Variable stepsize continuous two-step Runge-Kutta methods for ordinary differential equations*

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A general class of variable stepsize continuous two-step Runge-Kutta methods is investigated. These methods depend on stage values at two consecutive steps. The general convergence and order criteria are derived and examples of methods of order p and stage order $q = p$ or $q = p - 1$ are given for $p \leq 5$. Numerical examples are presented which demonstrate that high order and high stage order are preserved on nonuniform meshes with large variations in ratios between consecutive stepsizes.

Keywords: Continuous two-step Runge-Kutta method, convergence, order and stage order.

AMS (MOS) subject classification: 65L05, 65L06.

1. Introduction

Consider the initial-value problem for systems of ordinary differential equations (ODEs)

$$\begin{cases} y'(x) = f(y(x)), & x \in [x_0, X], \\ y(x_0) = y_0, \end{cases} \tag{1.1}$$

with sufficiently smooth function $f : R^s \rightarrow R^s$. To compute a numerical approximation to the solution $y(x)$ to (1.1) let there be given a nonuniform grid $x_0 < x_1 < \ldots < x_N$, $x_N \geq X$, and define $h_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, N-1$, and $\xi_i = h_i / h_{i-1}$, $i = 1, 2, \ldots, N$. We consider variable stepsize continuous m-stage two-step Runge-Kutta (TSRK) methods of the form

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$$\begin{cases} y_h(x_i+\theta\xi_i h_{i-1}) = \eta(\xi_i,\theta)y_h(x_{i-1}) + (1-\eta(\xi_i,\theta))y_h(x_i) \\ \qquad + h_{i-1}\displaystyle\sum_{j=1}^{m}(v_j(\xi_i,\theta)f(Y_{i-1}^j) + \xi_i w_j(\xi_i,\theta)f(Y_i^j)), \\ Y_i^j = u_j(\xi_i)y_h(x_{i-1}) + (1-u_j(\xi_i))y_h(x_i) \\ \qquad + h_{i-1}\displaystyle\sum_{j=1}^{m}(a_{js}(\xi_i)f(Y_{i-1}^j) + \xi_i b_{js}(\xi_i)f(Y_i^j)), \end{cases} \tag{1.2}$$

$i = 1, 2, \ldots, N-1$, $\theta \in [0, 1]$. Here $y_h(x)$ is an approximation to $y(x)$ and stage values Y_i^j are approximations (possibly of low order) to $y(x_i + c_j h_i)$, $j = 1, 2, \ldots, m$. The vector $c = [c_1, \ldots, c_m]^T$ may, in general, depend on ξ_i although we usually assume that it is constant. Usually the variable stepsize will be chosen according to some step selection scheme, i.e.,

$$h_i = h\Gamma(x_i, h),$$

where h is the maximum stepsize and for all $h > 0$ and $x \in [x_0, X]$ we have

$$0 < \Delta \leq \Gamma(x, h) \leq 1,$$

compare Gear [5].

We will represent the method (1.2) by the following table of the coefficients

$u(\xi_i)$	$A(\xi_i)$	$B(\xi_i)$
$\eta(\xi_i,\theta)$	$n^T(\xi_i,\theta)$	$w^T(\xi_i,\theta)$

where

$$u(\xi_i) = [u_1(\xi_i), \ldots, u_m(\xi_i)]^T,$$

$$v(\xi_i,\theta) = [v_1(\xi_i,\theta), \ldots, v_m(\xi_i,\theta)]^T,$$

$$w(\xi_i,\theta) = [w_1(\xi_i,\theta), \ldots, w_m(\xi_i,\theta)]^T,$$

$$A(\xi_i) = [a_{js}(\xi_i)]_{j,s=1}^m, \qquad B(\xi_i) = [b_{js}(\xi_i)]_{j,s=1}^m.$$

For $h_i = h = \text{const}$ and $\theta = 1$ the formulas (1.2) reduce to TSRK methods investigated recently by Jackiewicz and Tracogna [8]. Putting $A = 0$ they reduce to the methods investigated in [7,9]. This paper generalizes the work of [8] to the continuous methods (1.2) constructed directly on nonuniform meshes.

Let

$$Y_i = [Y_i^1, \ldots, Y_i^m]^T,$$
$$f(Y_i) = [f(Y_i^1), \ldots, f(Y_i^m)]^T,$$
$$e = [1, \ldots, 1] \in R^m.$$

Then the method (1.2) can be written in more compact form as

$$\begin{cases} y_h(x_i + \theta\xi_i h_{i-1}) = \eta(\xi_i,\theta) y_h(x_{i-1}) + (1 - \eta(\xi_i,\theta)) y_h(x_i) \\ \qquad + h_{i-1}(v^T(\xi_i,\theta) f(Y_{i-1}) + \xi_i w^T(\xi_i,\theta) f(Y_i)), \\ Y_i = u(\xi_i) y_h(x_{i-1}) + (e - u(\xi_i)) y_h(x_i) \\ \qquad + h_{i-1}(A(\xi_i) f(Y_{i-1}) + \xi_i B(\xi_i) f(Y_i)), \end{cases} \tag{1.3}$$

$i = 1, 2, \ldots, N-1$, $\theta \in [0, 1]$. We will always assume that the approximation $y_h(x)$ is given on the initial interval $[x_0, x_1]$. This starting function should be computed by some other self-starting method, for example, a continuous one-step method of appropriate order.

Similarly as in [4,8], the methods (1.2) can be divided into four types depending on the structure of the coefficient matrix B. For type 1 or type 2 methods, the matrix B has the form

$$B = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ b_{21} & \lambda & \cdots & 0 \\ b_{m_1} & b_{m_2} & \cdots & \lambda \end{bmatrix},$$

with $\lambda = 0$ or $\lambda \neq 0$, respectively. These methods are appropriate for nonstiff or stiff differential systems in a sequential computing environment. For type 3 or type 4 methods, the matrix B has the form

$$B = \operatorname{diag}(\lambda, \lambda, \ldots, \lambda),$$

with $\lambda = 0$ or $\lambda \neq 0$, respectively. These methods are appropriate for nonstiff or stiff systems in a parallel computing environment.

In the recent paper [10] the potential for efficient implementation of TSRK methods is investigated. The implementation issues addressed are the local error estimation and changing stepsize using Nordsieck and interpolation techniques. It is demonstrated that the constructed error estimates are very reliable in a fixed and variable stepsize environment.

2. Stability, convergence and order criteria

As in [8] we will use the recent approach by Albrecht [1–3] to investigate the convergence and order conditions for the method (1.3). Define the $(2m+2) \times (2m+2)$ matrices $\mathcal{A}(\xi_i, \theta)$ and $\mathcal{B}(\xi_i, \theta)$ by

$$\mathcal{A}(\xi_i,\theta) := \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & u(\xi_i) & e - u(\xi_i) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \eta(\xi_i,\theta) & 1 - \eta(\xi_i,\theta) \end{bmatrix},$$

$$\mathcal{B}(\xi_i,\theta) := \begin{bmatrix} 0 & 0 & 0 & 0 \\ A(\xi_i) & \xi_i B(\xi_i) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v^T(\xi_i,\theta) & \xi_i w^T(\xi_i,\theta) & 0 & 0 \end{bmatrix},$$

and let

$$z_h(x_i,\theta) := [Y_{i-1}^T, Y_i^T, y_h(x_i), y_h(x_i+\theta\xi_i h_{i-1})]^T,$$
$$f(z_h(x_i,\theta)) := [f(Y_{i-1})^T, f(Y_i)^T, f(y_h(x_i)), f(y_h(x_i+\theta\xi_i h_{i-1}))]^T.$$

Then (1.3) can be written in the form of A-method as defined by Albrecht [1]

$$z_h(x_i,\theta) = \mathcal{A}(\xi_i,\theta) z_h(x_{i-1},1) + h_{i-1}\mathcal{B}(\xi_i,\theta) f(z_h(x_i,\theta)), \tag{2.1}$$

$i = 1,2,\ldots,N-1$, $\theta \in [0,1]$, $z_h(x_0,1) = [0^T, Y_0^T, y_h(x_0), y_h(x_1)]^T$.

