \left(\frac{\gamma_{11}^2}{4(5+a)(3-a)} + \right.\right.$$

$$\left. + \frac{\gamma_{22}^2}{4a(1+5a)(3a-1)} + \frac{\gamma_{12}^2}{12a(a+1)}\right)\varkappa^2 \mp \frac{1}{2\sqrt{\varrho_{11}\varrho_{22}}}\cdot$$

$$\left.\cdot\sqrt{\left(\frac{\gamma_{11}\gamma_{12}}{(1+a)(3-a)} + \frac{\gamma_{22}\gamma_{12}}{(1+a)(3a-1)}\right)^2 \frac{\varkappa^4}{4a} - b^2\varrho_{11}\varrho_{22}}\,\right] \qquad (5.30.\mathrm{VI})$$

$\Omega_0 = 2a_0 = \dfrac{a-1}{2}$ no value of $\overline{\Omega}'$ and $\overline{\Omega}''$;

for $m = 3$ in second approximation no value of $\overline{\Omega}'$ and $\overline{\Omega}''$ is obtained.

6. Discussion of the results obtained.

Let us consider instabilities of the first order, that is for $m = 1$ (see (5.30.I,II,III)). In all three cases the mean value of the region of instability is equal to its Ω_0 value plus terms in second approximation due both to the variation in rigidity and to the damping, whereas the amplitude of the region depends on the terms in first approximation not in second. To discuss the amplitude of the regions of instability effectively it is useful to re-introduce the dimensional expressions given in the list of symbols. Using ω_0 to denote the angular frequencies in the neighborhood of which there is instability in the various cases and ω' ω'' to denote the limits of the region of instability, we have (6):

for $\omega_0 = 2\omega_1$,

$$\frac{\omega'' - \omega'}{\omega_1} = \sqrt{\left(\frac{\Delta K_{11}}{m_1\omega_1^2}\right)^2 - 4\left(\frac{R_{11}}{m_1\omega_1}\right)^2} \qquad (6.1)$$

for $\omega_0 = 2\omega_2$,

$$\frac{\omega'' - \omega'}{\omega_2} = \sqrt{\left(\frac{\Delta K_{22}}{m_1\omega_2^2}\right)^2 - 4\left(\frac{R_{22}}{m_1\omega_2}\right)^2} \qquad (6.2)$$

for $\omega_0 = \omega_1 + \omega_2$,

$$\frac{\omega'' - \omega'}{\sqrt{\omega_1\omega_2}} = \sqrt{\left(\frac{\Delta K_{12}}{m_1\omega_1\omega_2}\right)^2 \frac{1}{\varrho_{11}\varrho_{22}} - \left(\frac{R_{11} + R_{22}}{m_1\sqrt{\omega_1\omega_2}}\right)^2}. \qquad (6.3)$$

On comparing these relations with [1], we see that for the instabilities in the direct cases within the order of approximation adopted the results are the same as those for a single degree of freedom. There is no instability if

$$\left|\frac{\Delta K_{11}}{2\omega_1 R_{11}}\right| < 1, \qquad \left|\frac{\Delta K_{22}}{2\omega_2 R_{22}}\right| < 1. \qquad (6.4)$$

The instability of the mixed case, characteristic of systems with more than one degree of freedom, has a peculiar trend because the amplitude of the region of instability decreases with the mean damping $R_m = (R_{11} + R_{22})/2$ but increases as the ratios $\varrho_{11} = R_{11}/R_m$ and $\varrho_{22} = R_{22}/R_m$ move away from 1. Thus, given the same mean damping R_m, the more R_{11} and R_{22} differ the greater is the region of instability, that is to say, the more uneven the distribution of damping between the two main modes of vibration and the closer either R_{11} or R_{22} gets to vanishing point, the larger is the region of instability of the mixed type.

For the region of instability of the mixed type there is thus a possible destabilising effect due to damping, as Ziegler [7], Bolotin [8] and Herrmann [9] found for non-conservative elastic systems. Further, it is clear that the expressions for the case without damping are obtained for vanishing damping only when ϱ_{11} and ϱ_{22} are equal to one another (and hence equal to 1), whereas other expressions are obtained if ϱ_{11} and ϱ_{22} differ from one another (we remind the reader that $\varrho_{11} + \varrho_{22} = 2$).

Given the same ϱ_{11}, ϱ_{22}, the curves for vanishing mean damping contain all the ones with mean nonzero damping, and this does not occur if the ϱ values vary too; the case without damping ($\varrho_{11} = \varrho_{22}$) may be less unstable than a case with damping but with $\varrho_{11} \neq \varrho_{22}$.

