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USING THE REFINEMENT EQUATION FOR THE CONSTRUCTION OF PRE-WAVELETS III: ELLIPTIC SPLINES

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The purpose of this paper is to provide multiresolution analysis, stationary subdivision and pre-wavelet decomposition on $L^2(\mathbb{R}^d)$ based on a general class of functions which includes polyharmonic B-splines.

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1. Introduction

The purpose of this paper is to provide multiresolution analysis, stationary subdivision, and pre-wavelet decompositions of $L^2(\mathbb{R}^d)$ based on a general class of functions which includes *polyharmonic* B-splines (a definition of polyharmonic B-spline will be given later). For a detailed study of these interesting and useful functions, see [7], and also [3] for related matters.

We begin by recalling the multiresolution setup. Given a function $\varphi \in L^2 = L^2(\mathbb{R}^d)$ which satisfies the stability inequality

$$m_2 \| c \|_2 \leqslant \| [c, \varphi] \|_2 \leqslant M_2 \| c \|_2 \tag{1.1}$$

valid for all $c = (c_\alpha : \alpha \in \mathbb{Z}^d) \in l^2 = l^2(\mathbb{Z}^d)$. In this case we say that φ has

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l^2-stable integer translates. Here m_2, M_2 are constants such that $0 < m_2 < M_2$, $[c, \varphi]$ is the function

$$[c, \varphi](x) := \sum_{\alpha \in \mathbb{Z}^d} c_\alpha \varphi(x - \alpha), \quad x \in \mathbb{R}^d \tag{1.2}$$

and $\|\cdot\|_2$ denotes the standard norm(s) on l^2, L^2 (Generally, we use $\|\cdot\|_p$ for the usual norm(s) on l^p, L^p, $1 \leqslant p \leqslant \infty$). With φ we associate an infinite scale of closed subspaces of L^2 defined by

$$V^k(\varphi) := \{sc^k[c, \varphi]: c \in l^2\} = sc^k(V^0(\varphi)), \quad k \in \mathbb{Z} \tag{1.3}$$

where sc^k: $L^2 \to L^2$ is the *scaling* operator

$$(sc^k f)(x) := f(2^k x), \quad x \in \mathbb{R}^d. \tag{1.4}$$

We say that φ admits multiresolution provided that, in addition to (1.1), we have

$$\overline{\bigcup_{k \in \mathbb{Z}} V^k} = L^2, \tag{1.5}$$

$$\bigcap_{k \in \mathbb{Z}} V^k = \{0\} \tag{1.6}$$

and

$$V^k \subseteq V^{k+1}, \quad k \in \mathbb{Z}. \tag{1.7}$$

Following [1], [6], we say that $\psi \in L^2$ is a *pre-wavelet*, if the functions

$$sc^k\, sh^\alpha\, \psi, \quad k \in \mathbb{Z}, \ \alpha \in \mathbb{Z}^d \tag{1.8}$$

where sh^α: $L^2 \to L^2$ is the *shift* operator

$$(sh^y f)(x) := f(x - y), \quad x, y \in \mathbb{R}^d \tag{1.9}$$

are orthogonal on *different* scales, that is,

$$(sc^k\, sh^\alpha\, \psi, sc^{k'}\, sh^\beta\, \psi) = 0 \tag{1.10}$$

for all k, $k' \in \mathbb{Z}$, $k \neq k'$ and α, $\beta \in \mathbb{Z}^d$. Here we also use standard notation for the inner product on L^2, viz.

$$(f, g) := \int_{\mathbb{R}^d} f(x)\overline{g(x)}\, \mathrm{d}x. \tag{1.11}$$

The class of functions $\mathscr{R}_{m,n} = \mathscr{R} \subset L^2$ for which we build multiresolution and pre-wavelets are best described in terms of their Fourier transform

$$(\hat{f})(\omega) := \int_{\mathbb{R}^d} f(x)\, \mathrm{e}^{-\mathrm{i}\omega \cdot x}\, \mathrm{d}x, \quad \omega \in \mathbb{R}^d.$$

A function φ is in $\mathscr{R}_{m,n}$ provided that

$$\hat{\varphi} = T/q \tag{1.12}$$

where T is a trigonometric polynomial

$$T(\omega) := \sum_{\beta \in \mathbb{Z}^d} t_\beta \, \mathrm{e}^{-\mathrm{i}\beta\cdot\omega}, \quad \omega \in \mathbb{R}^d \tag{1.13}$$

and q is a homogeneous polynomial

$$q(\omega) := \sum_{|\beta| = m} q_\beta \omega \beta, \quad \omega \in \mathbb{R}^d \tag{1.14}$$

$\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d$, $|\beta| := \beta_1 + \ldots + \beta_d$, of degree m with $m > d$. For later use, we establish the notational convention of associating with every element $\rho = (\rho_\alpha \colon \alpha \in \mathbb{Z}^d) \in l^1$ the absolutely convergent trigonometric series

$$(\mathit{trig}\ \rho)(\omega) := \sum_{\alpha \in \mathbb{Z}^d} \rho_\alpha \, \mathrm{e}^{\mathrm{i}\alpha\cdot\omega}, \quad \omega \in \mathbb{R}^d \tag{1.15}$$

Obviously *trig* is a bounded linear map (of norm one) from l^1 into $C(Q^d)$ (continuous functions on Q^d), where $Q^d = [-\pi, \pi]^d$, the d-dimensional torus. For $\rho \in l^2(\mathbb{Z}^d)$, we also use (1.15) to define *trig* ρ, a.e., $\omega \in \mathbb{R}^d$. In this case, *trig* is an isometry from $l^2(\mathbb{Z}^d)$ onto $L^2(Q^d)$ with (normalized) Lebesgue measure.

The homogeneous polynomial q is required to be *elliptic*, in other words,

$$q(\omega) = 0, \ \omega \in \mathbb{R}^d \text{ implies that } \omega = 0. \tag{1.16}$$

We couple together the trigonometric polynomial T and the homogeneous polynomial q by requiring that there exists a nonnegative integer n such that

$$T(\omega) - q(\omega) = 0(\|\omega\|_\infty^{m+1+n}), \quad \omega \to 0 \tag{1.17}$$

where $\|\cdot\|_\infty$ is the maximum norm on $\mathbb{R}^d$. Finally, we suppose, analogous to (1.16), that

$$T(\omega) = 0, \ \|\omega\|_\infty \leqslant \pi \text{ implies that } \omega = 0. \tag{1.18}$$

For our main result, we set

$$E = \text{extreme points of } [0, 1]^d \tag{1.19}$$

We use $*\colon L^2 \to L^2$ for convolution, $\check{}\colon L^2 \to L^2$ for scaling by minus one ($\check{\varphi}(x) := \varphi(-x)$), and

$$\{f\}(x) := \sum_{\alpha \in \mathbb{Z}^d} f(x + 2\pi\alpha) \tag{1.20}$$

whenever the sum is convergent a.e., $\omega \in \mathbb{R}^d$. Also, for *every* function $\psi \in L^2(\mathbb{R}^d)$ we define the closed subspace of $L^2(\mathbb{R}^d)$

$$R(\psi) = \overline{\mathit{span}\{sh^\alpha\ \psi \colon \alpha \in \mathbb{Z}^d\}}.$$

Also, for all $k \in \mathbb{Z}$, we set

$$R^k(\psi) = sc^k\ R(\psi).$$

Thus for any $\varphi \in L^2(\mathbb{R}^d)$ with l^2-stable integer translates we have $R^k(\varphi) = V^k(\varphi)$.

