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ON BALANCE EQUATIONS FOR MATERIALS WITH AFFINE STRUCTURE

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SOMMARIO. Si deducono le equazioni di bilancio di massa, inerzia, quantità di moto e momento generalizzato della quantità di moto per materiali con struttura affine, come conseguenze dell'invarianza al cambiamento di osservatore di un conveniente principio di bilancio dell'energia.

SUMMARY. The equations of balance of mass, inertia, momentum and generalized moment of momentum for materials with affine structure are derived as consequences of the invariance under change of observer of a postulated law of energy balance.

1. INTRODUCTION

Materials with affine structure are a special, although quite inclusive, class of materials with internal structure. A fundamental assumption of standard continuum mechanics is that detailed microscopic deformations are adequately modeled by the smooth deformation of the entire continuum; the theory of continua with internal structure is intended to treat cases in which this modeling is not adequate, by the introduction, at each point of the continuum, of additional kinematic descriptors and by adjusting balance equations to account for these. In the theory of materials with affine structure, the additional kinematic descriptor is a second order tensor, representing an affine microdeformation; an additional inertia tensor occurs naturally in the specification of the kinetic energy associated with this microdeformation.

The form which the balance equations must take is the subject of a modest controversy. In [1] Capriz and Podio-Guidugli formulated the equations using as a model a finite system of mass points subject only to affine deformations and imagined to be attached, at its center of mass, to the material point in question. They elaborated the matter further in [2]. The balance equations which they advance are essentially the same as those previously presented by Toupin [3] and by Eringen [4]. Cowin and Leslie [5] suggest more far-ranging possibilities.

Our purpose here is to derive the equations of balance of mass, inertia, momentum and (generalized) moment of momentum for materials with affine structure as consequences of the invariance under change of observer of a postulated law of energy balance.

Substantially, we use the method devised by Green and Rivlin [6], [7] and Noll [8] (cfr. also [9], [10], [11], [12]). In fact, Green and Rivlin considered a case which includes ours [7, 13]. However, they chose to regard, as did Noll, the inertial effects as included in the external forces and hence could not arrive at a form for the inertial terms in the balance equations. Since it is precisely these terms which are in greatest question, we here expand their technique by separating, in the energy balance, the *kinetic energy* and the expression for the working of the external forces. A similar line of reasoning is followed by Allen, de Silva and Kline [17] and by Cowin [14, 18], under more special assumptions on the kinetic energy and the time rate of the microinertia tensor. The crucial point in our argument is the description of the transformation properties of the external forces, which in any case are not frame-indifferent in the usual sense. We deduce these transformation properties from the form taken in classical mechanics.

The equations which we derive in this way are exactly those of [2], that is, the usual equations of balance of mass and of linear momentum for the gross motion of the continuum, an equation of evolution for the micro-inertia tensor and an equation for the generalized moment of momentum. The latter involves the non-symmetric part of the stress tensor, a hyperstress (third order) tensor and the symmetric tensor which describes the moment of the interaction of the fine and the gross structure. Finally, we present the reduced, and frame-indifferent, form of the energy equation.

2. THE ENERGY PRINCIPLE

We identify a continuous material body $\mathscr{B}$ with a region $\mathscr{B}_R$ of Euclidean space $\mathscr{E}$, and denote by X a generic point of $\mathscr{B}_R$. If $\mathscr{B}$ has *affine structure*, a motion of $\mathscr{B}$ is a pair of smooth mappings on $\mathscr{B}_R \times \mathbb{R}$,

$$(X, \tau) \rightarrow \underset{\sim}{x}(X, \tau) \in \mathscr{E}, \qquad (X, \tau) \rightarrow G(X, \tau) \in \mathscr{F}, \tag{2.1}$$

where $\mathscr{F}$ is the set of second order tensors with positive determinant. We require that for each τ, $\underset{\sim}{x}(\cdot, \tau)$ is a bijection and that

$$F(X, \tau) = D_X \underset{\sim}{x}(X, \tau) \in \mathscr{F}. \tag{2.2}$$

We let $\underset{\sim}{X}(\cdot, \tau)$ denote the inverse of $\underset{\sim}{x}(\cdot, \tau)$ and a superposed dot denote the material time derivative. We define the