And likewise, as there is no instability in the case in point if

$$\left|\frac{\Delta K_{12}}{2\sqrt{\omega_1\omega_2\varrho_{11}\varrho_{22}}R_m}\right| = \left|\frac{\Delta K_{12}}{2\sqrt{\omega_1\omega_2 R_{11}R_{22}}}\right| < 1 \qquad (6.5)$$

when ϱ_{11} differs sufficiently from ϱ_{22} it is always possible with the same ΔK_{12} to find instability whatever be the value of mean damping $R_m = (R_{11} + R_{22})/2$.

On discussing the results for $m = 2$ (see (5.30.IV, V, VI)) we see that the mean value of the region of instability still depends on second approximation terms, but due only to the variation in rigidity. The amplitude of the region of instability depends also on terms, still in second approximation, due to damping. Thus, given the same damping, the amplitude of the regions of instability is smaller than in the cases in which $m = 1$ and instability begins when there are greater variations in rigidity. For mixed type instabilities there may be a destabilising effect due to damping and in any case, given the same mean damping, the amplitude of the region of instability increases the more ϱ_{11} differs from ϱ_{22}.

Note that in the direct cases the amplitudes of the regions of instability are not the same as for the corresponding cases with one degree of freedom, as they are affected also by variations in the coupling terms in the rigidities.

To complete the picture, we decided to find the conditions in the original system in Sec. 1 for which $\varrho_{11} = \varrho_{22}$. From the definition of ϱ_{11} and ϱ_{22}, taking into account

(6) Note that the various quantities are referred to the mass m_1 because the ΔK and the R for the particular choice of principal coordinates are referred to it too.

expressions (3.12) and (3.4) (3.5) (3.6) (3.7), with a few trasformations we find that $\varrho_{11} = \varrho_{22}$ if

$$\left(\frac{r_{11}}{m_1} - \frac{r_{22}}{m_2}\right)\left(\frac{k_{11}}{m_1} - \frac{k_{22}}{m_2}\right) + 4\,\frac{r_{12}k_{12}}{m_1 m_2} = 0. \qquad (6.6)$$

It will be noted that this condition is satisfied if, for example: $r_{11} : m_1 = r_{22} : m_2$ and $r_{12} = 0$ (6).

7. The approximation of the results.

When we reviewed the terms in second approximation we saw that some of them, in addition to increasing as $\varkappa$ increases, increase as a approaches some characteristic values which we will denote as a_c. As a is always assumed to be greater than 1 we have:

$$\begin{array}{lll} \text{for} & \Omega_0 = 2\,, & a_c = 1 \text{ and } 3 \\ \text{for} & \Omega_0 = 2a\,, & a_c = 1 \\ \text{for} & \Omega_0 = a+1\,, & a_c = 1 \\ \text{for} & \Omega_0 = 1, & a_c = 2 \\ \text{for} & \Omega_0 = (a+1)/2\,, & a_c = 3 \\ \text{for} & \Omega_0 = a\,, & \text{no value for } a_c\,. \end{array} \qquad (7.1)$$

Proceeding by successive approximations there appear other terms of a higher order in $\varkappa$ which introduce some other characteristic values a_c. In order not to overload this discussion we will merely quote the results without reporting the stages in the calculation, which are anyway based on the method we have described.

For $\Omega_0 = 2$, as the calculation is extended to approximations of the order $2k$ ($k = 1, 2, 3, \ldots$), there appear in the expression of the limit curve of stability terms in $\varkappa^{2k}$ that increase when a approaches the characteristic value $a_c = |1 \pm 2k|$, that is, in addition to 1 and 3, the values 5 and 7 and so on; for $\Omega_0 = 1$, on extending the calculation to approximations of the order $2k$ ($k = 1$, 2, 3, ...) terms in $\varkappa^{2k}$ appear that increase when a approaches $a_c = |1 \pm k|$, that is, in addition to 2, the values 1, 3, 4, ...; for $\Omega_0 = a$ and $\Omega_0 = (a+1)/2$ proceeding to the approximation of the fourth order we find that $a_c = 1$; in the other cases reviewed here there appear to be no new characteristic values of a.

The approximate formulas quoted are therefore sufficiently approximate when $\varkappa$ is sufficiently small, but the closer a gets to the above characteristic values the smaller must the $\varkappa$ value be.

The problem of the convergence of the proposed procedure is a difficult one to solve and still more difficult is the problem of determining at what value of $\varkappa$ depending upon a we should stop in order to obtain a given approximation. However, a review of the formulas quoted and of the successive approximations provide guidance as to the limits within which the regions of instability obtained by applying (5.30) are sufficiently approximate for technical purposes.