With this notation in hand, we introduce the family of 2^d functions

$$\psi_e = sh^{e/2}\,\psi_0, \quad e \in E, \tag{1.21}$$

where

$$\hat{\psi}_0 = 2^{-d}\,\bar{q}\,sc^{-1}\big(|\check{\varphi}|^2/\{|\check{\varphi}|^2\}\big), \tag{1.22}$$

and the scale of subspaces of V^k,

$$W^k := \sum_{e \in E \setminus \{0\}} R^k(\psi_e), \quad k \in \mathbb{Z}. \tag{1.23}$$

THEOREM 1.1

Let $\varphi \in \mathscr{R}_{m,n}$, with $m > d$ then

$$W^k \perp W^{k'}, \quad k \neq k' \tag{1.24}$$

and

$$\overline{\bigoplus_{k \in \mathbb{Z}} W^k} = L^2. \tag{1.25}$$

Along with this theorem, which provides an orthogonal decomposition of L^2 (using translates and scales of *one* function in V^1, albeit the translates are taken over the fine lattice $2^{-1}\mathbb{Z}^d$, we will show that there is a subdivision scheme for computing elements of $f \in V^0$.

The example which motivated us to consider the class $\mathscr{R}$ is the polyharmonic B-spline, [7], which is defined as follows. For every $r \in \mathbb{Z}_+$, set

$$K_{r,d}(x) = c_{r,d}\begin{cases} \|x\|_2^{2r-d}\log\|x\|_2, & d \text{ even} \\ \|x\|_2^{2r-d}, & d \text{ odd} \end{cases} \tag{1.26}$$

where

$$c_{r,d} = \begin{cases} \dfrac{1}{2^{2r}\pi^{d/2}}\,\dfrac{(-1)^{r-d/2+1}}{(r-d/2)!\,\Gamma(r)}, & d \text{ even} \\[2ex] \dfrac{1}{2^{2r}\pi^{d/2}}\,\dfrac{\Gamma(d/2-r)}{\Gamma(r)}, & d \text{ odd} \end{cases} \tag{1.27}$$

and $\|\cdot\|_2$ is the euclidean norm on $\mathbb{R}^d$. This function is the Green's function for the iterated Laplacian $(-1)^r\Delta^r$,

$$\Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}. \tag{1.28}$$

The polyharmonic B-spline is given by

$$B_{r,d} = (-1)^r\,\delta^r K_{r,d} \tag{1.29}$$

where δ is a discrete version of Δ defined by

$$(\delta f)(x) := \sum_{j=1}^{d} \left(f(x-e_j) - 2f(x) + f(x+e_j)\right) \tag{1.30}$$

and $e_1, \ldots, e_d \in \mathbb{R}^d$ are the coordinate vectors $(e_j)_k := \delta_{jk}$, $1 \leqslant j, k \leqslant d$. A somewhat involved calculation shows that for

$$q(\omega) = \| \omega/2 \|_2^{2r}, \quad T(\omega) = \left(\sum_{j=1}^{d} \sin^2 \omega_j/2 \right)^r \tag{1.31}$$

we have $\hat{B}_{r,d} = T/q$. Thus, $B_{r,d} \in \mathscr{R}_{2r,1}$ for each $r > d/2$. Because of this example, we call any element in $\mathscr{R}$ an *elliptic spline.*

We remark that a more sophisticated choice of T in (1.31) leads to what are called level k Polyharmonic B-splines (see [7]). These functions are in our class for an appropriate choice of n.

It is interesting to note that the *cube spline* also has a Fourier transform of the form (1.12). Specifically, for any set of vectors $x^1, \ldots, x^n \in \mathbb{Z}^d \backslash \{0\}$ which span $\mathbb{R}^d$ we set

$$q(\omega \mid X) := \prod_{j=1}^{n} \mathrm{i}\omega \cdot x^j, \; T(\omega \mid X) := \prod_{j=1}^{n} (1 - \mathrm{e}^{-\mathrm{i}\omega \cdot x^j}) \tag{1.32}$$

then $\hat{c}(\cdot \mid X) = T(\cdot \mid X)/q(\cdot \mid X)$. Equivalently, $c(\cdot \mid X)$ can be defined by the equation

$$\int_{[0,1]^d} f(Xt) \, \mathrm{d}t = \int_{\mathbb{R}^d} c(x \mid X) f(x) \, \mathrm{d}x \tag{1.33}$$

valid for all $f \in C(\mathbb{R}^d)$. It was shown in [6] that Theorem 1.1 holds for $\varphi = c(\cdot \mid X)$ if the matrix X is unimodular, i.e., every $s \times s$ nonsingular submatrix of X has determinant ± 1. However, the cube spline is not in $\mathscr{R}$ for $d > 1$ since the homogeneous polynomial, $q(\cdot \mid X)$ is characteristically *hyperbolic.*

Much of what we say below holds if T is an absolutely convergent trigonometric series. However, we do not pursue this issue here.

2. Multiresolution and subdivision for elliptic splines

In this section, we demonstrate that every $\varphi \in \mathscr{R}$ admits multiresolution, and that there is an associated stationary subdivision scheme in the sense of [2] which can be used to compute elements $f \in V^0$ iteratively. We begin with

PROPOSITION 2.1

Suppose $\varphi \in \mathscr{R}_{m,n}$ with $m > d$, $n \geqslant 0$ and

$$\hat{\varphi} = T/q. \tag{2.1}$$

Define

$$a_\alpha = \frac{2^{d-m}}{(2\pi)^2} \int_{Q^d} \frac{T(2\omega)}{T(\omega)} e^{i\alpha\cdot\omega}\, d\omega, \quad \alpha \in \mathbb{Z}^d. \tag{2.2}$$

Then

$$\varphi(x) = o(\| x \|_\infty^{-(d+n)}), \quad x \to \infty, \tag{2.3}$$

and

$$a_\alpha = o(\| \alpha \|_\infty^{-(d+n)}), \quad \alpha \to \infty. \tag{2.4}$$

Proof

First, let us observe that for every $\varphi \in \mathscr{R}$ and for ω near zero

$$\hat{\varphi}(\omega) = 1 + \frac{T(\omega) - q(\omega)}{q(\omega)} = 1 + O(\| \omega \|_\infty^{n+1}) \tag{2.5}$$

while at infinity

$$\hat{\varphi}(\omega) = O(\| \omega \|_\infty^{-m}). \tag{2.6}$$

Thus we conclude that

$$\lim_{\omega \to 0} \hat{\varphi}(\omega) = 1 \tag{2.7}$$

and

$$\hat{\varphi} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \tag{2.8}$$

To prove (2.3) we will next estimate the derivatives of φ at the origin. For this purpose, we use the next lemma.