The product of matrices $\mathcal{A}(\xi_i,1)$ determines stability properties of (2.1). The method (2.1) is said to be zero-stable if the product $\prod_{\nu=0}^{s}\mathcal{A}(\xi_{i-\nu},1)$ is uniformly bounded with respect to i and s. For constant stepsize ($\xi_i = 1$) this is satisfied if and only if $-1 < \eta(1,1) \le 1$ since the matrix $\mathcal{A}(1,1)$ has eigenvalues $1, -\eta(1,1)$, and eigenvalue 0 of multiplicity $2m$ (compare [8]). A similar criterion is also true for variable stepsize methods. We have the following sufficient condition for zero-stability of the methods (2.1).

Theorem 1
Assume that $m \le \eta(\xi_i,1) \le 1$, $i = 1,2,\ldots,N$, where $-1 < m < 0$. Then the TSRK method (2.1) is zero-stable.

Proof
We will investigate the uniform boundedness of the product of matrices

$$\prod_{\nu=s}^{1} A(\xi_\nu,1),$$

where ξ_ν stands for the ratio of stepsizes $h_\nu/h_{\nu-1}$. Consider the 2×2 matrix N_i consisting of the last two rows and columns of $A(\xi_i,1)$. Applying to this matrix the transformation

$$N_i \longrightarrow \tilde{N}_i = S^{-1}N_iS,$$

where

$$S = \begin{bmatrix} 1-m & 1+m \\ 1-m & -1-m \end{bmatrix},$$

it is easy to show that $||\tilde{N}_i||_1 \le 1$. Put

$$\tilde{S} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & S \end{bmatrix}$$

and define

$$\tilde{A}(\xi_\nu, 1) = \tilde{S}^{-1} A(\xi_\nu, 1)\tilde{S}.$$

It can be verified by using induction that for $s \geq 3$

$$\prod_{\nu=s}^{1} \tilde{A}(\xi_\nu, 1) = \begin{bmatrix} 0 & 0 & U(\xi_s)\tilde{N}_{s-2,s-1,\dots,1} \\ 0 & 0 & U(\xi_{s-1})\tilde{N}_{s-1,s-2,\dots,1} \\ 0 & 0 & \tilde{N}_{s,s-1,\dots,1} \end{bmatrix},$$

where

$$U(\xi_\nu) = [u(\xi_\nu), e - u(\xi_\nu)]$$

and

$$\tilde{N}_{r,r-1,\dots,1} = \tilde{N}_r \tilde{N}_{r-1} \dots \tilde{N}_1.$$

Since

$$||\tilde{N}_{r,r-1,\dots,1}||_1 \leq ||\tilde{N}_r||_1 ||\tilde{N}_{r-1}||_1 \dots ||\tilde{N}_1||_1 \leq 1$$

the theorem follows. □

We define the local discretization vector $h_{i-1}d(x_i, \theta)$ as a residuum obtained by replacing $z_h(x_i, \theta)$ in (2.1) by the exact value function $z(x_i, \theta)$ defined by

$$z(x_i, \theta) := [y(x_{i-1} + ch_{i-1})^T, y(x_i + \xi_i ch_{i-1})^T, y(x_i), y(x_i + \theta\xi_i h_{i-1})]^T,$$

where

$$y(x_{i-1} + ch_{i-1}) = [y(x_{i-1} + c_1 h_{i-1}), \dots, y(x_{i-1} + c_m h_{i-1})]^T.$$

This gives

$$h_{i-1}d(x_i, \theta) = z(x_i, \theta) - \mathcal{A}(\xi_i, \theta)z(x_{i-1}, 1) - h_{i-1}\mathcal{B}(\xi_i, \theta)f(z(x_i, \theta)), \tag{2.2}$$

and similarly as in [8], using the Taylor series technique, we obtain

$$h_{i-1}d(x_i, \theta) = h_{i-1}[0^T, d(x_i)^T, 0, \hat{d}(x_i, \theta)]^T,$$

where

$$h_{i-1}d(x_i) = \sum_{\mu=1}^{\infty} C_\mu(\xi_i)y^{(\mu)}(x_i)h_{i-1}^{\mu}, \tag{2.3}$$

$$h_{i-1}\hat{d}(x_i, \theta) = \sum_{\mu=1}^{\infty} \hat{C}_\mu(\xi_i, \theta)y^{(\mu)}(x_i)h_{i-1}^{\mu}, \tag{2.4}$$

and where $C_\mu(\xi_i)$ and $\hat{C}_\mu(\xi_i, \theta)$ are defined by

$$C_\mu(\xi_i) = \frac{\xi_i^\mu c^\mu}{\mu!} - \frac{(-1)^\mu u(\xi_i)}{\mu!} - \frac{A(\xi_i)(c-e)^{\mu-1}}{(\mu-1)!} - \frac{\xi_i^\mu B(\xi_i)c^{\mu-1}}{(\mu-1)!}, \tag{2.5}$$

$$\hat{C}_\mu(\xi_i,\theta) = \frac{\xi_i^\mu \theta^\mu}{\mu!} - \frac{(-1)^\mu \eta(\xi_i,\theta)}{\mu!} - \frac{v^T(\xi_i,\theta)(c-e)^{\mu-1}}{(\mu-1)!} - \frac{\xi_i^\mu w^T(\xi_i,\theta)c^{\mu-1}}{(\mu-1)!}. \tag{2.6}$$

Definition 1
The method is said to have order of consistency p if

$$\sup_i |h_{i-1}\hat{d}(x_i,\theta)| = O(h^{p+1})$$

and stage order of consistency q if

$$\sup_i \|h_{i-1}d(x_i)\| = O(h^q),$$

where $h = i \to \max\{h_i\}$.

Definition 2
The method has order of convergence p if

$$\sup_i |y_h(x_i + \theta\xi_i h_{i-1}) - y(x_i + \theta\xi_i h_{i-1})| = O(h^p)$$

and stage order of convergence q if

$$\sup_i \|y(x_i + \xi_i c h_{i-1}) - Y_i\| = O(h^q).$$

Remark
It follows from the form of the local discretization error $h_{i-1}d(x_i,\theta)$ that the method has order of consistency p if and only if $\hat{C}_1(\xi_i,\theta) = \hat{C}_2(\xi_i,\theta) = \ldots = \hat{C}_p(\xi_i,\theta) = 0$ and it has stage order of consistency q if and only if $C_1(\xi_i) = C_2(\xi_i) = \cdots = C_{q-1}(\xi_i) = 0$.

