Products of B-patches

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Products and tensor products of multivariate polynomials in B-patch form are viewed as linear combinations of higher degree B-patches. Univariate B-spline segments and certain regions of simplex splines are examples of B-patches. A recursive scheme for transforming tensor product B-patch representations into B-patch representations of more variables is presented. The scheme can also be applied for transforming an n-fold product of B-patch expansions into a B-patch expansion of higher degree. Degree raising formulas are obtained as special cases. The scheme calculates the blossom of the (tensor) product surface and generalizes the pyramidal recursive scheme for B-patches.

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1. Introduction

B-patches, introduced in [13], form a basis for multivariate polynomials and are defined by a set of vector valued knots similar to how Bernstein polynomials over a triangle are defined using the corners of the triangle. In [6] B-patches are shown to coincide with certain regions of simplex splines. The univariate analog of B-patches are B-spline segments. The B-patch control points of a polynomial are found by evaluating its blossom. Moreover, the blossom of a polynomial given in B-patch form as well as the polynomial itself can be evaluated by a pyramidal scheme. See [3,11] for blossoms, also called polar forms.

In this paper we mainly study how to convert products and tensor products of polynomials in B-patch form into linear combinations of B-patches of higher degree. This includes degree raising. In the univariate case the problem of finding the B-spline expansion of a product of two B-spline curves was studied in [10]. To achieve our goal we consider homogeneous B-patches together with their multi-

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linear blossoms. The connection between blossoming and differentiation is particularly apparent in the homogeneous setting, and we utilize that blossoming algorithms can be interpreted as successive directional differentiations using the knots as directions of differentiation.

An outline of this paper follows. In the rest of this section we introduce some notation. In section 2 we formulate our problem. Homogeneous B-patches are introduced in section 3. In section 4 we emphasize the connection between differentiation and blossoming. An explicit expression for the blossom of a product of polynomials in terms of the blossoms of the factors is given, theorem 2. In section 6 we present a recursive scheme for transforming tensor product B-patch representations into linear combinations of B-patches of more variables. This can be used for converting tensor product B-spline surfaces into triangular Bézier patches. The scheme, given in theorem 5, calculates the blossom of the tensor product surface and generalizes the pyramidal recursive scheme for B-patches. The scheme may also be applied for transforming n-fold products of B-patch expansions into a B-patch expansion of higher degree. As a special case we obtain degree raising formulas given in [4]. In the univariate case the presented scheme includes schemes for B-splines developed in [10]. Finally, we have also included a section discussing some further properties of B-patches, section 5.

In this paper the following notation is used. Let $\mathbb{Z}^s_+ = \{\beta = (\beta_1, \dots, \beta_s) : \beta_i \in \mathbb{Z}, \beta_i \geqslant 0\}$. For $x \in \mathbb{R}^s, \alpha, \beta \in \mathbb{Z}^s_+$ we set $|\alpha| = \alpha_1 + \dots + \alpha_s, x^\alpha = x_1^{\alpha_1} \dots x_s^{\alpha_s}$, $\alpha! = \alpha_1! \dots \alpha_s!$, $\binom{|\alpha|}{\alpha} = |\alpha|!/\alpha!$, $\binom{\alpha}{\beta} = \alpha!/\beta!(\alpha-\beta)!$ when $\beta \leqslant \alpha$, i.e. $\beta_i \leqslant \alpha_i, i = 1, \dots, s$, and e^j denotes the jth coordinate vector in $\mathbb{R}^s$. The cardinality of a set E is denoted $|E|$. Let

$$\Gamma_{k,s} = \left\{\beta \in \mathbb{Z}^s_+ : |\beta| = \sum_{i=1}^s \beta_i = k\right\}.$$

For the homogeneous polynomials of degree k over $\mathbb{R}^s$ we write

$$\mathrm{H}^s_k = \left\{\sum_{\alpha \in \Gamma_{k,s}} a_\alpha x^\alpha : a_\alpha \in \mathbb{R}, x \in \mathbb{R}^s\right\},$$

and for the polynomials of degree $\leqslant k$ over $\mathbb{R}^{s-1}$ we write

$$\Pi^{s-1}_k = \left\{\sum_{|\alpha| \leqslant k} a_\alpha x^\alpha : a_\alpha \in \mathbb{R}, x \in \mathbb{R}^{s-1}\right\}.$$

The set of linear transformations from $\mathbb{R}^s$ into $\mathbb{R}^q$ is denoted $\mathrm{L}(\mathbb{R}^s, \mathbb{R}^q)$.

2. Problem formulation

In this paper we will mainly focus on the following conversion problem.

P: For $i = 1, \dots, n$ let $\{\hat{p}_{\alpha,i}\}_{\alpha\in\Gamma_{k_i,s_i}}$ be a basis for the polynomials $\Pi_{k_i}^{s_i-1}$. Furthermore, let affine maps $\phi_1, \dots, \phi_n$ be given, where $\phi_i : \mathbb{R}^{s-1} \to \mathbb{R}^{s_i-1}$. Consider the $s-1$ variate polynomial defined as the following sum of products;

$$\hat{h} = \sum c_{\alpha^1,\dots,\alpha^n} \prod_{i=1}^{n} \hat{p}_{\alpha^i,i} \circ \phi_i, \tag{1}$$

where the sum is taken over all tuples $(\alpha^1, \dots, \alpha^n)$ such that $\alpha^i \in \Gamma_{k_i,s_i}$. Given a basis $\{\hat{p}_\alpha\}_{\alpha\in\Gamma_{|k|,s}}$ for $\Pi_{|k|}^{s-1}$, where $k = (k_1, \dots, k_n)$. We want to find the representation of $\hat{h}$ relative to this basis. In this paper we will only consider the case that all the $n+1$ bases involved are bases of B-patches.