LEMMA 2.1

Let q be a homogeneous elliptic polynomial of degree m on $\mathbb{R}^d$. Suppose U is some neighborhood of the origin and $f \in C^N(U\setminus\{0\})$, for some $N \geqslant 0$. If there exists a constant $c > 0$ and a $\rho \in \mathbb{Z}$ such that

$$|(D^\beta f)(x)| \leqslant C \| x \|_\infty^{\rho - |\beta|}$$

for all $|\beta| \leqslant N$ and $x \in U\setminus\{0\}$. Then for some constant $D > 0$ we have for $x \in U\setminus\{0\}$

$$\left| D^\beta \left(\frac{f}{q} \right)(x) \right| \leqslant D \| x \|_\infty^{\rho - m - |\beta|}.$$

Proof

Set $R = f/q$. Since q is homogeneous and elliptic, there exist positive constants A, B such that

$$A\|x\|_\infty^m \leqslant |q(x)|, \quad |(D^\alpha q)(x)| \leqslant B\|x\|_\infty^{m-|\alpha|}, \quad x \neq 0.$$

We will prove the claim by induction on β. First, for $\beta = 0$, we clearly have

$$|R(x)| \leqslant \frac{C}{A}\|x\|_\infty^{\rho-m}, \quad x \in U\setminus\{0\}.$$

Assume that the claim is true for all $\beta \leqslant \gamma$, but $\beta \neq \gamma$ where $|\gamma| \leqslant N$.

By Leibnitz's rule

$$(D^\gamma f)(x) = \sum_{0 \leqslant \alpha \leqslant \gamma} \binom{\gamma}{\alpha} (D^\alpha R)(x)(D^{\gamma-\alpha} q)(x).$$

Hence

$$\begin{aligned} |(D^\gamma R)(x) q(x)| &\leqslant C\|x\|_\infty^{\rho-|\gamma|} \\ &\quad + \sum_{0 \leqslant \alpha < \gamma} \binom{\gamma}{\alpha} D\|x\|_\infty^{\rho-m-|\alpha|} B\|x\|_\infty^{m-|\gamma|+|\alpha|} \\ &= (C + BD2^{|\gamma|})\|x\|_\infty^{\rho-|\gamma|} \end{aligned}$$

and so

$$|(D^\gamma R)(x)| \leqslant \left(\frac{C + BD2^{|\gamma|}}{A}\right)\|x\|_\infty^{\rho-m-|\gamma|}$$

which advances the induction and proves the theorem.

Using this lemma with $f = T - q$, $\rho = m + n + 1$ and $N = m + 1 + n$ we get for $0 < |\alpha| \leqslant N$

$$|(D^\alpha \hat{\varphi})(x)| \leqslant D\|x\|_\infty^{n+1-|\alpha|}$$

and so $D^\alpha \hat{\varphi} \in L^1(\mathbb{R}^d)$ for $|\alpha| \leqslant n + d$. Consequently (2.3) follows by the Riemann Lebesgue lemma and a standard integration by parts argument applied to the integral formula

$$\varphi(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\omega} \hat{\varphi}(\omega)\, d\omega$$

which defines φ.

The equation (2.4) follows similarly by noting that for ω near zero

$$\frac{T(2\omega)}{T(\omega)} = \frac{2^m + \dfrac{T(2\omega) - q(2\omega)}{q(\omega)}}{1 + \dfrac{T(\omega) - q(\omega)}{q(\omega)}}$$

$$= \frac{2^m + O(\| \omega \|_\infty^{n+1})}{1 + O(\| \omega \|_\infty^{n+1})}$$

$$= 2^m + O(\| \omega \|_\infty^{n+1}).$$

Thus, as before, $D^\alpha(T(2\,\cdot\,)/T(\cdot)) \in L^1(Q^d)$ for $|\alpha| \leqslant d + n$ which easily proves (2.4).

REMARK 2.1

For the polyharmonic B-spline (2.3) gives $B_{r,d}(\omega) = O(\| \omega \|_\infty^{-d-1})$ as $\omega \to \infty$. However, it is known that

$$B_{r,d}(\omega) = O(\| \omega \|_\infty^{-d-2}), \quad \omega \to \infty,$$

cf., [3], [7]. A similar remark holds for the estimate (2.4) in this case. This proposition leads us to

THEOREM 2.1

Every $\varphi \in \mathscr{R}_{m,n}$, $m > d$, $n \geqslant 0$ admits multiresolution.

Proof

First, let us prove the stability estimate (1.1). For this purpose, we observe that the function $\{|\hat{\varphi}|^2\}$ is continuous on Q^d since the series that defines it is absolutely and uniformly continuous. This function plays a central role in the proof of the stability estimate. To review these facts, we first point out that for every $c \in l^2$, the function $[c, \varphi]$ is continuous on $\mathbb{R}^d$. This follows from the fact that φ is continuous on $\mathbb{R}^d$ and by (2.3), the series

$$\sum_{\alpha \in \mathbb{Z}^d} |\varphi(x - \alpha)|^2$$

is absolutely and uniformly continuous on any compact subset of $\mathbb{R}^d$ (see the proof of Theorem 2.2 for an upper bound on this sum). Moreover, $[c, \varphi]$ is in $L^2(\mathbb{R}^d)$ because for any $d = (d_\alpha\colon \alpha \in \mathbb{Z}^d)$ of finite support

$$\| [d, \varphi] \|_2^2 = \frac{1}{(2\pi)^d} \| (\mathit{trig}\ d) \hat{\varphi} \|_2^2$$

$$= \frac{1}{(2\pi)^d} \| (\mathit{trig}\ d)^2 \{ |\hat{\varphi}|^2 \} \|_{L^1(Q^d)}.$$

Thus it follows for all $c \in l^2$ that

$$\min_{\omega \in Q^d} |\{|\hat{\varphi}|^2\}(\omega)| \, \|c\|_2 \leqslant \|[c, \varphi]\|_2 \leqslant \max_{\omega \in Q^d} |\{|\hat{\varphi}|^2\}(\omega)| \, \|c\|_2.$$

Hence $[c, \varphi]$ is in $L^2(\mathbb{R}^d)$ and to show (1.1) requires showing that there is no $\omega \in \mathbb{R}^d$ such that

$$\hat{\varphi}(\omega + 2\pi\alpha) = 0, \text{ for all } \alpha \in \mathbb{Z}^d, \tag{2.9}$$

see [4], [5] for related results. Suppose, to the contrary, $\omega \in \mathbb{R}^d$ satisfies (2.9). Choose a $\beta \in \mathbb{Z}^d$ so that $\|\omega + 2\pi\beta\|_\infty \leqslant \pi$. If $\omega + 2\pi\beta \neq 0$ then (2.9) implies that

$$\frac{T(\omega + 2\pi\beta)}{q(\omega + 2\pi\beta)} = 0$$

and so by (1.18) we get $\omega + 2\pi\beta = 0$, nonetheless. Thus, indeed, $\omega = -2\pi\beta$. Now choose $\alpha = \beta$ in (2.9) to conclude $\hat{\varphi}(0) = 0$. However (2.7) implies that $\hat{\varphi}(0) = 1$. This contradiction proves the claim.