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following fields over $\mathscr{B}_\tau = \underset{\sim}{x}(\mathscr{B}_R, \tau)$:

the *velocity*

$$v(x, \tau) := \underset{\sim}{x}^{\cdot}(\underset{\sim}{X}(x, \tau), \tau), \tag{2.3}$$

the *velocity gradient*

$$V(x, \tau) := F^{\cdot}(\underset{\sim}{X}(x, \tau), \tau) F^{-1}(\underset{\sim}{X}(x, \tau), \tau) \tag{2.4}$$

and the *whorl*, i.e., the tensor W such that

$$W^T(x, \tau) := G^{\cdot}(\underset{\sim}{X}(x, \tau), \tau) G^{-1}(\underset{\sim}{X}(x, \tau), \tau). \;(1) \tag{2.5}$$

$G(X, \tau)$, which describes the microdeformation at X, may be any element of $\mathscr{F}$, as in director theories, where it can be used to describe the location of the directors. For rigid granular materials, G is orthogonal, while for materials with voids it is isotropic. Similarly, W is unrestricted in general, but is restricted in form in those special cases.

To construct the energy equation we assume the following fields as given: a scalar heat supply σ, a vector heat flux h, a vector body force b, a second-order tensor body moment (2) B, a Cauchy (second-order) stress tensor T, and a hyperstress (third-order) tensor $\mathfrak{T}$, and we take the energy source and flux into a part $\mathscr{P}$ of $\mathscr{B}_\tau$ to have the standard form (cfr. [7], [13])

$$\int_{\mathscr{P}} \rho(b \cdot v + B \cdot W + \sigma)\, dv + \int_{\partial\mathscr{P}} (Tn \cdot v + \mathfrak{T} n \cdot W + h \cdot n)\, da, \tag{2.6}$$

where n is the outward normal and ρ is the mass density.

Likewise the internal energy is simply modeled as

$$\int_{\mathscr{P}} \rho\epsilon\, dv, \tag{2.7}$$

where the density ϵ is a scalar field.

Finally, for the kinetic energy we shall assume a density

$$\kappa := \frac{1}{2} v \cdot v + \frac{1}{2} IW \cdot W. \tag{2.8}$$

The first term is standard and represents the kinetic energy (per unit mass) of the gross motion. The second term introduces the inertia due to the affine structure through a micro-inertia tensor field I, supposed symmetric and positive semi-definite. The form of this term can be motivated from particle mechanics (cfr. [1, 2]), but in any case it corresponds to the simplest assumption which meets the requirements of positivity and quadratic form and which includes the classical director theories.

Balance of energy for the body is then expressed by

$$\frac{d}{d\tau}\int_{\mathscr{P}} \rho\left(\epsilon + \frac{1}{2} v \cdot v + \frac{1}{2} IW \cdot W\right) dv = \int_{\mathscr{P}} \rho(b \cdot v + B \cdot W + \sigma) dv + \int_{\partial\mathscr{P}} (Tn \cdot v + \mathfrak{T} n \cdot W + h \cdot n)\, da, \tag{2.9}$$

which must hold for each time and each part $\mathscr{P}$ of the body.

3. CHANGE IN OBSERVER

Two motions $(\underset{\sim}{x}^+, G^+)$ and $(\underset{\sim}{x}, G)$ of a body $\mathscr{B}$ with affine structure are related by a *change in observer* if, for any $(X, \tau) \in \mathscr{B}_R \times \mathbb{R}$,

$$\underset{\sim}{x}^+(X, \tau) = q(\tau) + Q(\tau)\underset{\sim}{x}(X, \tau) \tag{3.1}$$

and

$$G^+(X, \tau) = Q(\tau) G(X, \tau), \tag{3.2}$$

where $q(\tau)$ is a vector and $Q(\tau)$ is a proper orthogonal tensor.