First of all in technical problems we may suppose that the elastic energy stored in the system (in nondimensional form)

$$\frac{1}{2}(1 - \gamma_{11}\varkappa\cos 2\varphi)w_1^2 + \frac{1}{2}(a^2 - \gamma_{22}\varkappa\cos 2\varphi)w_2^2 - - \gamma_{12}\varkappa\cos 2\varphi \cdot w_1 w_2 = 0$$

is positive definite for any value of φ. It is necessary that:

$$\begin{gathered} 1 - \gamma_{11}\varkappa > 0 \qquad a^2 \pm \gamma_{22}\varkappa > 0 \\ (1 - \gamma_{11}\varkappa\cos 2\varphi)(a^2 - \gamma_{22}\varkappa\cos 2\varphi) - \gamma_{12}{}^2\varkappa^2\cos^2 2\varphi > 0. \end{gathered} \qquad (7.2)$$

This amounts to assuming that the system is stable in static condition, that is for Ω vanishing.

In the particular case of the numerical examples, that will be given in Sec. 9, in which γ_{11}, and γ_{22} are both positive, the formulas above written become:

$$\begin{gathered} 1 - \gamma_{11}\varkappa > 0 \qquad a^2 - \gamma_{22}\varkappa > 0 \\ (1 - \gamma_{11}\varkappa)(a^2 - \gamma_{22}\varkappa) - \gamma_{12}{}^2\varkappa^2 > 0\,. \end{gathered} \qquad (7.3)$$

On noting then the corrective terms, we may conclude generally that we may rely upon a sufficient approximation

$$\begin{array}{ll} \text{for} \quad |\gamma_{22}| \geqslant \gamma_{11}\,, & \text{if} \quad \varkappa^2 \leqslant 4(a - a_c)^2/(\gamma_{12}{}^2|\gamma_{22}| : \gamma_{11}) \\ \text{for} \quad |\gamma_{22}| \leqslant \gamma_{11}\,, & \text{if} \quad \varkappa^2 \leqslant 4(a - a_c)^2/(\gamma_{12}{}^2\gamma_{11} : |\gamma_{22}|)\,, \end{array} \qquad (7.4)$$

considering the corresponding characteristic a_c values for each limit curve. Ceteris paribus, the approximation is better for $\Omega_0 = 2a$ and $\Omega_0 = a$ than for $\Omega_0 = 2$ and $\Omega_0 = 1$, if $\gamma_{22} > \gamma_{11}$ and viceversa for $\gamma_{11} < \gamma_{22}$. The approximation improves for a_c values corrisponding to larger values of k.

In any case a precise check on the approximation of the results may be obtained by determining the stability or instability by the exact method presented in the next section. The availability of an exact method in no way detracts from the importance of the results obtained with the approximate method described, for the exact method requires repeated numeric integration with different initial conditions of the differential system (3.13) and also repeated calculations to determine the conditions of transition from stability to instability. It therefore demands a great deal of computer time and, moreover, it obviously supplies no analytical expression. The exact method is, on the other hand, very useful for checking the approximate results obtained. Clearly, since the error increases as $\varkappa$ increases, one need only carry out the check in the conditions of maximum $\varkappa$ considered to be sure that there are smaller errors for smaller $\varkappa$ values. This justifies the determination for a first orientation of a maximum value for $\varkappa$ as stated in this paragraph.

8. Checking the limits of stability with numeric integration.

As a check on the procedure adopted and to gauge the degree of approximation, as just stated, we determined the limits of instability exactly by applying Floquet's theory directly. This required numeric integration of the

(6) In this particular case the condition $\varrho_{11} = \varrho_{22}$ coincides with the one for which the damped system with constant coefficients still allows the principal modes [10], but generally the two conditions do not coincide.

system (3.13) with assigned initial conditions, which was done by the normal method of Runge and Kutta on an IBM 7040 digital computer at the Milan Polytechnic Computing Center.

In accordance with Floquet's theory [3] the 16 values for $\varphi = \pi$ of the functions w_1 and w_2 and of their derivatives were determined, that is the 16 values

$$a_{1i} = w_{1i}(\pi), \qquad a_{2i} = w'_{1i}(\pi),$$
$$a_{3i} = w_{2i}(\pi), \qquad a_{4i} = w'_{2i}(\pi)$$

$(i = 1, 2, 3, 4)$

with the four quaterns of fundamental initial conditions

$$w_{1i}(0) = \delta_{1i}, \qquad w'_{1i}(0) = \delta_{2i},$$
$$w_{2i}(0) = \delta_{3i}, \qquad w'_{2i}(0) = \delta_{4i}$$

$(\delta_{ji} = 1$ for $j = i$, $\delta_{ji} = 0$ for $j \neq i)$.