To establish the nesting of the spaces V^k, $k \in \mathbb{Z}$ we observe that φ satisfies the refinement equation

$$\varphi = sc[a, \varphi]. \tag{2.10}$$

Both sides of equation (2.10) are continuous functions in $L^2(\mathbb{R}^d)$ since by (2.4), $a \in l^2$. Taking the Fourier transform of both sides shows that (2.10) is equivalent to the equation

$$2^d(sc\ \hat{\varphi})/\hat{\varphi} = trig\ a$$

which follows immediately from (2.2). From these observations, we conclude that each V^k is a closed subspace of L^2 and the refinement equation (2.10) implies the spaces are nested in the sense of (1.7). Here we used the fact that $trig\ a = 2^{d-m} sc\ T/T \in L^\infty(Q^d)$ which guarantees that convolution with a acts as a bounded linear operator on l^2.

The remaining claims (1.5) and (1.6) will follow from the following lemma.

LEMMA 2.2

Let $\varphi \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Define the linear operator

$$(T_k f)(x) := \sum_{\alpha \in \mathbb{Z}^d} f\left(\frac{\alpha}{2^k}\right) \varphi(2^k x - \alpha). \tag{2.11}$$

If $f \in C_0(\mathbb{R}^d)$ (continuous functions of compact support on $\mathbb{R}^d$), $\hat{\varphi} \in L^\infty(\mathbb{R}^d)$, and $\hat{\varphi}(0) = 1$ then $\lim_{k\to\infty} T_k f = f$ weakly in $L^2(\mathbb{R}^d)$, that is $\lim_{k\to\infty}(T_k f, g) = (f, g)$ for all $g \in L^2(\mathbb{R}^d)$.

Proof

First we show that $\| T_k f \|_2$ is bounded independent of $k \in \mathbb{Z}_+$. To this end, we suppose that M is a positive integer such that $f(x) = 0$ whenever $\| x \|_\infty \geqslant M$. Therefore, using (1.1) we have

$$\| T_k f \|_2^2 = \int_{\mathbb{R}^d} \Big| \sum_{\| \alpha \|_\infty \leqslant 2^k M} f\Big(\frac{\alpha}{2^k}\Big) \varphi(2^k x - \alpha) \Big|^2 \, \mathrm{d}x$$

$$= 2^{-dk} \int_{\mathbb{R}^d} \Big| \sum_{\| \alpha \|_\infty \leqslant 2^k M} f\Big(\frac{\alpha}{2^k}\Big) \varphi(x - \alpha) \Big|^2 \, \mathrm{d}x$$

$$\leqslant \left((2^{k+1}M + 1)/2^k \right)^d M_2^2 \, \| f \|_\infty^2$$

and so

$$\| T_k f \|_2^2 \leqslant J^2 := (2M+1)^d \, \| f \|_\infty^2 M_2^2.$$

Given $g \in L^2(\mathbb{R}^d)$, and $\epsilon > 0$, choose a $h \in L^2(\mathbb{R}^d)$ of compact support such that

$$\| \hat{g} - h \|_2^2 \leqslant (2\pi)^d \epsilon^2 / (J + \| f \|_2)^2.$$

Let $\hat{g}_0 := k$ and note that

$$|(T_k f, g) - (f, g)| \leqslant |(T_k f - f, g - g_0)| + |(T_k f, g_0) - (f, g_0)|$$
$$\leqslant \epsilon + |(T_k f, g_0) - (f, g_0)|$$

and also

$$(T_k f, g_0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \bar{\hat{g}}_0(\omega) 2^{-kd} \sum_{\| \alpha \|_\infty \leqslant 2^k M} f\Big(\frac{\alpha}{2^k}\Big) \, \mathrm{e}^{-\mathrm{i}\alpha\cdot\omega/2^k} \hat{\varphi}(\omega/2^k) \, \mathrm{d}\omega.$$

Call the integrand $f_k(\omega)$; Then

$$| f_k(\omega) | \leqslant | h(\omega) | (2M+1)^d \, \| f \|_\infty \, \| \hat{\varphi} \|_\infty.$$

Since $h \in L^2(\mathbb{R}^d)$ and was chosen to be of compact support, the upper bound is in $L^1(\mathbb{R}^d)$. Moreover, since $f \in C_0(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} \varphi(\omega) \, \mathrm{d}\omega = 1$$

it follows that $\lim_{k\to\infty} f_k(\omega) = \bar{h}(\omega) \hat{f}(\omega)$, for $\omega \in \mathbb{R}^d$. Hence, by the Lebesgue dominated convergence theorem

$$\lim_{k\to\infty} (T_k f, g_0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \bar{h}(\omega) \hat{f}(\omega) \, \mathrm{d}\omega = (f, g_0).$$

This proves the result.

Returning to the proof of Theorem 2.1, we note that any $\varphi \in \mathscr{R}$ satisfies the hypothesis of Lemma 2.2. Thus, we conclude that every $f \in C_0(\mathbb{R}^d)$ is in the weak closure of the subspace $V^\infty := \bigcup_{k \in \mathbb{Z}} V^k$. This implies that V^∞ has a (strong) closure which is equal to $L^2(\mathbb{R}^d)$. This proves (1.5).

For the remaining claim (1.6) we use the fact that $\varphi \in L^\infty(\mathbb{R}^d)$ and the asymptotic estimate (2.3) to conclude that for some positive constant Q

$$|\varphi(x)|^2 \leqslant \frac{Q}{(1+\|x\|_\infty)^{d+1}}, \quad x \in \mathbb{R}^d.$$

Hence

$$\begin{aligned}\sum_{\alpha \in \mathbb{Z}^d} |\varphi(x-\alpha)|^2 &\leqslant Q\left(1+\sum_{l=0}^{\infty}(1+l)^{-d-1}\left(\sum_{\|\alpha\|_\infty = l+1} 1\right)\right)\\ &\leqslant Q\left(1+\sum_{l=0}^{\infty}\frac{4^d(l+1)^{d-1}}{(1+l)^{d+1}}\right)\\ &\leqslant 2\cdot 4^d Q\sum_{l=1}^{\infty} l^{-2} := R, \quad x \in \mathbb{R}^d.\end{aligned}$$

Now let $g \in \bigcap_{k \in \mathbb{Z}} V^k$ then for each $k \in \mathbb{Z}$ there is a $c \in l^2$ such that $g = sc^k[c, \varphi]$. Using the stability estimate (1.1) and the above inequality, gives

$$|g(x)|^2 \leqslant R\|c\|_2^2 \leqslant m_2^{-2} R 2^{kd} \|g\|_2^2, \quad x \in \mathbb{R}^d,$$

which goes to zero as $k \to -\infty$. This completes the proof of Theorem 2.1.