Some instances of **P** are:

P1: A product $\hat{h} = \prod_{i=1}^{n} \hat{f}_i$ can be written in the form (1) whenever $s_i = s$ and $\hat{f}_i = \sum_{\alpha^i} a_{\alpha^i}\hat{p}_{\alpha^i,i}$ by choosing $c_{\alpha^1,\dots,\alpha^n} = \prod_{i=1}^{n} a_{\alpha^i}$ and ϕ_i equal to the identity on $\mathbb{R}^{s-1}$ all i.

P2: Setting $n = 2$ and picking $\hat{f}_2 \equiv 1$ in **P1**, we obtain $\hat{h} = 1\hat{f}_1$. Finding the coefficients of $\hat{h}$ in this case is commonly referred to as raising the degree of $\hat{f}_1$ from k_1 to $k_1 + k_2$.

P3: If $s - 1 = \sum_{i=1}^{n}(s_i - 1)$ in **P** and $\phi_1, \dots, \phi_n$ are coordinate projections such that $\phi_i(x) = x^i$ where $x = (x^1, \dots, x^n)$, $x^i \in \mathbb{R}^{s_i-1}$, then

$$\hat{h}(x) = \sum c_{\alpha^1,\dots,\alpha^n}\hat{p}_{\alpha^1,1}(x^1)\dots\hat{p}_{\alpha^n,n}(x^n).$$

Hence $\hat{h}$ is the typical element in the tensor product space $\Pi_{k_1}^{s_1-1} \otimes \dots \otimes \Pi_{k_n}^{s_n-1}$.

Note that any composition $\hat{g} = \hat{h} \circ \psi$ of $\hat{h}$ in (1) with an affine map ψ will again be of the form (1). For example, let L be a line in $\mathbb{R}^{s-1}$ and choose $\psi : \mathbb{R} \to \mathbb{R}^{s-1}$ such that $\psi(\mathbb{R}) = L$, then $\hat{g}$ is the curve obtained by restricting the surface $\hat{h}$ to the line L.

In our treatise of **P** we prefer to work with homogeneous polynomials. Recall, for k an integer, that Π_k^{s-1} and H_k^s are isomorphic vector spaces and that the homogenization $f \in \mathrm{H}_k^s$ of $\hat{f} \in \Pi_k^{s-1}$ is given by

$$f(z,t) = t^k\hat{f}(t^{-1}z), \quad z \in \mathbb{R}^{s-1}, t \in \mathbb{R}.$$

In this paper we will continue using a $\hat{}$ notation to indicate that $\hat{f}$ is an inhomogeneous polynomial while the $\hat{}$ will be omitted for its homogeneous version f. We also let s denote the number of variables for homogeneous polynomials and $s-1$ for the corresponding inhomogeneous polynomials, rather than $s+1$ and s.

Let $h \in \mathrm{H}_{|k|}^s$, where $k \in \mathbb{Z}_+^n$, be the homogenization of the polynomial $\hat{h}$ in eq. (1), and let $p_{\alpha,i}, f_i \in \mathrm{H}_{k_i}^{s_i}$ be the homogenizations of $\hat{p}_{\alpha,i}, \hat{f}_i$. In the homogeneous version **H** of the problem **P** the polynomial h is given as follows.

H:

$$h = \sum c_{\alpha^1,\dots,\alpha^n} \prod_{i=1}^{n} p_{\alpha^i,i} \circ A_i, \tag{2}$$

where the sum is taken over the same set of indices as in (1), and where $A_i \in L(\mathbb{R}^s, \mathbb{R}^{s_i})$ is defined by

$$A_i x = t(\phi_i(t^{-1}z), 1), \quad x = (z, t) \in \mathbb{R}^{s-1} \times \mathbb{R}.$$

Corresponding to **P1, P2, P3** we have the following instances of **H**.

H1: $h = \prod_{i=1}^{n} f_i$.
H2: $h(x) = (x_s)^{k_2} f_1(x)$ where $x = (x_1, \dots, x_s) \in \mathbb{R}^s$.
H3: $A_i x = (z^i, t)$ in (2) where $x = (z^1, \dots, z^n, t) \in \mathbb{R}^s$, $z^i \in \mathbb{R}^{s_i - 1}$, $t \in \mathbb{R}$.

3. B-patches

Let us recapture some basic properties of B-patches. We follow the presentation in [6,13] except that we prefer to work with homogeneous polynomials. In addition we make use of some vector/matrix notation, originally developed for spline curves in [8].

Let k, s be positive integers. The knot vector

$$X = \{x^{i,j} \in \mathbb{R}^s : i = 1, \dots, s, j = 0, \dots, k-1\}$$

consists of ks points such that $X_\beta = \{x^{i,\beta_i}\}_{i=1}^{s}$ is a set of s linearly independent vectors for all $\beta = (\beta_1, \dots, \beta_s) \in \mathbb{Z}_+^s$, $0 \leqslant |\beta| \leqslant k-1$. One may think of X as a set of s clouds associated with the points $x^{1,0}, \dots, x^{s,0}$ in $\mathbb{R}^s$, and where each cloud consists of k points.

Define $\lambda_{\beta,1,X}, \dots, \lambda_{\beta,s,X}$ to be the coordinate functions relative to the basis X_β for $\mathbb{R}^s$;

$$x = \sum_{j=1}^{s} \lambda_{\beta,j,X}(x) x^{j,\beta_j}, \quad x \in \mathbb{R}^s, \tag{3}$$

whenever $\beta \in \Gamma_{l,s}$, $0 \leqslant l \leqslant k-1$. We note that these coordinate functions are linear.