The characteristic equation with the symbols adopted is thus valid

$$\det \left|\left| a_{ji} - \delta_{ji}\sigma \right|\right| = 0 \tag{8.1}$$

where σ is the characteristic factor (7). Solving the 4th degree equation (8.1) we obtain the four σ_i values whose modulus $|\sigma_i|$ is easy to calculate. Even when a single $|\sigma_i| > 1$, there is instability. By keeping h and $\varkappa$ fixed and varying Ω, by doing the calculation a few times in the neighborhood of the values obtained by the approximate method and interpolating as required, we arrive at the values for Ω at which the transition from stability to instability occurs.

We would point out that on occasion it is of interest, for checking the curves of $\bar{\Omega}'$ and $\bar{\Omega}''$ obtained by the proposed approximate method, to determine not only when one of the $|\sigma_i|$ becomes > 1 but also when a second becomes so.

The applications will be set forth in the next paragraph.

9. Numeric applications.

Some numeric applications were carried out. As the main purpose of these applications is to evidence some aspects of the results that are obtained by applying the proposed method and to check their degree of approximation, we have assigned the data for an unspecified system, already referred to the principal coordinates, and we may in successive works go into the behavior of particular vibrating systems that arise in the technical applications. We first of all considered the following cases by means of the second approximation formulas (5.30):

a) $a = 2.5$, $\quad \gamma_{11} = 1$, $\quad \gamma_{12} = \sqrt{a} = 1.58115$,
$\gamma_{22} = a = 2.5$,

b) $a = 1.5$, $\quad \gamma_{11} = 1$, $\quad \gamma_{12} = \sqrt{a} = 1.22475$,
$\gamma_{22} = a = 1.5$,

$$\varrho_{11} = \varrho_{22} = 1, \qquad \varrho_{12} = 0.5$$
$$h = 0, \qquad 0.025, \qquad 0.05,$$

that is in the absence of damping or in the presence of damping with equal direct dampings.

We would point out that in this particular case in which

$$\frac{\gamma_{11}}{1} = \frac{\gamma_{12}}{\sqrt{a}} = \frac{\gamma_{22}}{a},$$

the amplitudes of the regions of instability of the first order (that is for $m = 1$) are equal to one another.

The calculations were bounded by the more restrictive of the two conditions:

— that $\varkappa$ should not exceed the value at which static instability occurs (that is at Ω vanishing) as for (7.3);

— that $\varkappa$ should be such that we can rely on a sufficient degree of approximation as expounded in Sec. 7.

Therefore:

for $a = 2.5$ the limit curves of stability for $\Omega_0 = 2a$, $a + 1$, a have been limited up to maximum value $\varkappa = 0.715$, which satisfies the 3rd of the (7.3), for $\Omega_0 = 2$, $(a + 1)/2$, 1 up to the maximum value $\varkappa = 2\,|\,2.5 - 3\,|\,/(\gamma_{12}\sqrt{\gamma_{22}/\gamma_{11}}) = 0.4$ which satisfies (7.4) with $a_c = 3$, and for $\Omega_0 = 1$ likewise up to $\varkappa = 2\,|\,2.5 - 2\,|\,/(\gamma_{12}\sqrt{\gamma_{22}/\gamma_{11}})$ whose value is still 0.4, which is also obtained from (7.4) with $a_c = 2$;

for $a = 1.5$ the limit curves of stability have also been limited to the maximum value $\varkappa = 0.6$ which satisfies the 3rd of the expression (7.3), while the limitation of (7.4) is less and less restrictive.

The results have been plotted in Fig. 2 for the case $a = 2.5$ and in Fig. 3 for the case $a = 1.5$. These figures show the limit curves of instability and are drawn up the afore said limit value of $\varkappa$: the zones lying between the two branches of the limit curves are unstable. The mean lines of the regions of instability are also shown (these actually differ for the various values of h but this difference, being very small, is not shown on the diagram). In the cases $m = 1$ (that is for $\Omega_0 = 2a$, $a + 1$, 2) the mean curve in first approximation would be constant equal to Ω_0 (while the amplitude would be the same): the difference in the mean line from Ω_0 thus gives an indication of the magnitude of the terms in 2nd approximation. In the cases $m = 2$ (that is for $\Omega_0 = a, (a + 1)/2, 1$), as already stated, there is no instability in 1st approximation, and both the mean line and the amplitude depend on the terms in 2nd approximation.