Our final comments in this section pertain to the stationary subdivision scheme defined by

$$(S\lambda)_\alpha := \sum_{\beta \in \mathbb{Z}^d} a_{\alpha-2\beta}\lambda_\beta, \quad \alpha \in \mathbb{Z}^d, \tag{2.12}$$

$\lambda = (\lambda_\alpha: \alpha \in \mathbb{Z}^d)$. We now assume that $n > 0$. In this case, $a \in l^1$ (see (2.4)), and S is a bounded linear operator from l^∞ into itself. We are interested in the convergence of the iterates of S. It is convenient for this purpose to note the following fact.

LEMMA 2.3

Let $\varphi \in \mathscr{R}_{m,n}$ for $m > d$ and $n > 0$. Then

$$\sum_{\alpha \in \mathbb{Z}^d} \varphi(x-\alpha) = 1, \quad x \in \mathbb{R}^d \tag{2.13}$$

and

$$\lim_{k \to \infty} \|T_k f - f\|_\infty = 0 \tag{2.14}$$

if f is uniformly continuous on $\mathbb{R}$.

Proof

The proof is based on noting that for some $K>0$

$$|\varphi(x)| \leqslant \frac{K}{(1+\|x\|_\infty)^{d+1}}, \quad x \in \mathbb{R}^d,$$

and so, as before in the proof of Lemma 2.2, we have

$$\sum_{\alpha \in \mathbb{Z}^d} |\varphi(x-\alpha)| \leqslant 2 \cdot 4^d K \sum_{l=1}^{\infty} l^{-2} := L, \quad x \in \mathbb{R}^d.$$

In a similar way, we have for any $x \in \mathbb{R}^d$ and integer $\rho \in \mathbb{R}_+$

$$\sum_{\|x-\alpha\|_\infty \geqslant \rho} |\varphi(x-\alpha)| \leqslant (1+\rho)^{-1/2} \sum_{\|x-\alpha\|_\infty \geqslant \rho} \frac{K}{(1+\|x-\alpha\|_\infty)^{d+1/2}}$$
$$\leqslant 2 \cdot 4^d K (1+\rho)^{-1/2} \sum_{l=1}^{\infty} l^{-3/2} := W(1+\rho)^{-1/2}.$$

Also, since for any $\alpha \in \mathbb{Z}^d \setminus \{0\}$

$$\hat{\varphi}(2\pi\alpha) = \frac{T(2\pi\alpha)}{q(2\pi\alpha)} = \frac{T(0)}{q(2\pi\alpha)} = 0$$

we get by the Poisson summation formula

$$\sum_{\alpha \in \mathbb{Z}^d} \varphi(x-\alpha) = 1, \quad x \in \mathbb{R}^d.$$

Hence, if $\omega(f;\delta) := \sup\{|f(x)-f(y)| : \|x-y\|_\infty \leqslant \delta\}$ is the modulus of continuity of f we have for any $x \in \mathbb{R}^d$

$$|(T_k f)(x) - f(x)| \leqslant \sum_{\|x-2^{-k}\alpha\|_\infty \leqslant \delta} \left| f\left(\frac{\alpha}{2^k}\right) - f(x) \right| |\varphi(2^k x - \alpha)|$$
$$+ \sum_{\|x-2^{-k}\alpha\|_\infty \geqslant \delta} \left| f\left(\frac{\alpha}{2^k}\right) - f(x) \right| |\varphi(2^k x - \alpha)|$$
$$\leqslant L\omega(f;\delta) + 2W \|f\|_\infty (1+\delta 2^k)^{-1/2}$$

which goes to zero as $k \to \infty$ and then $\delta \to 0^+$. This proves the lemma.

THEOREM 2.2

Suppose $\varphi \in \mathscr{R}_{m,n}$ *with* $n>0$, $m>d$. *Then for every* $\lambda \in l^1$, $f_\lambda := [\lambda, \varphi]$ *is uniformly continuous on* $\mathbb{R}^d$ *and*

$$\lim_{k\to\infty} \|S^k\lambda - ssc^{-k} f_\lambda\|_\infty = 0 \tag{2.15}$$

(*here* $ssc^{-k} f_\lambda$ *is the sequence* $(f_\lambda(\alpha/2^k)\colon \alpha \in \mathbb{Z}^d)$).

Proof

We begin by noting that any $\varphi \in \mathscr{R}_{m,n}$ with $n > 0$ admits an L^∞ stability estimate

$$m_\infty \| \lambda \|_\infty \leqslant \| [\lambda, \varphi] \|_\infty \leqslant M_\infty \| \lambda \|_\infty \tag{2.16}$$

for some constants $0 < m_\infty < M_\infty < \infty$. For the proof, we observe that we can choose $M_\infty = L$ for the upper bound. To obtain a lower bound, we use the function $\theta \in V^0$ defined by

$$\hat{\theta} = \{ |\hat{\varphi}|^2 \}^{-1} \hat{\varphi}.$$

It is apparent that the sequence

$$g_\alpha := (\varphi * \check{\varphi})(\alpha) = \int_{\mathbb{R}^d} \varphi(x) \bar{\varphi}(x - \alpha) \, \mathrm{d}x, \quad \alpha \in \mathbb{Z}^d$$

is in l^1, in fact, $\| g \|_1 \leqslant L \| \varphi \|_1$, and moreover, by the Poisson summation formula *trig* $g = \{ |\hat{\varphi}|^2 \}$. Thus, by Wiener's lemma, cf. [8, p. 266] there is a $\mu = (\mu_\alpha \colon \alpha \in \mathbb{Z}^d) \in l^1$ such that *trig* $\mu = (\textit{trig}\ g)^{-1}$. Hence, we get

$$\theta = [\mu, \varphi]$$

and, in particular, $\theta \in L^1(\mathbb{R}^d)$, since $\| \theta \|_1 \leqslant \| \mu \|_1 \| \varphi \|_1 < \infty$, because $\varphi \in L^1(\mathbb{R}^d)$. Moreover, for any $\alpha \in \mathbb{Z}^d$

$$(sh^\alpha \varphi, \theta) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\hat{\varphi}(\omega)|^2}{\{ |\hat{\varphi}|^2 \}(\omega)} \mathrm{e}^{-\mathrm{i}\omega \cdot \alpha} \, \mathrm{d}\omega = \frac{1}{(2\pi)^d} \int_{Q^d} \mathrm{e}^{-\mathrm{i}\omega \cdot \alpha} \, \mathrm{d}\omega = \delta_{0\alpha}.$$

Therefore, we get for any $c \in l^\infty$ and $\alpha \in \mathbb{Z}^d$

$$|c_\alpha| = |([c, \varphi], sh^\alpha \theta)| \leqslant \| [c, \varphi] \|_\infty \| \theta \|_1.$$

This proves that we can choose $m_\infty^{-1} := \| \theta \|_1$ in the L^∞ stability estimate (2.16).

As for the uniform continuity of f_λ we note first that since φ is continuous and goes to zero at infinity, it is uniformly continuous. Moreover, we have $\omega(f_\lambda; \delta) \leqslant \| \lambda \|_1 \omega(\varphi; \delta)$ and, therefore, f_λ is uniformly continuous too.