The following recursive, pyramidal algorithm calculates homogeneous B-patches $b^l_{\beta,X}$ of degree l;

$$b^l_{\beta,X}(x) = \sum_{j=1}^{s} \lambda_{\beta-e^j,j,X}(x) b^{l-1}_{\beta-e^j,X}(x), \quad \beta \in \Gamma_{l,s}, l = 1, \dots, k, \tag{4}$$

where $b^0_{o,X}(x) = 1$, and where o is the origin of $\mathbb{R}^s$. Here $\lambda_{\beta,j,X}$ and $b^l_{\beta,X}$ are defined to be zero for all β with a strictly negative component. Algorithm (4) is a generalization of the recurrence relation for multivariate Bernstein polynomials on simplices.

We want to give a vector/matrix formulation of (4). We fix an ordering of $\Gamma_{l,s}$, i.e. for every $\beta \in \Gamma_{l,s}$ there is a corresponding number $\bar{\beta} \in \{1, 2, \ldots, |\Gamma_{l,s}|\}$. Let

$$b_X^l = \{b_{\beta,X}^l\}_{\beta \in \Gamma_{l,s}} \tag{5}$$

be the *row* vector determined by this ordering.

Equation (4) may now be written for $l = 1, 2, \ldots, k$

$$b_X^l(x) = b_X^{l-1}(x) T_X^{l-1}(x), \tag{6}$$

where $T_X^l(x)$ for $l = 0, \ldots, k-1$ is a $|\Gamma_{l,s}| \times |\Gamma_{l+1,s}|$ matrix with elements

$$T^l_{\bar{\alpha},\bar{\beta},X}(x) = \begin{cases} \lambda_{\alpha,j,X}(x) & \text{for } \beta = \alpha + e^j, j = 1, \ldots, s, \\ 0 & \text{for } \beta \notin \alpha + \{e^1, \ldots, e^s\}, \end{cases}$$

$$(\alpha, \beta) \in \Gamma_{l,s} \times \Gamma_{l+1,s} . \tag{7}$$

The B-patches of degree k over X form a basis for H_k^s, so $p \in \mathrm{H}_k^s$ can be written

$$p = \sum_{\beta \in \Gamma_{k,s}} c_\beta b_{\beta,X}^k, \tag{8}$$

or equivalently as the product $p(x) = b_X^k(x)c$, where

$$c = \{c_\beta\}_{\beta \in \Gamma_{k,s}}$$

is a column vector with ordering determined by $\Gamma_{k,s}$. It follows that p can be evaluated by the following algorithm which is sometimes called the dual to (6). For $l = 1, \ldots, k$,

$$c_X^l(x) = T_X^{k-l}(x) c_X^{l-1}(x), \tag{9}$$

where $c_X^0(x) = c$. Clearly $p(x) = c_X^k(x)$. In component form (9) is written

$$c_{\beta,X}^l(x) = \sum_{j=1}^{s} \lambda_{\beta,j,X}(x) c_{\beta+e^j,X}^{l-1}(x), \quad \beta \in \Gamma_{k-l,s}, l = 1, \ldots, k, \tag{10}$$

where the lth degree homogeneous polynomial $c_{\beta,X}^l$ is component $\bar{\beta}$ in the *column* vector c_X^l. This scheme generalizes the de Casteljau algorithm for Bézier patches on triangles.

The B-patch coefficients in (8) for the polynomial p is found by blossoming p at certain subsets of the knot vector X;

$$c_\beta = \mathcal{B}(p)(x^{1,0}, \ldots, x^{1,\beta_1-1}, \ldots, x^{s,0}, \ldots, x^{s,\beta_s-1}). \tag{11}$$

The blossom $\mathcal{B}(p)$ of p is defined according to the

BLOSSOMING PRINCIPLE

The vector space V of k-linear, symmetric forms from $\overbrace{\mathbb{R}^s \times \ldots \times \mathbb{R}^s}^{k}$ into $\mathbb{R}$ is isomorphic to H_k^s. In particular, for every $f \in \mathrm{H}_k^s$ there exists uniquely a k-linear, symmetric form $\mathcal{B}(f) \in V$ coinciding with f on its diagonal;

$$f(x) = \mathcal{B}(f)(x, \ldots, x) \quad \text{for all } x \in \mathbb{R}^s.$$

See [11,3] or standard textbooks on algebra.

The blossom of the polynomial p, and thereby the coefficients of p in (8), is found by multilinear versions of the schemes (6) and (9), see [13,6];

$$\mathcal{B}(b_X^l)(u^1, \ldots, u^l) = \mathcal{B}(b_X^{l-1})(u^1, \ldots, u^{l-1}) T_X^{l-1}(u^l), \tag{12}$$

for $l = 1, 2, \ldots, k$, and

$$\mathcal{B}(c_X^{k-l})(u^{l+1}, \ldots, u^k) = T_X^l(u^{l+1}) \mathcal{B}(c_X^{k-l-1})(u^{l+2}, \ldots, u^k), \tag{13}$$

for $l = k-1, k-2, \ldots, 0$. Duality of the two algorithms is revealed through the equation

$$\begin{aligned}\mathcal{B}(p)(u^1, \ldots, u^k) &= T_X^0(u^1) \ldots T_X^{k-1}(u^k) c \\ &= \mathcal{B}(b_X^l)(u^1, \ldots, u^l) \mathcal{B}(c_X^{k-l})(u^{l+1}, \ldots, u^k),\end{aligned} \tag{14}$$

for $l = 1, \ldots, k$.

From properties of homogeneous B-patches we easily derive properties for inhomogeneous B-patches. The *inhomogeneous B-patches* $\{\hat{b}_{\beta,X}^k\}_{\beta \in \Gamma_{k,s}}$, where

$$\hat{b}_{\beta,X}^k(z) = b_{\beta,X}^k(z, 1), \quad z \in \mathbb{R}^{s-1},$$

form a basis for Π_k^{s-1}. Moreover, the B-patch coefficients of an inhomogeneous $\hat{f}$ are found by blossoming the homogenization f.