The results obtained with the approximate formulas (5.30) have been checked for the maximum values of $\varkappa$ by the exact method expounded in Sec. 8. The conditions for the transition from stability to instability obtained with this method are indicated with small circles in Figs. 2 and 3.

(7) As is known, the characteristic factors σ are related to the corresponding characteristic exponents $\bar{\nu}$ by $\sigma = e^{\pi\bar{\nu}}$.

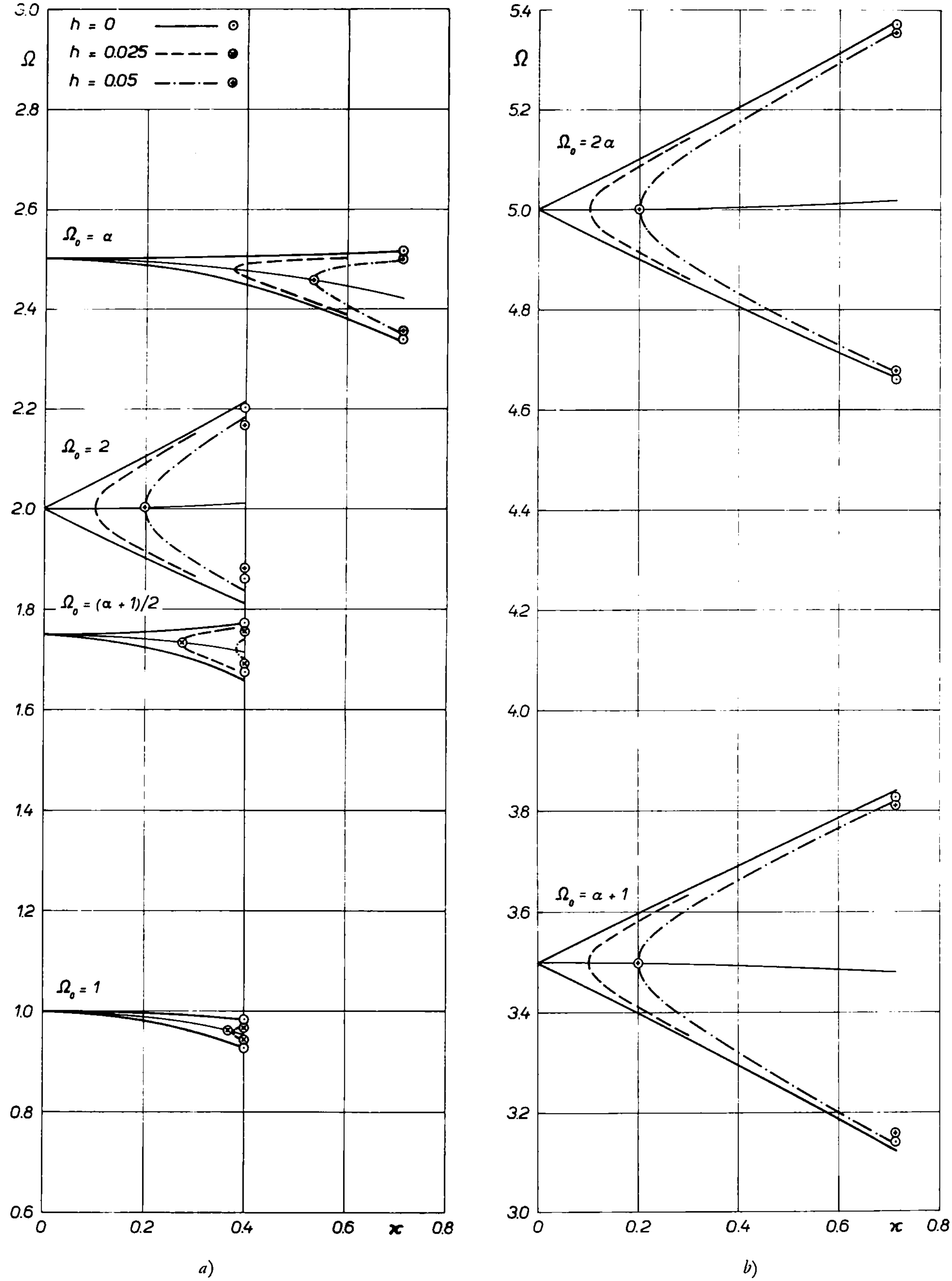

Fig. 2. Regions of instability for $\alpha = 2.5$, $\gamma_{11} = 1$, $\gamma_{12} = \sqrt{\alpha} = 1.5812$, $\gamma_{22} = \alpha = 2.5$, $\varrho_{11} = \varrho_{22} = 1$, $\varrho_{12} = 0.5$ without and with damping.