Now, the result (2.15) follows from arguments used in [2], namely, for the operator T_k defined in (2.11) we have

$$f_\lambda - T_k f = sc^k \left([S^k \lambda - ssc^{-k} f_\lambda, \varphi] \right) \tag{2.17}$$

and so (2.16) implies that

$$\| S^k \lambda - ssc^{-k} f_\lambda \|_\infty \leqslant m_\infty^{-1} \| T_k f_\lambda - f_\lambda \|_\infty.$$

Using Lemma 2.3, this inequality proves the result.

In the terminology of [2] the subdivision operator converges to f_λ and φ is the refinable function for the subdivision scheme (2.12).

3. Orthogonal pre-wavelet decomposition in L^2

This section is devoted to the proof of Theorem 1.1. Some of the steps in the argument parallel the case of cube spline analyzed in [6]. However, there are important differences. We start with an observation which generalizes a special case considered in [6].

THEOREM 3.1

Let $\varphi \in L^2$ *such that*

$$\| [c, \varphi] \|_2 \leqslant M \| c \|_2, \ M < \infty \tag{3.1}$$

so that $V^k(\varphi) \subset L^2$, $k \in \mathbb{Z}$. *If,*

$$\overline{\bigcup_{k \in \mathbb{Z}} V^k(\varphi)} = L^2 \tag{3.2}$$

then for any $\psi = [d, \varphi]$, $d \in l^2$, *with trig* $d \neq 0$ (a.e). *Then*

$$\overline{\bigcup_{k \in \mathbb{Z}} R^k(\psi)} = L^2 \tag{3.3}$$

Proof

We need the following fact which also appears in [5].

LEMMA 3.1

If $f, g \in L^2$ *then* $f * sh^\alpha g = 0$ *for all* $\alpha \in \mathbb{Z}^d$ *if and only if* $\{ \hat{f} \overline{\hat{g}} \} = 0$.

Proof

By Plancherel's formula, our hypothesis implies that for all $\beta \in \mathbb{Z}^d$

$$0 = \lim_{N \to \infty} \int_{Q^d} \sum_{\| \alpha \|_\infty \leqslant N} \hat{f}(\omega + 2\pi\alpha) \overline{\hat{g}(\omega + 2\pi\alpha)} \, \mathrm{e}^{-i\beta \cdot \omega} \, \mathrm{d}\omega. \tag{3.4}$$

Call the integrand $H_N(\omega)$. When we have the pointwise inequality

$$| H_N | \leqslant \left\{ | \hat{f} | \cdot | \hat{g} | \right\} \leqslant H := \left\{ | \hat{f} |^2 \right\}^{1/2} \left\{ | \hat{g} |^2 \right\}^{1/2}. \tag{3.5}$$

However, by the Cauchy Schwarz inequality

$$\| H \|_{L^1(Q^d)} \leqslant \| \{ | \hat{f} |^2 \} \|_{L^2(Q^d)} \| \{ | \hat{g} |^2 \} \|_{L^2(Q^d)}. \tag{3.6}$$

Moreover, using Fatou's lemma, we see that each term on the right of this inequality is finite since

$$\| \{ | \hat{f} |^2 \} \|^2_{L^2(Q^d)} \leqslant \lim_{N \to \infty} \int_{Q^d} \sum_{\| \alpha \|_\infty \leqslant N} | \hat{f}(\omega + 2\pi\alpha) |^2 \, \mathrm{d}\omega = (2\pi)^d \| f \|_2^2 < \infty.$$

For the equality above, we used Plancherel's formula once again. Thus we conclude by (3.6) that $H \in L^1(Q^d)$ and so (3.5) implies by the Lebesgue dominated convergence theorem that

$$0 = \int_{Q^d} e^{i\beta \cdot \omega} \left\{ \hat{f}\bar{\hat{g}} \right\}(\omega) \, d\omega, \quad \beta \in \mathbb{Z}^d,$$

whence $\{\hat{f}\bar{\hat{g}}\} = 0$. The converse follows by merely reversing the steps.

Returning to the proof of the Theorem, we suppose $f \in L^2$ satisfies $0 = (f, sc^k\, sh^\alpha\, \psi) = 2^{-dk}(sc^{-k}\, f, sh^\alpha\, \psi)$ for all $k \in \mathbb{Z}$, $\alpha \in \mathbb{Z}^d$. Then by Lemma 3.1

$$0 = \left\{ (sc^{-k}\, f)^\wedge \bar{\hat{\psi}} \right\} = \overline{trig\ d} \left\{ (sc^{-k}\, f)^\wedge \bar{\hat{\varphi}} \right\}$$

from which we conclude that $\{(sc^{-k}f)^\wedge \bar{\hat{\varphi}}\} = 0$. Using the lemma in the reverse direction implies $0 = (f, sc^k sh^\alpha \varphi)$, $k \in \mathbb{Z}$, $\alpha \in \mathbb{Z}^d$. But now, invoking the hypothesis (3.2), we conclude $f = 0$. This proves the result.

Let us now turn to the proof of Theorem 1.1. First we make some preliminary observations about the function ψ_0 defined by (1.22). We introduce the sequence $f = (f_\alpha \colon \alpha \in \mathbb{Z}^d)$ defined by

$$trig\ f = 2^m \frac{\bar{T}}{\{|\hat{\varphi}|^2\}} = 2^m \bar{T}\ trig\ \mu$$

so that

$$\psi_0 = sc[f, \varphi] \in V^1$$

and $trig\ f \in L^\infty(Q^d)$. Therefore it folows that for all $e \in E$

$$\| [c, \psi_e] \|_2 \leqslant \| trig\ f \|_{L^\infty(Q^d)} \| c \|_2 \tag{3.7}$$

Next, we prove the following.

PROPOSITION 3.1

Let $\varphi \in \mathscr{R}$. Then

$$sc^{-1}\, \psi_0 - 2^m \psi_0 \in W^0. \tag{3.8}$$

where W^0 is defined in (1.23).