Subtracting (2.1) and (2.2), we obtain

$$q(x_i,\theta) = \mathcal{A}(\xi_i,\theta)q(x_{i-1},1) + h_{i-1}\mathcal{B}(\xi_i,\theta)t(x_i,\theta) + h_{i-1}d(x_i,\theta), \tag{2.7}$$

$i = 1,2,\ldots,N$, $\theta \in [0,1]$, where

$$q(x_i,\theta) := z(x_i,\theta) - z_h(x_i,\theta),$$
$$t(x_i,\theta) := f(z(x_i,\theta)) - f(z_h(x_i,\theta)).$$

It is convenient to partition $q(x_i,\theta)$ and $t(x_i,\theta)$ into four parts as follows

$$q(x_i,\theta) = \begin{bmatrix} q(x_{i-1}) \\ q(x_i) \\ \hat{q}(x_{i-1},1) \\ \hat{q}(x_i,\theta) \end{bmatrix} = \begin{bmatrix} y(x_{i-1} + ch_{i-1}) - Y_{i-1} \\ y(x_i + \xi_i c h_{i-1}) - Y_i \\ y(x_i) - y_h(x_i) \\ y(x_i + \theta\xi_i h_{i-1}) - y_h(x_i + \theta\xi_i h_{i-1}) \end{bmatrix}$$

and

$$t(x_i,\theta) = \begin{bmatrix} t(x_{i-1}) \\ t(x_i) \\ \hat{t}(x_{i-1},1) \\ \hat{t}(x_i,\theta) \end{bmatrix} = \begin{bmatrix} f(y(x_{i-1}+ch_{i-1})) - f(Y_{i-1}) \\ f(y(x_i+\xi_i ch_{i-1})) - f(Y_i) \\ f(y(x_i)) - f(y_h(x_i)) \\ f(y(x_i+\theta\xi_i h_{i-1})) - f(y_h(x_i+\theta\xi_i h_{i-1})) \end{bmatrix}.$$

We can assume without loss of generality that

$$q(x_0,\theta) = [0^T, q(x_0)^T, \hat{q}(x_0), \hat{q}(x_0,\theta)]^T.$$

Observe that $\hat{q}(x_i,\theta)$ is the discretization error on the interval $[x_i, x_{i+1}]$ while the vector $q(x_i)$ represents the error of the stage values Y_i computed at the current step from x_i to x_{i+1}. In what follows we will write $\hat{q}(x_{i+1})$ instead of $\hat{q}(x_i,1)$ and $\hat{d}(x_{i+1})$ instead of $\hat{d}(x_i,1)$, and we define $h = \max_i\{h_i\}$.

After these preparations, we can now formulate the convergence and order criteria.

Theorem 2

Assume that the TSRK method (2.1) is zero-stable, $\hat{q}(x_0) = O(h^p)$, $\hat{q}(x_1) = O(h^p)$, $\hat{d}(x_l) = O(h^p)$, $\hat{d}(x_l,\theta) = O(h^{p-1})$, $l = 1,2,\dots,N-1$, $\theta \in [0,1]$, $h \to 0$, and that

$$v^T(\xi_l,1)t(x_{l-1}) + \xi_l w^T(\xi_l,1)t(x_l) = O(h^p), \tag{2.8}$$

$$v^T(\xi_l,\theta)t(x_{l-1}) + \xi_l w^T(\xi_l,\theta)t(x_l) = O(h^{p-1}), \tag{2.9}$$

$l = 1,2,\dots,N-1$, $\theta \in [0,1]$, as $h \to 0$. Then the method (2.1) is convergent with uniform order p, i.e.,

$$\hat{q}(x_i,\theta) = O(h^p), \qquad h \to 0,$$

$i = 1,2,\dots,N-1$, $\theta \in [0,1]$, and the errors $q(x_i) = y(x_i+\xi_i ch_{i-1}) - Y_i$ of the stages Y_i are given by

$$q(x_i) = h_{i-1}(A(\xi_i)t(x_{i-1}) + \xi_i B(\xi_i)t(x_i)) + h_{i-1}d(x_i) + O(h^p), \tag{2.10}$$

where $h_{i-1}d(x_i)$ is defined by (2.3) with $C_\mu(\xi_i)$ given by (2.5).

Proof

The equation (2.7) for $\theta = 1$ takes the form

$$q(x_i,1) = \mathcal{A}(\xi_i,1)q(x_{i-1},1) + h_{i-1}\mathcal{B}(\xi_i,1)t(x_i,1) + h_{i-1}d(x_i,1)$$

and its solution is

$$q(x_i,1) = \prod_{\nu=0}^{i-1}\mathcal{A}(\xi_{i-\nu},1)q(x_0,1) + \sum_{\mu=1}^{i-1} h_\mu \prod_{\nu=0}^{i-\mu-2}\mathcal{A}(\xi_{i-\nu},1)(\mathcal{B}(\xi_{\mu+1},1)t(x_{\mu+1},1) + d(x_{\mu+1},1)),$$

$i = 1, 2, \ldots, N-1$. Substituting the above formula into (2.7), we get

$$\begin{aligned} q(x_i, \theta) = {} & \mathcal{A}(\xi_i, \theta) \prod_{\nu=0}^{i-2} \mathcal{A}(\xi_{i-1-\nu}, 1) q(x_0, 1) \\ & + h_{i-1} d(x_i, \theta) + h_{i-2} \mathcal{A}(\xi_i, \theta) d(x_{i-1}, 1) \\ & + \mathcal{A}(\xi_i, \theta) \sum_{\mu=0}^{i-3} h_\mu \prod_{\nu=0}^{i-\mu-3} \mathcal{A}(\xi_{i-1-\nu}, 1) d(x_{\mu+1}, 1) \\ & + h_{i-1} \mathcal{B}(\xi_i, \theta) t(x_i, \theta) + h_{i-2} \mathcal{A}(\xi_i, \theta) \mathcal{B}(\xi_{i-1}, 1) t(x_{i-1}, 1) \\ & + \mathcal{A}(\xi_i, \theta) \sum_{\mu=0}^{i-3} h_\mu \prod_{\nu=0}^{i-\mu-3} \mathcal{A}(\xi_{i-1-\nu}, 1) \mathcal{B}(\xi_{\mu+1}, 1) t(x_{\mu+1}, 1), \end{aligned} \tag{2.11}$$

$i = 1, 2, \ldots, N-1$, $\theta \in [0, 1]$. It can be verified that for $i \geq 2$ and $s \leq i-1$ the products $\mathcal{A}(\xi_i, \theta) \prod_{\nu=0}^{i-1-s} \mathcal{A}(\xi_{i-1-\nu}, 1)$ have the following structure

$$\mathcal{A}(\xi_i, \theta) \prod_{\nu=0}^{i-1-s} \mathcal{A}(\xi_{i-1-\nu}, 1) = \begin{bmatrix} 0 & 0 & \alpha_{11}^{(i,s)} & \alpha_{12}^{(i,s)} \\ 0 & 0 & \alpha_{21}^{(i,s)} & \alpha_{22}^{(i,s)} \\ 0 & 0 & \beta_{11}^{(i,s)} & \beta_{12}^{(i,s)} \\ 0 & 0 & \beta_{21}^{(i,s)}(\theta) & \beta_{22}^{(i,s)}(\theta) \end{bmatrix},$$

where $\alpha_{kl}^{(i,s)}$, $k, l = 1, 2$, $\beta_{11}^{(i,s)}$, and $\beta_{12}^{(i,s)}$ depend on the ratios of stepsizes $\xi_i, \xi_{i-1}, \ldots, \xi_s$, and $\beta_{21}^{(i,s)}(\theta)$ and $\beta_{22}^{(i,s)}(\theta)$ depend on $\xi_i, \xi_{i-1}, \ldots, \xi_s$, and θ. Since by assumption the TSRK method (2.1) is zero-stable, these elements are uniformly bounded with respect to i and s. We have also