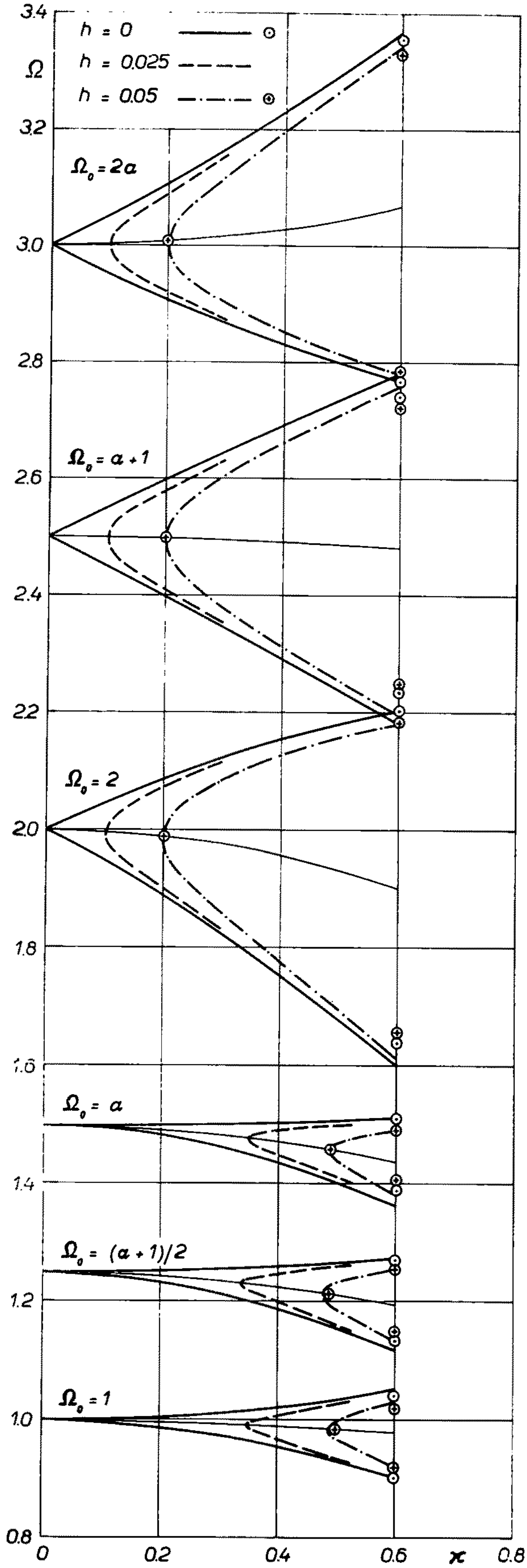

Fig. 3. Regions of instability for $a = 1.5$, $\gamma_{11} = 1$, $\gamma_{12} = \sqrt{a} = 1.2247$, $\gamma_{22} = a = 1.5$, $\varrho_{11} = \varrho_{22} = 1$, $\varrho_{12} = 0.5$ without and with damping.