Proof

The idea of the proof is to introduce the function L defined by

$$\hat{L} = |\hat{\varphi}|^2 / \{|\hat{\varphi}|^2\}.$$

First, recall that we show in Section 2 that the 2π-continuous function $\{|\hat{\varphi}|^2\}$ strictly positive. We draw two conclusions from this fact. First, by (2.5), (2.6),

$$\left(|\hat{\varphi}|^2/\{|\hat{\varphi}|^2\}\right)(\omega)=1+O\left(\|\omega\|_\infty^{n+1}\right), \quad \omega\to 0 \tag{3.9}$$

and

$$\left(|\hat{\varphi}|^2/\{|\hat{\varphi}|^2\}\right)(\omega)=O\left(\|\omega\|_\infty^{-m}\right), \quad \omega\to\infty. \tag{3.10}$$

Thus

$$L(x)=o\left(\|x\|_\infty^{-(d+n)}\right), \quad x\to\infty. \tag{3.11}$$

We introduce again the sequence $\mu\in L^2$ such that

$$\left\{|\hat{\varphi}|^2\right\}^{-1}=trig\ \mu \tag{3.12}$$

and so

$$L=[\mu,\varphi*\check{\varphi}]. \tag{3.13}$$

Obviously, L is continuous since $\hat{L}\in L^1(\mathbb{R}^d)$. Moreover, by the Fourier inversion formula, one can show that

$$L(\alpha)=\delta_{0\alpha},\ \alpha\in\mathbb{Z}^d. \tag{3.14}$$

For this reason, we call L a *Lagrange function* for the space $V^0(\varphi*\check{\varphi})$. Moreover, the sum on the right hand side of (3.13) converges to a continuous function. To see this, we observe that $\varphi*\check{\varphi}\in\mathscr{R}_{2m,n}$ and so applying the inequality we used to prove (1.6) in Theorem 2.1 to $\varphi*\check{\varphi}$ we get that the function

$$\sum_{\alpha\in\mathbb{Z}^d}|(\varphi*\check{\varphi})(x-\alpha)|^2$$

is bounded for all $x\in\mathbb{R}^d$.

Next, we claim that L satisfies the refinement equation

$$L=sc\left[l_{1/2},L\right] \tag{3.15}$$

where $l_{1/2}:=(L(\alpha/2)\colon \alpha\in\mathbb{Z}^d)\in l^2(\mathbb{Z}^d)$. By what we have already proved, both sides of (3.15) are in $C(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and so we can replace it by the equivalent identity

$$2^d sc\ \hat{L}=trig\ l_{1/2}\cdot\hat{L}. \tag{3.16}$$

By the Poisson summation formula we have

$$trig\ l_{1/2}=2^d\{sc\ \hat{L}\} \tag{3.17}$$

which, by the refinement equation (2.10) and the definition (3.5), becomes

$$\begin{aligned} trig\ l_{1/2}&=2^d\left\{|sc\ \hat{\varphi}|^2\right\}/sc\left\{|\hat{\varphi}|^2\right\}\\ &=2^{-d}|trig\ a|^2\left\{|\hat{\varphi}|^2\right\}/sc\left\{|\hat{\varphi}|^2\right\}. \end{aligned} \tag{3.18}$$

Similarly, we have

$$2^d sc\ \hat{L}/\hat{L} = 2^d \frac{|sc\ \hat{\varphi}|^2}{|\hat{\varphi}|^2} \frac{\{|\hat{\varphi}|^2\}}{sc\{|\hat{\varphi}|^2\}}$$

$$= 2^{-d} |trig\ a|^2 \{|\hat{\varphi}|^2\}/sc\{|\hat{\varphi}|^2\}. \tag{3.19}$$

Combining (3.18) and (3.19) proves the claim.

The reason we introduced L is that it directly relates to ψ_0 as defined in (1.22). Specifically we recall that

$$\hat{\psi}_0 = 2^{-d} \bar{q} sc^{-1}\ \hat{L}. \tag{3.20}$$

Hence,

$$\psi_0 = (-2i)^m\ sc\ \bar{q}(D)L. \tag{3.21}$$

We now apply the differential operator $\bar{q}(D)$ to both sides of (3.15) to obtain from (3.7) and (3.14) that

$$\bar{q}(D)L = 2^m\ sc\ \bar{q}(D)L + \Omega \tag{3.22}$$

where $\Omega \in W^0$ (see (1.23) for the definition of the space W^0). This proves the proposition.

We are now ready to prove that the space

$$\mathscr{S} := \bigoplus_{k \in \mathbb{Z}} W^k \tag{3.23}$$

is dense in L^2 when $\varphi \in \mathscr{R}$. This is the second claim of Theorem 1.1.

We introduce a subset $\mathscr{L}$ of $\mathbb{Z}^d$ by setting

$$\mathscr{L} = \{\beta \in \mathbb{Z}^d\colon sc^k\ sh^\beta\ \bar{q}(D)L \in \mathscr{S}, \text{ for all } k \in \mathbb{Z}\}. \tag{3.24}$$

From (3.22) we get

$$sc^k\ sh^\beta\ \bar{q}(D)L = 2^m\ sc^k\ sh^\beta\ sc\ \bar{q}(D)L + sc^k\ sh^\beta\ \Omega$$

$$= 2^m\ sc^{k+1}\ sc^{2\beta}\ q(D)L + sc^k\ sh^\beta\ \Omega. \tag{3.25}$$

Since by definition $sc^k\ sh^\beta\ \Omega \in \mathscr{S}$, we get from (3.25)

$$2\mathscr{L} \subseteq \mathscr{L}. \tag{3.26}$$

One final comment is needed about the set $\mathscr{L}$. By definition, for $k \in \mathbb{Z}$, $\beta \in \mathbb{Z}^d$, and $e \in E\backslash\{0\}$

$$\mathscr{S} \ni sc^k\ sh^\beta\ \psi_e = sc^k\ sh^{\beta+e/2}\ \psi_0$$

$$= (-2i)^m\ sc^k\ sh^{\beta+e/2}\ sc\ \bar{q}(D)L$$

$$= (-2i)^m\ sc^{k+1}\ sh^{2\beta+e}\ \bar{q}(D)L.$$

In this computation, we used (3.21) and the fact that $\psi_e = sh^{e/2}\psi_0$. Thus we ahve shown that $2\beta + e \in \mathscr{L}$ for all $e \in E\backslash\{0\}$ and $\beta \in \mathbb{Z}^d$. Since every $\alpha \in \mathbb{Z}^d \backslash\{0\}$ can be written as

$$\alpha = 2^r(2\beta + e), \quad e \in E\backslash\{0\}, \quad \beta \in \mathbb{Z}^d$$

we conclude that $\mathscr{L} = \mathbb{Z}^d\backslash\{0\}$.

To complete the argument, we recall that

$$\psi_0 = sc[f, \varphi] \tag{3.27}$$

where $trig\, f := 2^m \overline{T}\, trig\, \mu$. Obviously $f \in l^2$ and $trig\, f \neq 0$ (a.e.). Moreover, by (3.21) we obtain

$$\bar{q}(D)L = (-2i)^{-m}[f, \varphi]. \tag{3.28}$$

We can now rephrase the fact that $\mathscr{L} = \mathbb{Z}^d\backslash\{0\}$ by saying that

$$[f, \varphi](2^k \cdot -\alpha) \in \mathscr{S}, \quad \alpha \in \mathbb{Z}^d\backslash\{0\},\ k \in \mathbb{Z}. \tag{3.29}$$

But, according to Lemma 3.1, since $\varphi \in \mathscr{R}$ admits multiresolution it suffices to prove $[f, \varphi] = sc^{-1}\psi_0 \in \overline{\mathscr{S}}$. For this purpose, we use (3.8) iteratively to conclude that

$$sc^{-1}\psi_0 - (2^{-m})^k\, sc^{-k-1}\psi_0 \in W^{-1} + \ldots + W^{-k} \subseteq \mathscr{S}$$

for any $k \in \mathbb{Z}_+\backslash\{0\}$. Since

$$\|(2^{-m})^k sc^{-k-1}\psi_0\|_2^2 = 2^{d+k(d-2m)}\|\psi_0\|_2^2$$

goes to zero as $k \to \infty$ we conclude that $sc^{-1}\psi_0 \in \overline{\mathscr{S}}$. This proves the second claim of Theorem 1.1.