4. Blossoms of products

In view of (11) we can solve our problem **P** if we are able to evaluate the blossom of the polynomial h in **H**. In this section we concentrate on the case $h = \prod_{i=1}^n f_i \circ A_i$. This includes **H1**, **H2** and the essential part of **H3**. We describe the blossom of h in terms of the blossoms of the polynomials f_i.

We shall repeatedly make use of the following correspondence between blossoming and differentiation of homogeneous polynomials. For $f \in \mathrm{H}_k^s$ and $0 \leqslant l \leqslant k$ it is shown in [11] that

$$\mathrm{d}_{u^1} \ldots \mathrm{d}_{u^l} f(x) = \frac{k!}{(k-l)!} \mathcal{B}(f)(\overbrace{x, \ldots, x}^{k-l}, u^1, \ldots, u^l), \quad u^i \in \mathbb{R}^s, \tag{15}$$

where $\mathrm{d}_u f$ is the directional derivative of f in direction u. In particular for any $x \in \mathbb{R}^s$

$$\mathcal{B}(f) = \frac{1}{k!}\mathrm{D}^k f(x),$$

where $\mathrm{D}^m g$ denotes the mth order total derivative of $g \in C^m(\mathbb{R}^s)$, so that

$$\mathrm{D}^m g(x)(v^1, \dots, v^m) = \mathrm{d}_{v^1} \dots \mathrm{d}_{v^m} g(x), \quad v^i \in \mathbb{R}^s.$$

We will need Leibniz' formula for the mth derivative of a product of n functions.

LEMMA 1

Assume $f_1, \dots, f_n \in C^m(\mathbb{R}^s)$. Then

$$\mathrm{d}_{v^1} \dots \mathrm{d}_{v^m} \prod_{i=1}^n f_i = \sum_{\substack{I_1 \cup \dots \cup I_n = \{1,\dots,m\} \\ I_i \cap I_j = \emptyset}} \prod_{i=1}^n \left(\prod_{j \in I_i} \mathrm{d}_{v^j} \right) f_i. \tag{16}$$

Here the sum is taken over ordered n-tuples $(I_1, \dots, I_n)$ of sets $I_i \subseteq \mathbb{Z}$.

Proof

The lemma follows by induction on m using the product rule $\mathrm{d}_v(fg) = g\mathrm{d}_v f + f\mathrm{d}_v g$. □

In addition to the product rule for differentiation the chain rule will be useful. For $g \in C^m(\mathbb{R}^q)$ and $A \in \mathrm{L}(\mathbb{R}^s, \mathbb{R}^q)$ the chain rule takes on the form

$$\mathrm{d}_v(g \circ A)(x) = (\mathrm{d}_{Av} g) \circ A(x),$$

which yields

$$\mathrm{D}^m(g \circ A)(x)(u^1, \dots, u^m) = \mathrm{D}^m g(Ax)(Au^1, \dots, Au^m), \quad u^i \in \mathbb{R}^s. \tag{17}$$

We can now blossom a product of polynomials by blossoming its factors.

THEOREM 2

Let $k \in \mathbb{Z}_+^n$. Assume $f_i \in \mathrm{H}_{k_i}^{s_i}$ and $A_i \in \mathrm{L}(\mathbb{R}^s, \mathbb{R}^{s_i})$ for $i = 1, 2, \dots, n$. The $|k|$-th degree homogeneous s-variate polynomial $\prod_{i=1}^n f_i \circ A_i$ has $|k|$-linear blossom given by

$$\mathcal{B}\left(\prod_{i=1}^n f_i \circ A_i \right)(u^1, \dots, u^{|k|}) = \frac{1}{\binom{|k|}{k}} \sum_{\substack{I_1 \cup \dots \cup I_n = \{1,\dots,|k|\} \\ |I_i| = k_i}} \prod_{i=1}^n \mathcal{B}(f_i)(A_i U_{I_i}), \tag{18}$$

where $A_i U_{I_i}$ is the sequence $\{A_i u_j\}_{j \in I_i}$.

Proof

Assume first that $s_i = s$ and A_i is the identity on $\mathbb{R}^s$ all i. Then $f_1, \dots, f_n$ will satisfy (16). The polynomial f_i is of degree k_i, so that $\mathrm{D}^j f_i(x) \equiv 0$ for $j > k_i$. Consequently, if $m = |k|$, the nonzero terms in the sum (16) appear only for $|I_i| = k_i$ all i. Hence

$$D^{|k|}\left(\prod_{i=1}^{n} f_i\right)(x)(v^1,\ldots,v^{|k|}) = \sum_{\substack{I_1\cup\ldots\cup I_n=\{1,\ldots,|k|\}\\ |I_i|=k_i}} \prod_{i=1}^{n} D^{k_i} f_i(x)(\{v^j\}_{j\in I_i}),$$

when $f_i \in \mathrm{H}^s_{k_i}$ all i.

Suppose then $f_i \in \mathrm{H}^{s_i}_{k_i}$ and $A_i \in \mathrm{L}(\mathbb{R}^s, \mathbb{R}^{s_i})$ so that $f_i \circ A_i \in \mathrm{H}^s_{k_i}$. The previous equation remains valid if we replace f_i by $f_i \circ A_i$. Application of the chain rule (17) yields the conclusion of the theorem. □

Consider the conversion problems **P1, P2, P3** when the number of factors equals $n = 2$. For the respective problems let h, f_1, f_2 be the homogenizations of $\hat{h} \in \Pi^{s-1}_{k_1+k_2}$, $\hat{f}_1 \in \Pi^{s_1-1}_{k_1}$ and $\hat{f}_2 \in \Pi^{s_2-1}_{k_2}$ as in **H1, H2, H3**. For these problems theorem 2 specializes as follows.