$$\mathcal{A}(\xi_i, \theta)\mathcal{B}(\xi_{i-1}, 1) = \begin{bmatrix} A(\xi_{i-1}) & \xi_{i-1} B(\xi_{i-1}) & 0 & 0 \\ (e - u(\xi_i)) v^T(\xi_{i-1}, 1) & \xi_{i-1}(e - u(\xi_i)) w^T(\xi_{i-1}, 1) & 0 & 0 \\ v^T(\xi_{i-1}, 1) & \xi_{i-1} w^T(\xi_{i-1}, 1) & 0 & 0 \\ (1 - \eta(\xi_i, \theta)) v^T(\xi_{i-1}, 1) & \xi_{i-1}(1 - \eta(\xi_i, \theta)) w^T(\xi_{i-1}, 1) & 0 & 0 \end{bmatrix},$$

and it follows from equation (2.11) that

$$\begin{aligned} q(x_i) = {} & \alpha_{21}^{(i,1)} \hat{q}(x_0) + \alpha_{22}^{(i,1)} \hat{q}(x_1) \\ & + h_{i-1} d(x_i) + h_{i-2}(e - u(\xi_i)) \hat{d}(x_i) + \sum_{\mu=0}^{i-3} h_\mu \alpha_{22}^{(i,\mu+2)} \hat{d}(x_{\mu+2}) \\ & + h_{i-1} (A(\xi_i) t(x_{i-1}) + \xi_i B(\xi_i) t(x_i)) \\ & + h_{i-2}(e - u(\xi_i))(v^T(\xi_{i-1}, 1) t(x_{i-2}) + \xi_{i-1} w^T(\xi_{i-1}, 1) t(x_{i-1})) \\ & + \sum_{\mu=0}^{i-3} h_\mu \alpha_{22}^{(i,\mu+2)} (v^T(\xi_{\mu+1}, 1) t(x_\mu) + \xi_{\mu+1} w^T(\xi_{\mu+1}, 1) t(x_{\mu+1})) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\hat{q}(x_i,\theta) = {} & \beta_{21}^{(i,1)}(\theta)\hat{q}(x_0) + \beta_{22}^{(i,1)}(\theta)\hat{q}(x_1) \\ & + h_{i-1}\hat{d}(x_i,\theta) + h_{i-2}(1-\eta(\xi_i,\theta))\hat{d}(x_i) + \sum_{\mu=0}^{i-3} h_\mu \beta_{22}^{(i,\mu+2)}(\theta)\hat{d}(x_{\mu+2}) \\ & + h_{i-1}(v^T(\xi_i,\theta)t(x_{i-1}) + \xi_i w^T(\xi_i,\theta)t(x_i)) \\ & + h_{i-2}(1-\eta(\xi_i,\theta))(v^T(\xi_{i-1},1)t(x_{i-2}) + \xi_{i-1}w^T(\xi_{i-1},1)t(x_{i-1})) \\ & + \sum_{\mu=0}^{i-3} h_\mu \beta_{22}^{(i,\mu+2)}(\theta)(v^T(\xi_{\mu+1},1)t(x_\mu) + \xi_{\mu+1}w^T(\xi_{\mu+1},1)t(x_{\mu+1})),\end{aligned} \tag{2.13}$$

$i = 1,2,\ldots,N-1$, $\theta \in [0,1]$. These equations and relations (2.8) and (2.9) imply $\hat{q}(x_i,\theta) = O(h^p)$ and formula (2.10) for $q(x_i)$. This completes the proof. □

We will not attempt to use this theorem to derive the more convenient form of order conditions for (2.1) as was done in [8] for constant stepsize methods but concentrate instead on derivation of methods with stage order q equal to p or $p-1$, where p stands for the order. We believe that such methods are the most useful in practical computations. Methods of high stage order are characterized by the following result.

Theorem 3
Assume that (2.1) is zero-stable, $\hat{q}(x_0) = O(h^p)$, $\hat{q}(x_1) = O(h^p)$, and that it has order of consistency $p-1$ and stage order of consistency p, i.e.,

$$\hat{C}_\mu(\xi_i,\theta) = 0, \qquad C_\mu(\xi_i) = 0,$$

for $\mu = 1,2,\ldots,p-1$, $\theta \in [0,1]$ and that $\hat{C}_p(\xi_i,1) = 0$. Then the method has order of convergence p and stage order of convergence p, i.e.,

$$\hat{q}(x_i,\theta) = O(h^p) \qquad \text{and} \qquad q(x_i) = O(h^p)$$

as $h \to 0$ for $i = 1,2,\ldots,N-1$, $\theta \in [0,1]$.

Proof
It follows from the definition of $q(x_i)$ and $t(x_i)$ that

$$||t(x_i)|| \le L||q(x_i)||,$$

where L is the Lipschitz constant for the function f. This inequality, the zero-stability of the method (2.1), and equation (2.12) imply that there exists a constant M independent of h and ratios of stepsizes such that

$$||q(x_i)|| \le M\sum_{\mu=0}^{i} h_\mu ||q(x_\mu)|| + O(h^p),$$

$i = 1, 2, \ldots, N$. Hence, using standard arguments we obtain $\|q(x_i)\| = O(h^p)$. Using (2.13) we have also $|\hat{q}(x_i, \theta)| = O(h^p)$ which completes the proof. □

3. Examples of methods

Consider first one-stage TSRK methods of the form

$$\begin{array}{c|c|c} u(\xi) & a(\xi) & \lambda(\xi) \\ \hline \eta(\xi,\theta) & v(\xi,\theta) & w(\xi,\theta) \end{array}$$

Solving $\hat{C}_1(\xi,\theta) = 0$ and $\hat{C}_2(\xi,\theta) = 0$ for $v(\xi,\theta)$ and $w(\xi,\theta)$, $\hat{C}_3(\xi,\theta) = 0$ for $\eta(\xi,\theta)$, $C_1(\xi) = 0$ for $a(\xi)$ and $C_2(\xi) = 0$ for $u(\xi)$ we obtain a two parameter family of methods depending on c and λ. The coefficients are

$$u(\xi) = \frac{\xi(2c - 2c^2 - 2\lambda + 2c\lambda + c^2\xi - 2c\lambda\xi)}{2c - 1},$$

$$a(\xi) = \frac{\xi(c - \lambda + c^2\xi - 2c\lambda\xi)}{2c - 1},$$

$$\eta(\xi,\theta) = \frac{\xi^2\theta(6c - 6c^2 - 3\theta + 3c\theta + 3c\theta\xi - 2\theta^2\xi)}{6c^2\xi - 3c\xi + 3c - 1},$$

$$v(\xi,\theta) = \frac{\xi^2\theta(1 + \xi\theta)(2c - \theta + 3c^2\xi - 2c\xi\theta)}{(1 - c + c\xi)(6c^2\xi - 3c\xi + 3c - 1)},$$

$$w(\xi,\theta) = \frac{\theta(1 + \xi\theta)(1 - 4c + 3c^2 + \theta\xi - 2c\xi\theta)}{(1 - c + c\xi)(1 - 3c + 3c\xi - 6c^2\xi)}.$$

These methods are zero-stable if

$$m \le \eta(\xi_i, 1) \le 1, \tag{2.14}$$

where $-1 < m < 0$ (compare theorem 1).