As a consequence, the kinematic quantities introduced in Section 2 transform as follows under a change in observer:

$$F^+ = QF; \qquad v^+ = Qv + q^{\cdot} + Q^{\cdot}p, \tag{3.3}$$

where $p := x - o$ is the position vector of a point $x \in B_\tau$ with respect to a fixed origin $o \in \mathscr{E}$;

$$V^+ = QVQ^T + Q^{\cdot}Q^T; \qquad W^+ = QWQ^T - Q^{\cdot}Q^T. \tag{3.4}$$

Moreover, the transformation laws for the acceleration $v^{\cdot}$ and the rate of whorl $W^{\cdot}$ are

$$v^{\cdot +} = Qv^{\cdot} + 2Q^{\cdot}v + Q^{\cdot\cdot}p + q^{\cdot\cdot}, \qquad W^{\cdot +} = QW^{\cdot}Q^T + Q^{\cdot}WQ^T + QWQ^{\cdot T} - Q^{\cdot\cdot}Q^T - Q^{\cdot}Q^{\cdot T}. \tag{3.5}$$

The body force and body moment must transform in a rather complicated way for a non-Galilean transformation. First, the standard form for b (cfr. [15]) is

$$b^+ = Qb + 2Q^{\cdot}v + Q^{\cdot\cdot}p + q^{\cdot\cdot}. \tag{3.6}$$

The interpretation, of course, is that if Newton's laws are to be valid in each frame, then in non-inertial frames b must be regarded as including «inertial forces». To determine the corresponding results for B we again turn to particle mechanics. Thus for a system of particles, if p_α is the location of the α^{th} particle relative to the mass center; v_α its velocity; μ_α its mass, with $\mu := \Sigma\mu_\alpha$; and $\mu_\alpha b_\alpha$ the external force acting on it, with b_α obeying (6), we have

$$\mu B = \Sigma p_\alpha \otimes \mu_\alpha b_\alpha, \tag{3.7}$$

so that

$$\mu B^+ = \Sigma(Qp_\alpha \otimes Q\mu_\alpha b_\alpha + Qp_\alpha \otimes 2\mu_\alpha Q^{\cdot} v_\alpha + Qp_\alpha \otimes \mu_\alpha Q^{\cdot\cdot} p_\alpha + Qp_\alpha \otimes \mu_\alpha \ddot{q}) = \mu QBQ^T + 2Q\Sigma\mu_\alpha p_\alpha \otimes v_\alpha Q^{\cdot T} + Q\,\Sigma\,\mu_\alpha p_\alpha \otimes p_\alpha Q^{\cdot\cdot T} \tag{3.8}$$

since $\Sigma\mu_\alpha p_\alpha = 0$. Now, in our system, we have

(1) Here we call «whorl» the transpose of the tensor $G^{\cdot}G^{-1}$; the latter was called «wrench» in [1] and «wrenching» in [2], and denoted by W in both those papers. Admittedly, none of these names is especially appealing; however, our present choice is motivated by some slight notational advantages.

(2) By moment we always mean generalized moment, that is, constructs of the form $x \otimes f$, as opposed to the corresponding skew product $x \wedge f$.

$$\mu I = \Sigma \mu_\alpha p_\alpha \otimes p_\alpha . \tag{3.9}$$

Moreover, when the motion is affine,

$$v_\alpha = W^T p_\alpha . \tag{3.10}$$

Therefore we finally obtain

$$B^+ = QBQ^T + 2QIWQ^{\cdot T} + QIQ^{\cdot\cdot T}, \tag{3.11}$$

a transformation law for the body moment that we assume valid in general.