Assume first that $\hat{h}$ is the *product* $\hat{h} = \hat{f}_1\hat{f}_2$ as in **P1**. The blossom of the polynomial h of **H1** is given by

$$\mathcal{B}(h)(u^1,\ldots,u^{k_1+k_2}) = \frac{1}{\binom{k_1+k_2}{k_1}} \sum_{\substack{I_1\cup I_2=\{1,\ldots,k_1+k_2\}\\ |I_i|=k_i}} \mathcal{B}(f_1)(U_{I_1})\mathcal{B}(f_2)(U_{I_2}).$$

In the *degree raising* problem **P2** we have $\hat{f}_2 = 1$ and $\hat{h} = 1\hat{f}_1$. In this case the blossom of h in **H2** is given by

$$\mathcal{B}(h)(u^1,\ldots,u^{k_1+k_2}) = \frac{1}{\binom{k_1+k_2}{k_1}} \sum_{\substack{I_1\cup I_2=\{1,\ldots,k_1+k_2\}\\ |I_i|=k_i}} \mathcal{B}(f_1)(U_{I_1}) \prod_{j\in I_2} u^j_s.$$

For the tensor product $\hat{h} = \hat{f}_1 \otimes \hat{f}_2$ of **P3** we have $h = (f_1 \circ A_1)(f_2 \circ A_2)$, where A_1, A_2 are given by **H3**, and

$$\mathcal{B}(h)(u^1,\ldots,u^{k_1+k_2}) = \frac{1}{\binom{k_1+k_2}{k_1}} \sum_{\substack{I_1\cup I_2=\{1,\ldots,k_1+k_2\}\\ |I_i|=k_i}} \mathcal{B}(f_1)(A_1U_1)\mathcal{B}(f_2)(A_2U_{I_2}).$$

The last two blossoming formulas also appear in [11] as propositions 11.2 and 13.2. In [11] the latter is spoken of as a *degree joining* formula.

5. Further properties of B-patches

In this section we discuss properties of inhomogeneous B-patches like rotation and translation. Also, some further properties of B-patches are derived as easy consequences of the close relation between blossoming and differentiation.

Differentiation formulas for B-patches can be found in [14,4]. In our matrix notation such a formula can be written

$$\mathrm{d}_{v^1}\ldots\mathrm{d}_{v^l}b_X^k(x)=\frac{k!}{(k-l)!}b_X^{k-l}(x)T_X^{k-l}(v^1)\ldots T_X^{k-1}(v^l). \tag{19}$$

The proof follows easily from the recurrence relation (12) and the diagonal property of blossoms if we observe the connection (15) between derivatives of a homogeneous polynomial and its blossom. A differentiation formula for inhomogeneous B-patches is immediate from the corresponding homogeneous formula;

$$\mathrm{d}_{u^1}\ldots\mathrm{d}_{u^l}\hat{b}_X^k(z)=\frac{k!}{(k-l)!}\hat{b}_X^{k-l}(z)T_X^{k-l}(u^1,0)\ldots T_X^{k-1}(u^l,0),$$

where $u^i, z\in\mathbb{R}^{s-1}$.

Using this interpretation of blossoming as differentiation we have a simple proof that the *monomials* are B-patches. To see this we differentiate the monomial $(|\alpha|!/\alpha!)x^\alpha$ along the coordinate axes obtaining

$$\frac{1}{k!}(\mathrm{d}_{e^1})^{\beta_1}\ldots(\mathrm{d}_{e^s})^{\beta_s}\frac{|\alpha|!}{\alpha!}x^\alpha=\delta_{\alpha,\beta},\quad \alpha,\beta\in\Gamma_{k,s}.$$

Hence according to eq. (11) the homogeneous s-variate monomials of degree k satisfy

$$\frac{|\alpha|!}{\alpha!}x^\alpha=b_{\alpha,X}^k(x),\quad x\in\mathbb{R}^s,\alpha\in\Gamma_{k,s}, \tag{20}$$

where

$$X=\{x^{i,j}:x^{i,j}=e^i, i=1,\ldots,s, j=0,\ldots,k-1\}.$$

We shall also see that translates of inhomogeneous monomials are B-patches. This is a consequence of the following, more general B-patch property.

LEMMA 3

Let $X=\{x^{i,j}\in\mathbb{R}^s : i=1,\ldots,s, j=0,\ldots,k-1\}$ be a knot set for homogeneous B-patches of degree k, and assume A is a nonsingular $s\times s$ matrix. Then

$$b_{\alpha,AX}^k(x)=b_{\alpha,X}^k(A^{-1}x),\quad x\in\mathbb{R}^s,\alpha\in\Gamma_{k,s},$$

where $AX=\{Ax^{i,j}: i=1,\ldots,s, j=0,\ldots,k-1\}$.

Proof

Let $p\in\mathrm{H}_k^s$. According to eq. (11) and theorem 2 for $n=1$ the polynomial $p\circ A$ has the following expansion in B-patches on X,

$$p\circ A=\sum_{\beta\in\Gamma_{k,s}}\mathcal{B}(p)(AV_\beta)b_{\beta,X}^k,$$

where $V_\beta = \{x^{1,0}, \ldots, x^{1,\beta_1-1}, \ldots, x^{s,0}, \ldots, x^{s,\beta_s-1}\}$. The knot set AX also defines a B-patch basis. In the above equation choose p equal to the basis polynomial $p = b^k_{\alpha,AX}$. Then according to eq. (11) we have $\mathcal{B}(p)(AV_\beta) = \delta_{\alpha,\beta}$. □

As a consequence of lemma 3 translates of a set of inhomogeneous B-patches are obtained by applying linear transformations to the knot set:

$$\hat{b}^k_{\alpha,AX}(z) = \hat{b}^k_{\alpha,X}(z-a), \quad a, z \in \mathbb{R}^{s-1} \tag{21}$$

whenever the $s \times s$ matrix A is given by

$$Ax = (z + ta, t), \quad x = (z, t) \in \mathbb{R}^s.$$

Similarly we may rotate a basis of B-patches by applying appropriate matrices to the knot set.