The function $\eta^2(\xi, 1) - 1$ can be expressed as

$$\frac{(\xi^2 - 1)(c - f_1)(c - f_2)(c - f_3)(c - f_4)}{(c - f_5)^2(c - f_6)^2},$$

where

$$f_1 = \frac{-3 + 6\xi + 3\xi^2 - \sqrt{3}\sqrt{\Delta_1}}{12\xi},$$

$$f_2 = \frac{-3 + 6\xi + 3\xi^2 + \sqrt{3}\sqrt{\Delta_1}}{12\xi},$$

$$f_3 = \frac{3 - 3\xi + 9\xi^2 + 3\xi^3 - \sqrt{3}\sqrt{\Delta_2}}{12\xi(\xi - 1)},$$

$$f_4 = \frac{3 - 3\xi + 9\xi^2 + 3\xi^3 + \sqrt{3}\sqrt{\Delta_2}}{12\xi(\xi - 1)},$$

$$\Delta_1 = 3 - 4\xi - 2\xi^2 - 4\xi^3 + 3\xi^4,$$

$$\Delta_2 = 3 + 2\xi + 13\xi^2 + 12\xi^3 + 13\xi^4 + 2\xi^5 + 3\xi^6.$$

It is easy to check that the inequality (2.14) is satisfied for (ξ, c) belonging to the region shaded in figure 1. Choosing, for example, $c = 5/4$, the condition (2.14) is satisfied if $0 < \xi < 2.32015$. In addition, if $9/14 < \xi < 6/5$, then

$$||\mathcal{A}(\xi, 1)||_\infty = 1.$$

In the next section we will demonstrate the effect of these restrictions on the ratio ξ on the accuracy of numerical computations for type 1 methods (i.e., $\lambda = 0$).

The functions f_1 and f_3, in figure 1, have both the horizontal asymptote $c = 2/3$ for $\xi \to +\infty$ and both approach this asymptote from below. The function f_2 has a slant asymptote for $\xi \to +\infty$. Therefore, there are no values of c for which the inequality (2.14) is satisfied for every $\xi > 0$.

In figure 2, c_{opt} stands for the value of c corresponding to the largest interval of ξ for which (2.14) is satisfied. This value is $c = 0.655858$ and the corresponding $\xi_{\max}$ is 11.7512.

Putting, for example, $\lambda = 1/2$ we obtain a one-parameter family of type 2 methods of order 3 and stage order 3 depending on c. We will next choose $c = (6 \pm \sqrt{6})/6$ so that for constant stepsize ($\xi = 1$) and for $\theta = 1$ these methods reduce to the A_0-stable methods constructed in [6]. It can be verified that if

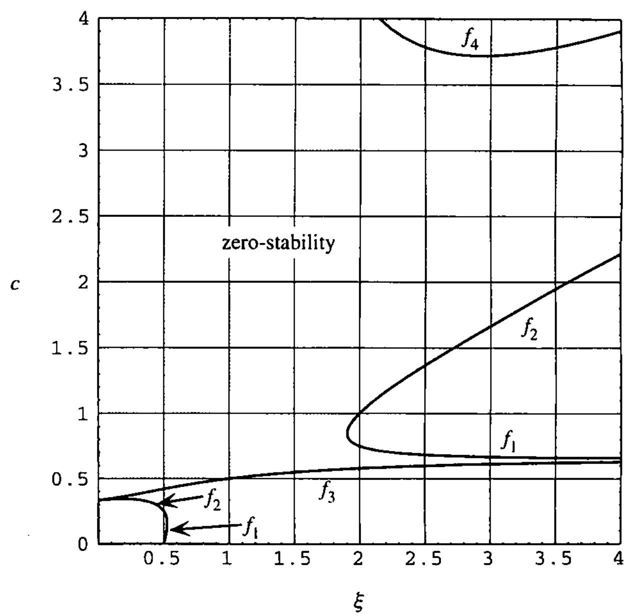

Figure 1.

$c = (6 - \sqrt{6})/6$ this method is zero-stable if $0 < \xi < 2.29041$ and if $c = (6 + \sqrt{6})/6$ this method is zero-stable if $0 < \xi < 2.56508$. In this last case, however, the method is not defined for $\xi = (\sqrt{6} - 1)/5$.

Consider next the two-stage methods given by

$$\begin{array}{c|cc|cc} u_1(\xi) & a_{11}(\xi) & a_{12}(\xi) & \lambda(\xi) & 0 \\ u_2(\xi) & a_{21}(\xi) & a_{22}(\xi) & b(\xi) & \lambda(\xi) \\ \hline 0 & v_1(\xi,\theta) & v_2(\xi,\theta) & w_1(\xi,\theta) & w_2(\xi,\theta) \end{array}$$

where we have chosen $\eta(\xi, \theta) = 0$ to guarantee that the resulting methods are zero-stable for any ratio of stepsizes. Solving the system of equations $\hat{C}_\mu(\xi, \theta) = 0$, $\mu = 1, 2, 3, 4$, for $v_i(\xi, \theta)$ and $w_i(\xi, \theta)$, the system $C_\mu(\xi) = 0$, $\mu = 1, 2$, for $a_{ij}(\xi)$ and $C_3(\xi) = 0$ for $u_i(\xi)$ we obtain a family of variable stepsize continuous methods of order 4 and stage order 4 which depend on the parameters c_1, c_2, λ, and b. The coefficients of these methods are quite complicated and are not reproduced here. We could choose the free parameters c_1, c_2, λ, and b in such a way that when the stepsize is kept constant and $\theta = 1$ the resulting discrete method has good stability properties. Choosing, for example, $\lambda = 0$, $c_1 = 2.15183$, $c_2 = -3.00706$, and $b = 2.87785$ we obtain a type 1 method of the form

$$u_1(\xi) = -\xi(1.73943 - 1.15774\xi - 0.581684\xi^2),$$
$$u_2(\xi) = \xi(4.75706 - 0.835826\xi - 3.92123\xi^2),$$
$$a_{11}(\xi) = \xi(0.48891 + 1.23582\xi + 0.395434\xi^2),$$

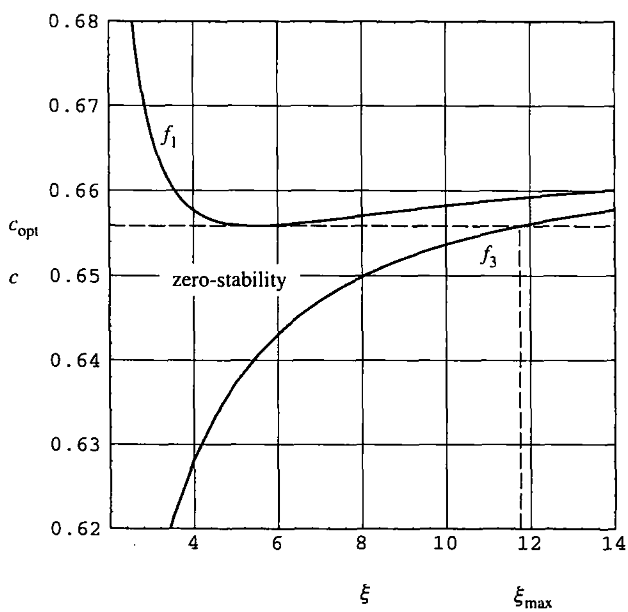

Figure 2.