The proof of the first claim depends on showing that

$$V^k(\varphi) \perp R^k(\psi_e) \tag{3.30}$$

for all $k \in \mathbb{Z}$ and $e \in E\backslash\{0\}$. In fact, suppose for the moment that (3.30) is true. We wish now to demonstrate that

$$R^k(\psi_e) \perp R^{k'}(\psi_{e'}) \tag{3.31}$$

for all $k \neq k'$ and $e, e' \in E\backslash\{0\}$. We assume without loss of generality that $k' > k$. From (3.27), we have $\psi_e \in V^1(\varphi)$ and so

$$R^k(\psi_e) \subseteq V^{k'}(\varphi)$$

and so (3.31) follows from (3.30).

To prove (3.30), it suffices to prove it for $k=0$ which we do by a computation:

$$(2\pi)^d \int_{\mathbb{R}^d} \psi_e(x)\overline{\varphi(x-\alpha)}\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\alpha\cdot\omega} \hat{\psi}_e(\omega)\overline{\hat{\varphi}(\omega)}\,\mathrm{d}\omega$$

$$= 2^{m-2d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(\alpha - e/2)\cdot\omega} \frac{\overline{T}(\omega/2)\hat{\varphi}(\omega/2)\overline{\mathit{trig}\ a}(\omega/2)\overline{\hat{\varphi}(\omega/2)}}{\sum_{\beta\in\mathbb{Z}^d} |\hat{\varphi}(\omega/2+2\pi\beta)|^2}\,\mathrm{d}\omega$$

which is zero, since $e \neq 0$. This proves Theorem 1.1.

We conclude with the following remark. Let $U^k(\varphi)$ be the orthogonal complement of $V^k(\varphi)$ in $V^{k+1}(\varphi)$. As shown above W^k is a closed subspace of U^k. Since φ admits multiresolution we have that $\oplus_{k\in\mathbb{Z}} U^k = L^2$. Hence $W^k = U^k$ for all $k\in\mathbb{Z}$, that is,

$$V^{k+1} = V^k \oplus W^k,\ V^k \perp W^k,\ k\in\mathbb{Z}.$$

This statement also applies to the analogous construction for the cube spline given in [6] and therefore the above remark corrects an oversight made in Remark 4.3 of [6].

We end this paper by demonstrating that the functions $\{\psi_e\}_{e\in E\setminus\{0\}}$ have l^2-stable integer translates, a fact which does not hold for the cube spline case studied in [6]. According to Theorem 4.1 of [5] we must show that there is no $\theta\in\mathbb{R}^d$ and $y = (y_e)_{e\in E\setminus\{0\}} \neq 0$ such that

$$\sum_{e\in E\setminus\{0\}} y_e \hat{\psi}_e(\theta + 2\pi\alpha) = 0, \qquad \alpha\in\mathbb{Z}^d. \tag{3.32}$$

Every $\alpha\in\mathbb{Z}^d$ can be expressed in the form $\alpha = 2\gamma + e'$ where $\gamma\in\mathbb{Z}^d$ and $e'\in E$. Thus (3.32) is equivalent to

$$\left(\sum_{e\in E\setminus\{0\}} \left(y_e\,\mathrm{e}^{-\mathrm{i}(e\cdot\theta)/2}\right)(-1)^{e\cdot e'}\right)\hat{\psi}_0(\theta + 4\pi\gamma + 2\pi e') = 0 \tag{3.33}$$

for all $\gamma\in\mathbb{Z}^d$ and $e'\in E$. To make use of this equation we first observe that $\hat{\phi}(\omega) = 0$ for some $\omega\in\mathbb{R}^d$ if and only if $\omega = 2\pi\alpha$, $\alpha\in\mathbb{Z}^d\setminus\{0\}$. This of course follows from our definition of $\hat{\phi}$ and our requirements (1.18), (1.16) on the trigonometric polynomial T and homogenous polynomial q. Consequently, by our definition of $\hat{\psi}_0$, see (1.22), we have $\hat{\psi}_0(\omega) = 0$ if and only if $\omega = 4\pi\alpha$, $\alpha\in\mathbb{Z}^d$. Returning to (3.33) we choose $\gamma = 0$. Observe that there is at most one $e'\in E$ such that $\hat{\psi}_0(\theta + 2\pi e') = 0$. In fact, if to the contrary there were two distinct values in E, say e_0', e_1' we would have $e_0' - e_1' = 2\mu$, for some $\mu\in\mathbb{Z}^d$

which is impossible unless $e_0' = e_1'$. Hence we conclude there is a $e_0 \in E$ such that

$$\sum_{e \in E \setminus \{0\}} \left(y_e \, \mathrm{e}^{-\mathrm{i}(e \cdot \theta)/2}\right)(-1)^{e \cdot e'} = 0, \qquad e' \in E / \{e_0\}. \tag{3.34}$$

We will finish the proof by showing that the matrix

$$\left((-1)^{e \cdot e'}\right)_{e \in E \setminus \{0\},\, e' \in E \setminus \{e_0\}} \tag{3.35}$$

is nonsingular. To this end, we consider the $2^d \times 2^d$ real symmetric matrix

$$\mathscr{A} = \left((-1)^{e \cdot e'}\right)_{e \in E,\, e' \in E}.$$

It is known (and easily verified) that $A^2 = 2^d I$ and so $A^{-1} = 2^d A$. Hence, since every element of A is nonzero we conclude every $2^d - 1$ minor of A is nonzero as well. In particular, the matrix in (3.35) is nonsingular for any $e_0 \in E$.

Thus we have established that the functions $\{\psi_e\}_{e \in E \setminus \{0\}}$ are stable. In particular it follows that ψ_0 has l^2-stable integer translates. However, $\{\psi_e\}_{e \in E}$ are unstable. To prove this latter fact we choose $y = (y_e)_{e \in E} \neq 0$ such that

$$\sum_{e \in E} y_e (-1)^{e \cdot e'} = 0, \qquad e' \in E \setminus \{0\}$$

and observe that

$$\sum_{e \in E} y_e \hat{\psi}_e(2\pi\alpha) = 0, \qquad \alpha \in \mathbb{Z}^d.$$

The above consideration can be used to identify a $2^d \times 2^d$ nonsingular matrix of trigonometric series which maps the functionals $\{\hat{\phi}\} \cup \{\hat{\psi}_e\}_{e \in E \setminus \{0\}}$ into $\{\hat{\phi}_e\}_{e \in E}$, $\phi_e := sc\ sh^{e/2}\phi$. The coefficients of the trigonometric series appearing in this matrix allow one to write any element in V^1 as a sum of elements in V^0 and W^0. The explicit form of this matrix will be provided at another occasion. In special cases this decomposition may be useful for data compression based on polyharmonic B-splines.

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