Choosing A to be as in (21) we find that the *translated $s-1$ variate monomials of degree $\leqslant k$ are inhomogeneous B-patches* defined by the knot set obtained by applying A to X of (20).

In general, applying an arbitrary nonsingular $s \times s$ matrix A to the knot set of the monomials yields a new B-patch knot set $Y = [a^1, \ldots, a^1, \ldots, a^s, \ldots, a^s]$ consisting of k copies of each column a^i of A. An example of such a basis is the Bernstein polynomials. If A has columns $a^i = (z^i, 1) \in \mathbb{R}^s$ all i, the inhomogeneous B-patches defined by the knot set Y are the $s-1$ variate Bernstein polynomials over the simplex in $\mathbb{R}^{s-1}$ with vertices $z^1, \ldots, z^s$.

Recognizing translates of the monomials as B-patches can be useful for the evaluation of polynomials given as linear combinations of B-patches. Suppose we want to evaluate the inhomogeneous polynomial $\hat{f}$ at quite a number of points in $\mathbb{R}^{s-1}$ and that we know the coefficients of $\hat{f}$ relative to some given B-patch basis, for example the Bernstein basis. Instead of using algorithm (10) for every point of evaluation, it would be more efficient to transform the representation of $\hat{f}$ into monomial basis centered at some point $a \in \mathbb{R}^{s-1}$ using algorithm (13) and then evaluate $\hat{f}$ at the given points. Stability considerations should determine the choice of a.

6. A blossoming scheme for (tensor) products

Given knot sets $X^1, \ldots, X^n$ such that X^i defines a basis $b^{k_i}_X$ of s_i-variate B-patches of degree k_i. In this section we give two recursive algorithms for calculating the blossom of the polynomial h where

$$h \text{ is given in } \mathbf{H} \text{ with } \{p_{\alpha,i}\}_{\alpha \in \Gamma_{k_i,s_i}} = b^{k_i}_{X^i} \quad \text{for } i = 1, \ldots, n. \tag{22}$$

This yields two algorithms for solving **P**, including **P1, P2, P3.** The algorithms are mutually dual.

In order to give algorithms for manipulating polynomials given in the form (22) we shall make use of the vector/matrix notation developed in section 3. In addition we need the Kronecker product of matrices, also called the tensor product of matrices.

Let A and B be $\mu \times \nu$ and $\xi \times \eta$ matrices respectively. The Kronecker-product of A and B, $A \otimes B$ is a $\mu\xi \times \nu\eta$ matrix defined as

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,\nu}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,\nu}B \\ \vdots & \vdots & & \vdots \\ a_{\mu,1}B & a_{\mu,2}B & \dots & a_{\mu,\nu}B \end{bmatrix}, \tag{23}$$

where $a_{i,j}$ is the typical element of A. Some properties of the Kronecker product will be useful. Let A, B, C, D, E be matrices of dimensions $\mu \times \nu, \xi \times \eta, \rho \times \sigma, \nu \times \rho$ and $\eta \times \sigma$ respectively. The Kronecker-product $\otimes$ has the following properties,

(1) bilinearity,
(2) associativity, $A \otimes (B \otimes C) = (A \otimes B) \otimes C$,
(3) $A^{\mathrm{T}} \otimes B^{\mathrm{T}} = (A \otimes B)^{\mathrm{T}}$,
(4) $(A \otimes B)(D \otimes E) = AD \otimes BE$.

See e.g. [7]. When the components in the matrices A, B are differentiable functions over $\mathbb{R}^s$, we have the product rule

$$\mathrm{d}_v(A \otimes B) = (\mathrm{d}_v A) \otimes B + A \otimes (\mathrm{d}_v B) \tag{24}$$

for the derivative of the matrix valued map $A \otimes B$. For the Kronecker product of n matrices we use the notation $\otimes_{i=1}^{n} E_i = E_1 \otimes E_2 \otimes \ldots \otimes E_n$.

The polynomial h of (22) may be written as a product

$$h(x) = B^k(x)C^k,$$

where $k = (k_1, \ldots, k_n)$, and where C^k are the coefficients in the sum (2) written as a column vector and the row vector B^k contains the associated products of B-patches. More formally, let the knot sets $X^1, \ldots, X^n$ be as in (22).Using the vector valued maps $b_{X^1}^{m_1}, \ldots, b_{X^n}^{m_n}$ such that $0 \leqslant m_i \leqslant k_i$ for $i = 1, \ldots, n$ we construct the vector valued map

$$B^m = \bigotimes_{i=1}^{n} (b_{X^i}^{m_i} \circ A_i), \tag{25}$$

where $A_1, \ldots, A_n$ are the linear maps used in $\mathbf{H}$ and $m = (m_1, \ldots, m_n)$. The map B^m is a *row* vector with $|\Gamma_{m_1,s_1}| \ldots |\Gamma_{m_n,s_n}|$ components, and the typical component $\prod_{i=1}^{n} (b_{\alpha^i, X^i}^{m_i} \circ A_i)$ is a homogeneous s-variate polynomial of degree $|m|$. Furthermore, let C^k be the *column* vector defined by

$$C^k = \sum c_{\alpha^1,\ldots,\alpha^n}\left(\bigotimes_{i=1}^{n} e^{\bar{\alpha}^i}\right), \tag{26}$$

where the sum is taken over all tuples $(\alpha^1,\ldots,\alpha^n)$ such that $\alpha^i \in \Gamma_{k_i,s_i}$. Here $e^{\bar{\alpha}^i}$ denotes the $(\bar{\alpha}^i)$th coordinate vector in $\mathbb{R}^{|\Gamma_{k_i,s_i}|}$, and $\bar{\alpha}^i$ is the integer associated with α^i in the ordering of Γ_{k_i,s_i} that is used for the vector/matrix notation introduced in section 3. The blossom of h can be written as the product

$$\mathcal{B}(h)(u^1,\ldots,u^{|k|}) = \mathcal{B}^k(u^1,\ldots,u^{|k|})C^k, \tag{27}$$

where for $O=(0,\ldots,0)\leqslant m\leqslant k$ we define $\mathcal{B}^m$ to be the componentwise blossom of B^m.