$$a_{12}(\xi) = -\xi(0.076508 + 0.078078\xi - 0.18625\xi^2),$$
$$a_{21}(\xi) = -\xi(1.33709 + 0.892194\xi + 2.66569\xi^2),$$
$$a_{22}(\xi) = \xi(0.209237 + 0.0563681\xi - 1.25554\xi^2),$$

$$\begin{aligned} v_1(\xi,\theta) = {} & -\xi^2\theta(0.776729\xi - 0.0513301\xi\theta - 0.0400127\xi\theta^2 \\ & + 0.0969201\xi^2\theta - 0.00853995\xi^2\theta^2 - 0.00748917\xi^2\theta^3)/ \\ & (0.205034 + 0.152237\xi - \xi^2), \\ v_2(\xi,\theta) = {} & -\xi^2\theta(0.223271\xi - 0.0147549\xi\theta - 0.0115017\xi\theta^2 \\ & - 0.0969201\xi^2\theta + 0.00853995\xi^2\theta^2 + 0.00748917\xi^2\theta^3)/ \\ & (2.48143 - 0.529613\xi - \xi^2), \\ w_1(\xi,\theta) = {} & \theta(0.581011 + 0.0966078\theta - 0.179714\xi\theta \\ & - 0.0398426\xi\theta^2 - 0.0419613\xi^2\theta^2 - 0.0104657\xi^2\theta^3)/ \\ & (0.996778 - 1.32688\xi - \xi^2), \\ w_2(\xi,\theta) = {} & \theta(0.212903 - 0.0494702\theta - 0.0658535\xi\theta \\ & + 0.0204023\xi\theta^2 - 0.0153761\xi^2\theta^2 + 0.00535919\xi^2\theta^3)/ \\ & (0.510423 + 0.949509\xi - \xi^2). \end{aligned}$$

For $\xi = 1$ and $\theta = 1$ these formulas reduce to the type 1 method

$$\begin{array}{c|cc|cc} 0 & 2.12016 & 0.031664 & 0 & 0 \\ 0 & -4.89497 & -0.989938 & 2.87785 & 0 \\ \hline 0 & 1.19223 & -0.122001 & -0.304963 & 0.234739 \end{array}$$

obtained in [8] whose interval of absolute stability is approximately $(-2.27, 0)$.

Choosing $\lambda = 3/4$, $c_1 = 0.716383$, $c_2 = 2.17828$, $b = 2.70983$ we obtain a type 2 method of the form

$$u_1(\xi) = -\xi(0.0251613 + 0.562435\xi - 0.587597\xi^2),$$
$$u_2(\xi) = -\xi(0.959201 + 2.40963\xi - 3.36883\xi^2),$$
$$a_{11}(\xi) = -\xi(0.0559807 + 0.453683\xi - 0.674571\xi^2),$$
$$a_{12}(\xi) = -\xi(0.00279765 + 0.108752\xi + 0.0869733\xi^2),$$
$$a_{21}(\xi) = -\xi(2.1341 + 1.94371\xi - 3.86747\xi^2),$$
$$a_{22}(\xi) = -\xi(0.106652 + 0.465923 + 0.498639\xi^2),$$

$$\begin{aligned} v_1(\xi,\theta) = {} & \xi^3\theta(0.805994 - 0.747551\theta + 0.172168\theta^2 \\ & - 0.342021\xi\theta + 0.422962\xi\theta^2 - 0.109588\xi\theta^3)/ \\ & (0.0515473 + 0.526104\xi + \xi^2), \end{aligned}$$

$$v_2(\xi,\theta) = \xi^3\theta(0.194006 - 0.179939\theta + 0.0414415\theta^2 + 0.342021\xi\theta - 0.422962\xi\theta^2 + 0.109588\xi\theta^3)/(0.889689 - 2.18568\xi + \xi^2),$$

$$w_1(\xi,\theta) = \theta(0.970258 - 0.222712\theta + 1.29878\xi\theta - 0.397494\xi\theta^2 - 0.967799\xi^2\theta^2 + 0.333221\xi^2\theta^3)/(0.651164 + 1.24886\xi - \xi^2),$$

$$w_2(\xi,\theta) = -\theta(0.0345129 - 0.0240883\theta + 0.0461988\xi\theta - 0.0429926\xi\theta^2 - 0.0344255\xi^2\theta^2 + 0.0360409\xi^2\theta^3)/(0.0704293 + 0.41072\xi - \xi^2).$$

For $\xi = 1$ and $\theta = 1$ these formulas reduce to the A-stable type 2 method

$$\begin{array}{c|cc|cc} 0 & 0.164905 & -0.198522 & 0.75 & 0 \\ 0 & -0.210337 & -1.07121 & 2.70983 & 0.75 \\ \hline 0 & 0.128015 & -0.284316 & 1.12692 & 0.0293846 \end{array}$$

obtained in [8].

However, the disadvantage of these methods is that they are not well defined for all positive values of ξ. For example, the type 1 method listed above is not defined for $\xi = 0.535279$ and $\xi = 1.33255$ and the type 2 method is not defined for $\xi = 0.540925$ and $\xi = 1.64476$ and in practical implementations we should restrict the ratio to stay away from these values by some safe margin.

We can avoid these complications choosing c_1 and c_2, $c_1 \neq c_2$, from the interval $[0, 1]$. In this case it can be verified that the methods are well defined for every $\xi > 0$. The example of such a method of type 1 is listed below.

$$c_1 = 1/3, \qquad c_2 = 2/3, \qquad \eta = 0,$$

$$u_1(\xi) = -\frac{\xi(12 + 9\xi + 2\xi^2)}{9},$$

$$u_2(\xi) = -\frac{\xi(6 + 9\xi + 7\xi^2)}{9},$$

$$a_{11}(\xi) = -\frac{\xi(3 + \xi)^2}{9},$$

$$a_{12}(\xi) = -\frac{\xi^2(3 + \xi)}{9},$$

$$a_{21}(\xi) = -\frac{\xi(9 + 12\xi + 7\xi^2)}{18},$$

$$a_{22}(\xi) = -\frac{\xi^2(6 + 7\xi)}{18},$$

$$v_1(\xi,\theta)=\frac{\theta\xi^3(-8+18\theta-12\theta^2-12\theta\xi+36\theta^2\xi-27\theta^3\xi)}{8(1+\xi)(2+\xi)},$$
$$v_2(\xi,\theta)=\frac{\theta\xi^3(16-36\theta+24\theta^2+12\theta\xi-36\theta^2\xi+27\theta^3\xi)}{4(1+\xi)(1+2\xi)},$$
$$w_1(\xi,\theta)=\frac{\theta(16-12\theta+36\theta\xi-36\theta^2\xi+24\theta^2\xi^2-27\theta^3\xi^2)}{4(1+\xi)(2+\xi)},$$
$$w_2(\xi,\theta)=\frac{\theta(-8+12\theta-18\theta\xi+36\theta^2\xi-12\theta^2\xi^2+27\theta^3\xi^2)}{8(1+\xi)(1+2\xi)}.$$

Its interval of stability for $\xi=1$ and $\theta=1$ is approximately $(-1.134,0)$.

Similarly we can obtain variable stepsize continuous TSRK methods of types 3 and 4 which reduce to the methods obtained in [8] when $\xi=1$ and $\theta=1$. The details of these methods are omitted.

We will conclude this section with the construction of the 3-stage TSRK method of type 1 of order 5 and stage order 5, where, again, we have chosen $\eta(\xi,\theta)=0$ to guarantee that the resulting methods are zero-stable for any ratio of stepsize. Solving the system $C_\mu(\xi)=0$, $\mu=1,2,3$ for $a_{ij}(\xi)$, $C_3(\xi)=0$ for $u_j(\xi)$ and $\hat{C}_\mu(\xi,\theta)=0$, $\mu=1,2,3,4,5$ for $v_j(\xi,\theta), j=1,2,3$ and $w_j(\xi,\theta), j=1,2$, we obtain a family of variable stepsize continuous methods which depend on the parameters $c_1,c_2,c_3,b_{21},b_{31},b_{32}$ and w_3.