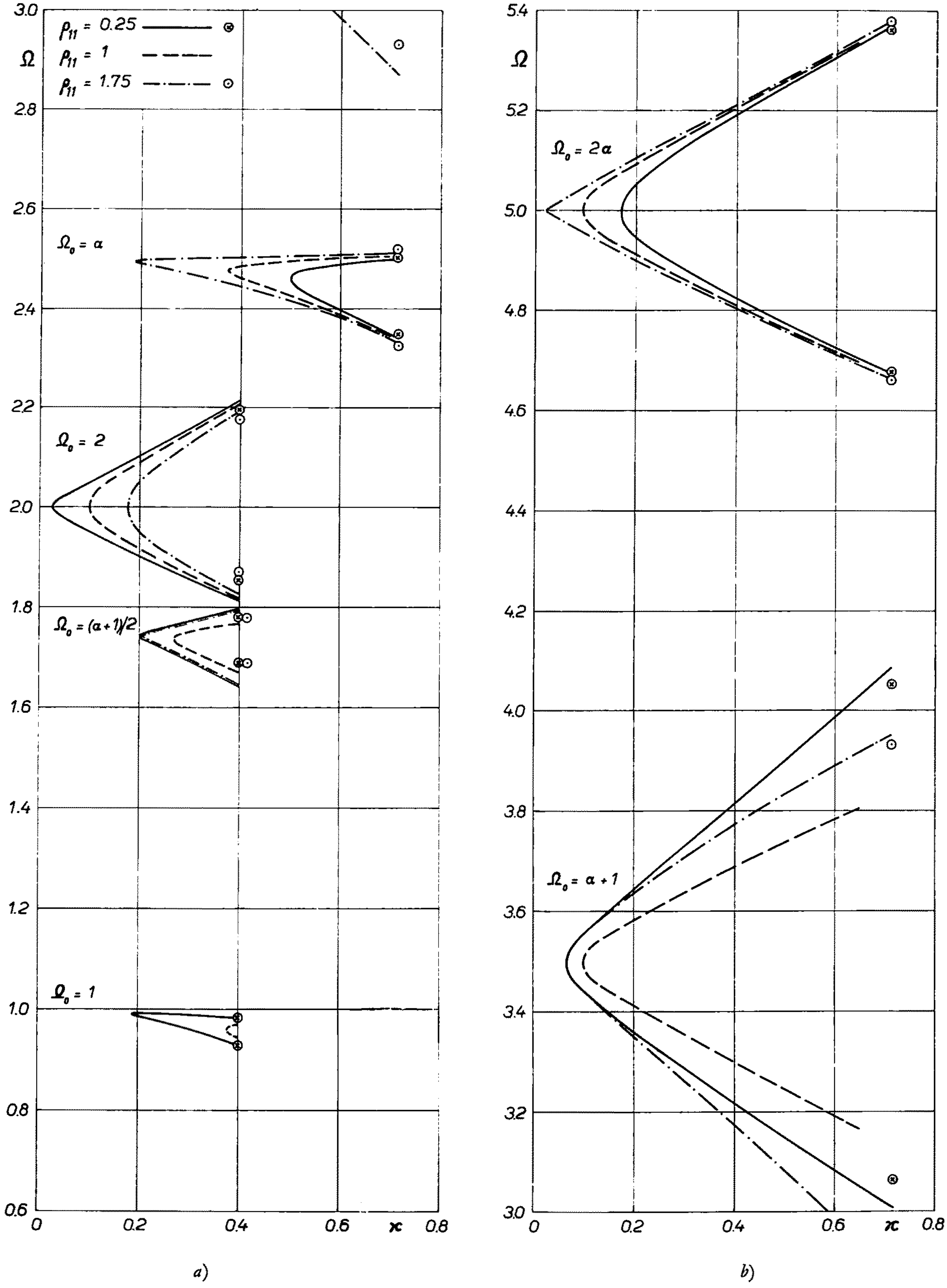

Fig. 4. Regions of instability for $\alpha = 2.5$, $\gamma_{11} = 1$, $\gamma_{12} = \sqrt{\alpha} = 1.5812$, $\gamma_{22} = \alpha = 2.5$, $b = 0.025$, $\varrho_{12} = 0.5$ and with unequal direct damping coefficients ϱ_{11}, ϱ_{22}.