In order to give recurrence relations for $\mathcal{B}^k$ we introduce the matrix valued map $\Lambda_{m,i}$ defined by

$$\Lambda_{m,i}(u) = \bigotimes_{j=1}^{i-1} I_{\mu_j} \otimes T^{m_i}_{X^i}(A_i u) \otimes \bigotimes_{j=i+1}^{n} I_{\mu_j}, \tag{28}$$

where $T^{m_i}_{X^i}$ is the matrix valued map defined in (7), and I_{μ_j} is the $\mu_j \times \mu_j$ identity matrix with $\mu_j = |\Gamma_{m_j,s_j}|$.

PROPOSITION 4

For $|m|\geqslant 1$ we have

$$\mathcal{B}^m(u^1,\ldots,u^{|m|}) = \sum_{\substack{i=1\\ m_i>0}}^{n} \frac{m_i}{|m|}\mathcal{B}^{m-e^i}(u^1,\ldots,u^{|m|-1})\Lambda_{m-e^i,i}(u^{|m|}), \tag{29}$$

where $\mathcal{B}^O(\,)=1$.

Proof

The chain rule together with eq. (19) for the derivative of $b^{m_j}_{X^j}$ yield

$$\mathrm{d}_u(b^{m_j}_{X^j}\circ A_j)(x) = (\mathrm{d}_{A_j u} b^{m_j}_{X^j})(A_j x) = m_j(b^{m_j-1}_{X^j}\circ A_j)(x)T^{m_j-1}_{X^j}(A_j u).$$

Using property 4 of the Kronecker product and eq. (24), this yields

$$\mathrm{d}_u B^m(x) = \sum_{\substack{j=1\\ m_j>0}}^{n} m_j B^{m-e^j}(x)\Lambda_{m-e^j,j}(u).$$

We set u equal to $u^{|m|}$ and differentiate this equation $|m|-1$ times with respect to x in the directions $u^1,\ldots,u^{|m|-1}$. This results in (29) utilizing the relation (15) between blossoming and differentiation. □

Proposition 4 offers a method to compute the blossom of the polynomial h in (22). Using (29) we may compute the blossoms of all the products of B-patches found in the sum in eq. (2). The blossom of h is then obtained by taking a linear combination. Alternatively we may derive a scheme for evaluating the blossom of h starting with the coefficients in (2). Define for $m, k \in \mathbb{Z}_+^n$ such that $O \leqslant m \leqslant k$ and $|k - m| > 0$ the column vector

$$\mathcal{C}^m(u^1, \ldots, u^{|k-m|}) = \sum_{\substack{i=1 \\ m_i < k_i}}^{n} \frac{k_i - m_i}{|k - m|} \Lambda_{m,i}(u^{|k-m|}) \mathcal{C}^{m+e^i}(u^1, \ldots, u^{|k-m|-1}), \quad (30)$$

where $\mathcal{C}^k(\ ) = C^k$ is given by (26). Equation (30) builds up a multilinear form recursively. The next theorem shows that this recurrence offers a method for evaluating the blossom of h which is dual to the one based on (29).

THEOREM 5

Suppose h is given by (22) and let $\mathcal{B}^m, \mathcal{C}^m$ be defined by (27), (30). Then for $l \in \{0, 1, \ldots, |k|\}$ we have

$$\mathcal{B}(h)(u^1, \ldots, u^{|k|}) = \sum_{\substack{|m|=l \\ O \leqslant m \leqslant k}} \frac{\binom{k}{m}}{\binom{|k|}{|m|}} \mathcal{B}^m(u^1, \ldots, u^{|m|}) \mathcal{C}^m(u^{|m|+1}, \ldots, u^{|k|}). \quad (31)$$

In particular we have $\mathcal{C}^O = \mathcal{B}(h)$.

Proof

We show eq. (31) by downward induction on l. Equation (31) reduces to eq. (27) for $l = |k|$. Suppose (31) is true for some $l \leqslant |k|$. Define $\mathcal{F}_j$ for $j = 1, \ldots, n$ by

$$\mathcal{F}_j(m) = \mathcal{B}^{m-e^j}(u^1, \ldots, u^{|m|-1}) \Lambda_{m-e^j,j}(u^{|m|}) \mathcal{C}^m(u^{|m|+1}, \ldots, u^{|k|})$$

for $e^j \leqslant m \leqslant k$. The recurrence relation (29) yields

$$\sum_{\substack{|m|=l \\ O \leqslant m \leqslant k}} \frac{\binom{k}{m}}{\binom{|k|}{|m|}} \mathcal{B}^m(u^1, \ldots, u^{|m|}) \mathcal{C}^m(u^{|m|+1}, \ldots, u^{|k|})$$