Choosing $c_1=1/8$, $c_2=1/4$, $c_3=3/4$, and b_{21},b_{31},b_{32} and w_3 in order to obtain satisfactory stability properties when the stepsize is kept constant and $\theta=1$, we obtain a type 2 method of the form

$$u_1(\xi)=-\frac{\xi(336+136\xi+20\xi^2+\xi^3)}{128},$$
$$u_2(\xi)=\frac{-\xi^3(5+\xi)}{16},$$
$$u_3(\xi)=-\frac{3\xi(42-34\xi+13\xi^3)}{8},$$
$$a_{11}(\xi)=-\frac{\xi(8+\xi)(24+8\xi+\xi^2)}{480},$$
$$a_{12}(\xi)=-\frac{\xi(6+\xi)(8+\xi)(14+\xi)}{768},$$
$$a_{13}(\xi)=-\frac{\xi(8+\xi)(588+196\xi+17\xi^2)}{3840},$$
$$a_{21}(\xi)=-\frac{\xi^3(4+\xi)}{60},$$
$$a_{22}(\xi)=-\frac{\xi^3(7+\xi)}{96},$$

$$a_{23}(\xi) = -\frac{\xi^3(83+17\xi)}{480},$$

$$a_{31}(\xi) = -\frac{\xi(24-22\xi+13\xi^3)}{10},$$

$$a_{32}(\xi) = -\frac{\xi(84-61\xi+13\xi^3)}{16},$$

$$a_{33}(\xi) = -\frac{\xi(588-539\xi+221\xi^3)}{80},$$

$$b_{21} = 1/4, \qquad b_{31} = -3, \qquad b_{32} = 3,$$

$$\begin{aligned}
v_1(\xi,\theta) = {} & 8\xi^3(-270+45\theta-270\theta^2+480\theta^3-1020\xi+120\theta^2\xi-960\theta^3\xi \\
& +1920\theta^4\xi-810\xi^2+80\theta^3\xi^2-720\theta^4\xi^2+1536\theta^5\xi^2)/(75(7+\xi)(7+2\xi)), \\
v_2(\xi,\theta) = {} & \xi^3(630-105\theta+630\theta^2-1120\theta^3+2430\xi-270\theta^2\xi+2160\theta^3\xi \\
& -4320\theta^4\xi+1620\xi^2-160\theta^3\xi^2+1440\theta^4\xi^2-3072\theta^5\xi^2)/ \\
& (30(3+\xi)(6+\xi)), \\
v_3(\xi,\theta) = {} & \xi^3(-1890+315\theta-1890\theta^2+3360\theta^3-3150\xi+390\theta^2\xi-3120\theta^3\xi \\
& +6240\theta^4\xi-1620\xi^2+160\theta^3\xi^2-1440\theta^4\xi^2+3072\theta^5\xi^2)/ \\
& (150(1+\xi)(2+\xi)), \\
w_1(\xi,\theta) = {} & 8(378+315\theta-630\theta^2+1836\xi+1020\theta^2\xi-2720\theta^3\xi+2430\xi^2 \\
& +1200\theta^3\xi^2-3600\theta^4\xi^2+972\xi^3+480\theta^4\xi^3-1536\theta^5\xi^3)/ \\
& (15(2+\xi)(6+\xi)(7+\xi)), \\
w_2(\xi,\theta) = {} & (-945-315\theta+1260\theta^2-4590\xi-1020\theta^2\xi+5440\theta^3\xi-6075\xi^2 \\
& -1200\theta^3\xi^2+7200\theta^4\xi^2-2430\xi^3-480\theta^4\xi^3+3072\theta^5\xi^3)/ \\
& (15(1+\xi)(3+\xi)(7+2\xi)), \\
w_3(\xi,\theta) = {} & 3/5.
\end{aligned}$$

The stability interval of this method with $\xi = \theta = 1$ is approximately $(-0.537, 0)$.

4. Numerical examples

In this section we report the results of some numerical experiments on the equation A1 in [6]

$$\begin{cases} y'(x) = -y^3(x)/2, & x \in [0,20], \\ y(0) = 1, \end{cases} \tag{4.1}$$

with the solution $y(x) = 1/\sqrt{1+x}$ and on the system B2 in [6]

$$\begin{cases} y_1' = -y_1 + y_2, & y_1(0) = 2, \\ y_2' = y_1 - 2y_2 + y_3, & y_2(0) = 0, \\ y_3' = y_2 - y_3, & y_3(0) = 1, \end{cases} \tag{4.2}$$

which is a model of a linear chemical reaction. We will investigate first the influence of the zero-stability condition on the accuracy of numerical computations. Consider the one-stage TSRK method of order 3 and stage order 3 corresponding to $c = 5/4$. The coefficients of this method are

$$\begin{aligned} u(\xi) &= \frac{5\xi(5\xi-2)}{24}, \\ a(\xi) &= \frac{5\xi(5\xi+4)}{24}, \\ \eta(\xi,\theta) &= -\frac{\xi^2\theta(15-6\theta-30\xi\theta+16\xi\theta^2)}{45\xi+22}, \\ v(\xi,\eta) &= \frac{2\xi^2\theta(40+75\xi-16\theta-40\xi\theta)(1+\xi\theta)}{(45\xi+22)(5\xi-1)}, \\ w(\xi,\theta) &= \frac{2\theta(24\xi\theta-11)(1+\xi\theta)}{(45\xi+22)(5\xi-1)}. \end{aligned}$$

We have integrated first the problem (4.1) and (4.2) with constant stepsize $h = 1/16$, $1/32$ and $1/64$ and the results are presented in tables 1a and 1b.

In these tables p stands for the average of the values

$$\log(e_h(x_i) \big/ e_{h/2}(x_i)) \big/ \log 2,$$

Table 1a
Constant stepsize, problem (4.1).

h	p	q	err	n
1/16	3.0167	3.0219	5.46E-6	320
1/32	3.0079	3.0166	7.29E-7	640
1/64	3.0037	3.0061	9.43E-8	1280

Table 1b
Constant stepsize, problem (4.2).

h	p	q	err	n
1/16	3.0385	3.0122	1.74E-12	320
1/32	3.0155	3.0031	2.19E-13	640
1/64	2.9427	2.9338	3.11E-14	1280

Table 2a
Variable stepsize, $r = 6/5$, problem (4.1).

h_0	p	q	err	n
1/16	3.0157	3.0212	2.16E-6	1242.50
1/32	3.0013	3.0096	7.45E-7	1184.50
1/64	2.9946	3.0011	2.35E-7	1907.50

Table 2b
Variable stepsize, $r = 6/5$, problem (4.2).

h_0	p	q	err	n
1/16	3.0540	3.0316	1.24E-11	391.85
1/32	3.0299	3.0107	7.98E-12	707.45
1/64	3.0189	3.0109	6.79E-12	872.60

where $e_h(x_i)$ is the absolute error at the grid point x_i; q stands for the average stage order computed by

$$\log(se_h(x_i) \big/ se_{h/2}(x_i)) \big/ \log 2,$$

where $se_h(x_i)$ is the average absolute error of the stage values; err stands for the average error and n for the number of steps.