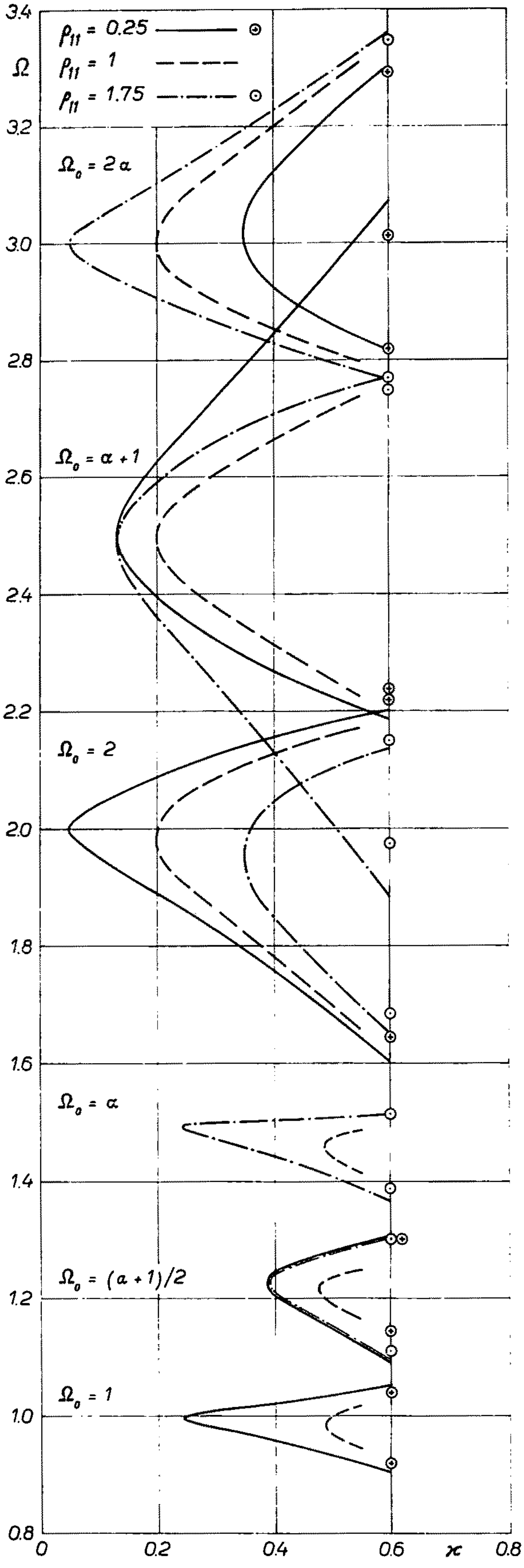

Fig. 5. Regions of instability for $\alpha = 1.5$, $\gamma_{11} = 1$, $\gamma_{12} = \sqrt{\alpha} = 1.2247$, $\gamma_{22} = \alpha = 1.5$, $h = 0.05$, $\varrho_{12} = 0.5$ and with unequal direct damping coefficients ϱ_{11}, ϱ_{22}.

It will be noted that the errors are fairly small, and as these errors depend upon terms in $\varkappa^k$ with $k \geqslant 3$, they decrease at least with the cube of $\varkappa$.

Using the same values of α, γ_{11}, γ_{12}, γ_{22} and ϱ_{12} as in the previous cases, in Fig. 4 for $\alpha = 2.5$, $h = 0.025$ and in Fig. 5 for $\alpha = 1.5$, $h = 0.05$, we have considered the cases in which ϱ_{11} and ϱ_{22} are not equal but have the values $\varrho_{11} = 0.25$ and $\varrho_{22} = 1.75$, or $\varrho_{11} = 1.75$ and $\varrho_{22} = 0.25$ (i.e. very different from one another) to show up the destabilising effect described earlier. For the purpose of comparison the curves for $\varrho_{11} = \varrho_{22} = 1$ are indicated by a dash-line in the same figures. The curves have been plotted up to the limit values of $\varkappa$ stated earlier and as a check on the adequacy of their approximation we have determined and indicated with small circles the transition from stability to instability by the exact method of Sec. 8 at the maximum values of $\varkappa$ considered.

It is clear that, in the direct cases, given the same mean damping, if ϱ_{11} increases (and ϱ_{22} decreases) the regions of instability of the direct cases $\Omega_0 = 2$ and $\Omega_0 = 1$, in which the incidence of ϱ_{11} is higher, decrease. If instead ϱ_{22} increases (and ϱ_{11} decreases) it is the regions of instability of the direct cases $\Omega_0 = 2\alpha$, $\Omega_0 = \alpha$, in which ϱ_{22} has a greater effect, that decrease. Whereas in the mixed cases the region of instability always increases, sometimes considerably, as the difference between ϱ_{11} and ϱ_{22} increases. The diagrams thus show quantitatively the destabilising effect of the viscous damping described earlier.

Received 21 September 1967.

REFERENCES

[1] E. Massa, *Sulla instabilità delle vibrazioni ad un grado di libertà a parametri variabili periodicamente nel tempo e con resistenza viscosa*, Istituto Lombardo di Scienze e Lettere, 1964.

[2] E. Mettler, *Allgemeine Theorie der Stabilitat erzwungene Schwingungen elastischer Korper*, Ingenieur Archiv, pp. 418-449, 1949.

[3] L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, Springer Verlag, Berlin-Göttinger-Heidelberg, 1959.

[4] C. S. Hsu, *On the parametric excitation of a dynamic system having multiple degrees of freedom*, Journal of Applied Mechanics, pp. 367-372, 1963.

[5] C. S. Hsu, *Further results on parametric excitation of a dynamic system*, Journal of Applied Mechanics, pp. 373-377, 1965.

[6] A. Tondl, *Some problems of rotor dynamics*, Chapmann and Hall, London, 1965.

[7] H. Ziegler, *Die Stabilitatskriterien der Elastomechanik*, Ingenieur Archiv, pp. 45-56, 1952.

[8] V. V. Bolotin, *Non conservative problems of the theory of elastic stability*, Pergamon Press, New York, 1963.

[9] G. Herrmann, Ing-Chang Jong, *On the destabilising effect of damping in non conservative elastic systems*, Journal of Applied Mechanics, pp. 592-597, 1965.

[10] T. K. Caughey, M. E. J. O'Kelly, *Classical normal modes in damped linear dynamic systems*, Journal of Applied Mechanics, pp. 583-592, 1965.