$$= \sum_{\substack{|m|=l \\ O \leqslant m \leqslant k}} \frac{\binom{k}{m}}{\binom{|k|}{|m|}} \sum_{\substack{j=1 \\ m_j > 0}}^{n} \frac{m_j}{|m|} \mathcal{F}_j(m) = \sum_{j=1}^{n} \sum_{\substack{|m|=l \\ e^j \leqslant m \leqslant k}} \mathcal{F}_j(m) \frac{\binom{k}{m}}{\binom{|k|}{|m|}} \frac{m_j}{|m|}$$

$$= \sum_{j=1}^{n} \sum_{\substack{|m|=l-1 \\ O \leqslant m \leqslant k-e^j}} \mathcal{F}_j(m + e^j) \frac{\binom{k}{m}}{\binom{|k|}{|m|}} \frac{k_j - m_j}{|k - m|} = \sum_{\substack{|m|=l-1 \\ O \leqslant m \leqslant k}}^{n} \frac{\binom{k}{m}}{\binom{|k|}{|m|}} \sum_{\substack{j=1 \\ m_j < k_j}} \frac{k_j - m_j}{|k - m|} \mathcal{F}_j(m + e^j)$$

$$= \sum_{\substack{|m|=l-1 \\ O \leqslant m \leqslant k}} \frac{\binom{k}{m}}{\binom{|k|}{|m|}} \mathcal{B}^m(u^1, \ldots, u^{|m|}) \mathcal{C}^m(u^{|m|+1}, \ldots, u^{|k|}).$$

For the last equality we have used eq. (30). The last expression is the right hand side of (31) with l replaced by $l - 1$, and the theorem is shown. □

For computational purposes it is useful to have the component versions of the recurrences (29), (30). Let $\mathcal{C}^m_{\alpha^1,\ldots,\alpha^n}$ denote the typical component in $\mathcal{C}^m$ and $\mathcal{B}^m_{\alpha^1,\ldots,\alpha^n} = \mathcal{B}(\prod_{i=1}^n (b^{m_i}_{\alpha^i,X^i} \circ A_i))$ the typical component of $\mathcal{B}^m$. With $\lambda_{\alpha,j,X}$ the coordinate function defined in (3) the recurrence relations (29) and (30) are given as follows in component form. For $|m| \geqslant 1$ and $(\alpha^1, \ldots, \alpha^n) \in \Gamma_{m_1,s_1} \times \ldots \times \Gamma_{m_n,s_n}$ we have

$$\mathcal{B}^m_{\alpha^1,\ldots,\alpha^n}(u^1, \ldots, u^{|m|}) = \sum_{\substack{i=1 \\ m_i>0}}^{n} \frac{m_i}{|m|} \sum_{j=1}^{s} \lambda_{\alpha^i-e^j,j,X^i}(A_i u^{|m|}) \mathcal{B}^{m-e^i}_{\alpha^1,\ldots,\alpha^i-e^j,\ldots,\alpha^n}(u^1, \ldots, u^{|m|-1}). \tag{32}$$

For $|k-m| \geqslant 1$ and $(\alpha^1, \ldots, \alpha^n) \in \Gamma_{m_1,s_1} \times \ldots \times \Gamma_{m_n,s_n}$ we have

$$\mathcal{C}^m_{\alpha^1,\ldots,\alpha^n}(u^1, \ldots, u^{|k-m|}) = \sum_{\substack{i=1 \\ m_i<k_i}}^{n} \frac{k_i - m_i}{|k-m|} \sum_{j=1}^{s} \lambda_{\alpha^i,j,X^i}(A_i u^{|k-m|}) \mathcal{C}^{m+e^i}_{\alpha^1,\ldots,\alpha^i+e^j,\ldots,\alpha^n}(u^1, \ldots, u^{|k-m|-1}). \tag{33}$$

In (32), (33) e^i is the ith coordinate vector in $\mathbb{R}^n$ when used as a superscript, and when used as a subscript e^j is the jth coordinate vector in $\mathbb{R}^{s_i}$.

By means of (32), (33) we can make algorithms for solving problems **P**, **P1**, **P2**, **P3**. Let us take a closer look at **P2**.

EXAMPLE (DEGREE RAISING)

Let k, q be nonnegative integers. Given an inhomogeneous B-patch basis $\hat{b}^k_X$ for Π^{s-1}_k. We want to express the elements in $\hat{b}^k_X$ in terms of an arbitrary given B-patch basis for Π^{s-1}_{k+q}. In order to achieve this let $Y = \{1, \ldots, 1\} \subseteq \mathbb{R}$ where the integer 1 is repeated q times. The univariate homogeneous B-patch basis b^q_Y of degree q consists of the single polynomial $b^q_{q,Y}(t) = t^q$. With $A : x \to x_s$, where $x = (x_1, \ldots, x_s)$, the typical element in $(b^q_Y \circ A) \otimes b^k_X$ is given by $(x_s)^q b^k_{\alpha,X}(x)$. Therefore, according to **H2** and eq. (11) we can raise the degree of $\hat{b}^k_{\alpha,X}$ from k to $k+q$ by evaluating the blossom of the product $(b^q_{q,Y} \circ A) b^k_{\alpha,X}$. Blossoming this product, eq. (32) specializes to

$$\mathcal{B}^{q,k}_{q,\alpha}(u^1, \ldots, u^{k+q}) = \frac{q}{k+q}(u^{k+q})_s \mathcal{B}^{q-1,k}_{q-1,\alpha}(u^1, \ldots, u^{k+q-1}) + \frac{k}{k+q} \sum_{j=1}^{s} \lambda_{\alpha-e^j,j,X}(u^{k+q}) \mathcal{B}^{q,k-1}_{q,\alpha-e^j}(u^1, \ldots, u^{k+q-1})$$

for $\alpha \in \Gamma_{k,s}$. Similarly, based on (33) we may derive degree raising formulas recurring on the coefficients. For the case $q = 1$ the formula (34) reduces to the degree raising formula given in [4], proposition 6.1.

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