Next, we integrated these problems with variable stepsize chosen according to the rule

$$h_{\text{new}} = h_{\text{old}} r^{2\text{rand}-1},$$

Table 3a
Variable stepsize, $r = 2$, problem (4.1).

h_0	p	q	err	n
1/16	2.8761	2.8921	9.81E-6	482.10
1/32	2.8950	2.9070	7.20E-6	501.95
1/64	2.8556	2.8760	4.083E-6	590.35

Table 3b
Variable stepsize, $r = 2$, problem (4.2).

h_0	p	q	err	n
1/16	3.0024	2.9932	9.63E-12	451.10
1/32	2.9934	2.9866	9.96E-12	458.05
1/64	2.9910	2.9875	1.06E-11	463.70

Table 4a
Variable stepsize, $r = 3$, problem (4.1).

h_0	p	q	err	n
1/16	2.6804	2.6913	9.31E-6	501.50
1/32	2.6682	2.6816	7.63E-6	492.15
1/64	2.7034	2.7247	7.21E-6	537.80

Table 4b
Variable stepsize, $r = 3$, problem (4.2).

h_0	p	q	err	n
1/16	2.8978	2.8874	9.01E-12	415.10
1/32	2.8806	2.8719	8.92E-12	426.50
1/64	2.8584	2.8545	9.33E-12	447.25

where rand stands for the random number with uniform distribution. In order to avoid the use of too small or too large steps, whenever h_{new} was less than 0.001 or greater than 0.3, it was automatically set as $2h_{\text{old}}$ or $\frac{1}{2}h_{\text{old}}$ respectively. We have chosen the initial stepsize $h_0 = 1/16$, $1/32$, and $1/64$, and the ratio r equal to 6/5, 2, and 3. In the variable stepsize case we have solved the problem (4.1) and (4.2) 20 times, each time computing the average order and stage order over the interval $[0, 20]$ according to the same rule followed for the constant stepsize case, as well as the error at the endpoint $x = 20$ and the number of steps. This error was computed using the continuous TSRK method for an appropriate value of the parameter θ. In tables 2a–4b, we present the averages of these results over 20 runs. In all these tables h_0 stands for initial stepsize, p for the order, q for the stage order, err for absolute error at $x = 20$, and n for the number of steps.

We can see from tables 1a, 1b, 2a and 2b that for constant stepsize and for $r = 6/5$ (which implies that $\|\mathcal{A}(\xi, 1)\|_\infty = 1$) the order and stage order are almost exactly 3 as predicted by theorem 3. We observe a slight reduction in order and stage order for example (4.1) and for $r = 2$ (compare table 3a) in spite of the fact that for this ratio the variable stepsize TSRK method is always zero-stable. We do not observe a similar reduction for the problem (4.2) (compare table 3b). For $r = 3$ the zero-stability condition is violated whenever

$$\text{rand} > \frac{1}{2}(\log_3 2.32015 + 1)$$

and this is reflected in a somewhat bigger reduction in order and stage order as given in table 4a and some reduction as given in table 4b.

We will demonstrate next that for the method of order p and stage order p the stage order is not affected significantly by the additional requirement that $C_p(\xi) = 0$, i.e., that the stage order of consistency is $p + 1$ (compare theorem 3).

To this end, consider the method 1 with $m=1$, $c=3/4$, $\hat{C}_1(\xi,\theta)=\hat{C}_2(\xi,\theta)=\hat{C}_3(\xi,\theta)=0$, $C_1(\xi)=C_2(\xi)=0$ (and $C_3(\xi)\neq 0$), and the method 2 with the additional requirement that $C_3(\xi)=0$. The coefficients of these methods are listed below.

Method 1:

$$u(\xi)=\frac{\xi(4+3\xi)}{8},$$
$$a(\xi)=\frac{\xi(8+3\xi)}{8}, \qquad \lambda=\frac{1}{4},$$
$$\eta(\xi,\theta)=\frac{\xi^2\theta(9-6\theta+18\xi\theta-16\xi\theta^2)}{10+9\xi},$$
$$v(\xi,\theta)=\frac{2\xi^2\theta(24+27\xi-16\theta-24\xi\theta)(1+\xi\theta)}{(1+3\xi)(10+9\xi)},$$
$$w(\xi,\theta)=\frac{2\theta(5+8\xi\theta)(1+\xi\theta)}{(1+3\xi)(10+9\xi)}.$$

Method 2:

$$u(\xi)=\frac{27\xi^2(1+\xi)}{8(10+9\xi)},$$
$$a(\xi)=\frac{9\xi^2(4+3\xi)^2}{8(1+3\xi)(10+9\xi)},$$
$$\lambda(\xi)=\frac{3(4+3\xi)(5+6\xi)}{8(1+3\xi)(10+9\xi)}.$$

For this method $\eta(\xi,\theta)$, $v(\xi,\theta)$, and $w(\xi,\theta)$ are defined as for method 1.

Table 5a
Method 1, $C_3(\xi)\neq 0$, problem (4.1).

h_0	p	q	err	n
1/16	2.8328	2.8194	1.51E-6	450.55
1/32	2.8206	2.8111	1.23E-6	580.40
1/64	2.7814	2.7709	5.61E-7	574.10

Table 5b
Method 1, $C_3(\xi)\neq 0$, problem (4.2).

h_0	p	q	err	n
1/16	2.8782	2.8590	5.60E-12	563.55
1/32	2.8846	2.8644	6.55E-12	464.70
1/64	2.8668	2.8402	5.79E-12	556.20

Table 6a
Method 2, $C_3(\xi) = 0$, problem (4.1).

h_0	p	q	err	n
1/16	2.8503	2.8562	5.26E-7	523.65
1/32	2.7920	2.8008	3.59E-7	519.90
1/64	2.8111	2.8239	3.83E-7	544.80

Table 6b
Method 2, $C_3(\xi) = 0$, problem (4.2).

h_0	p	q	err	n
1/16	2.7161	2.7156	5.51E-13	510.60
1/32	2.7303	2.7458	9.86E-7	483.95
1/64	2.7214	2.7372	4.37E-13	590.50

We have again integrated problem (4.1) and (4.2) 20 times with the ratio $r = 2$ using both methods and the averages of these results are presented in tables 5a–6b. They are again in agreement with theorem 3.

We will next test how well the order and stage order of the method are preserved in a variable stepsize environment for the methods which are zero-stable for any ratio of stepsizes ξ. This is true if, for instance, $\eta(\xi, \theta) \equiv 0$, and the examples of such methods are given in section 3. We present in tables 7a and 7b the results of numerical experiments on the problem (4.1) and (4.2) using the type 1 two-stage TSRK method of order 4 and stage order 4 with $c_1 = 1/3$, $c_2 = 2/3$, $b_{21} = 1/2$,

Table 7a
TSRK with $m = 2$, $p = q = 4$, problem (4.1).

h_0	p	q	err	n
1/16	3.8845	3.8823	2.71E-7	558.25
1/32	3.8497	3.8451	1.04E-7	547.80
1/64	3.8503	3.8439	9.17E-8	550.05

Table 7b
TSRK with $m = 2$, $p = q = 4$, problem (4.2).

h_0	p	q	err	n
1/16	3.8364	3.8443	5.19E-13	463.65
1/32	3.8416	3.8474	5.77E-13	508.40
1/64	3.8475	3.8513	4.80E-13	636.90

Table 8a
TSRK with $m = 3$, $p = q = 5$, problem (4.1).

h_0	p	q	err	n
1/16	4.8562	4.8619	7.09E-9	510.00
1/32	4.8905	4.8953	8.01E-9	562.20
1/64	4.8900	4.8972	3.27E-9	581.90

Table 8b
TSRK with $m = 3$, $p = q = 5$, problem (4.2).

h_0	p	q	err	n
1/16	4.8361	4.8539	1.48E-12	461.30
1/32	4.8516	4.8737	1.59E-12	440.55
1/64	4.8949	4.9126	1.51E-12	508.95

and with $u(\xi)$, $A(\xi)$, $v(\xi, \theta)$, and $w(\xi, \theta)$ listed in section 3. In tables 8a and 8b we present the selection of numerical experiments using the three-stage method of order 5 and stage order 5 given in section 3.

As before, we have solved the problem (4.1) and (4.2) 20 times with $r = 2$ and with the stepsize chosen according to the rule discussed at the beginning of this section. The results obtained are again in very good agreement with theorem 3.

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