Scalar, vector and (second-order) tensor quantities δ, d, and D, respectively, are said to be *indifferent* to changes in observer or, briefly, frame-indifferent, if

$$\delta^+ = \delta,\ d^+ = Qd,\ D^+ = QDQ^T, \tag{3.12}$$

while a third-order tensor $\mathbb{D}$ we call frame-indifferent if

$$\mathbb{D}^+ d^+ = Q\mathbb{D} d Q^T \tag{3.13}$$

for any frame-indifferent vector d. In the case of quantities not frame-indifferent we designate their nonconformant part as

$$[\![\delta]\!] = \delta^+ - \delta. \tag{3.14}$$

Notice that the external non-inertial body force $b - v^{\cdot}$ and body moment $B - ((IW)^{\cdot} - W^T IW)$ are indeed frame-indifferent quantities. We now stipulate that all of the remaining fields introduced in Section 2 are frame-indifferent. In particular,

$$I^+ = QIQ^T \tag{3.15}$$

implies that

$$I^{\cdot +} = QI^{\cdot}Q^T + Q^{\cdot}IQ^T + QIQ^{\cdot T}. \tag{3.16}$$

Finally, we require that the *equation of balance of energy* (2.7) *be indifferent to changes in observer.* The consequences of this requirement form the contents of the next Section.

4. BALANCE EQUATIONS

For our present purposes, there is no loss of generality if we choose in (3.1) and (3.2)

$$q(\tau) = \tau q_0,\quad Q(\tau) = e^{\tau\Omega}, \tag{4.1}$$

where q_0 is a fixed vector and Ω a fixed skew-symmetric tensor.

Firstly, we exploit the requirement that (2.9) be indifferent to changes of observer of the form (1) with $\Omega = 0$. In view of the invariance of the volume measure and the various assumptions listed in Section 3,

$$\left[\!\!\left[\left(\int_{\mathcal{P}} \rho(\epsilon + \kappa)\mathrm{d}v\right)^{\cdot}\right]\!\!\right] = q_0 \cdot \int_{\mathcal{P}_R} (\rho(\det F)v)^{\cdot}\mathrm{d}v_R + \tag{4.2}$$

$$+ \frac{1}{2}(q_0 \cdot q_0)\int_{\mathcal{P}_R} (\rho \det F)^{\cdot}\,\mathrm{d}v_R .$$

On the other hand,

$$\left[\!\!\left[\int_{\mathcal{P}} \rho(b\cdot v + B\cdot W + \sigma)\,\mathrm{d}v + \right.\right.$$

$$\left.\left. + \int_{\partial\mathcal{P}} (Tn\cdot v + \mathbb{C}n\cdot W + h\cdot n)\,\mathrm{d}a\right]\!\!\right] = \tag{4.3}$$

$$= q_0 \cdot \int_{\mathcal{P}} (\rho b + \operatorname{div} T)\,\mathrm{d}v$$

Since q_0 and $\mathcal{P}$ are arbitrary, under the usual blanket assumption of regularity of the integrands, it follows from (2) and (3) that the energy balance is invariant under such a change of frame if and only if

$$(\rho \det F)^{\cdot} = 0 \tag{4.4}$$

and

$$\rho v^{\cdot} = \rho b + \operatorname{div} T. \tag{4.5}$$

(5) is the familiar balance equation of linear momentum, which retains its form for structured bodies. Since

$$(\det F)^{\cdot} = (\det F)\operatorname{trace}(F^{\cdot}F^{-1}) = (\det F)\operatorname{div} v, \tag{4.6}$$

(4) becomes

$$\rho^{\cdot} + \rho \operatorname{div} v = 0, \tag{4.7}$$

which is an equally familiar form of the conservation law for mass. By use of (5) and (7) the energy balance equation (2.9) may be reduced to

$$\int_{\mathcal{P}} \rho(\epsilon + \frac{1}{2}IW\cdot W)^{\cdot}\mathrm{d}v = \int_{\mathcal{P}} (\rho(B\cdot W + \sigma) + T\cdot V)\,\mathrm{d}v + \tag{4.8}$$

$$+ \int_{\partial\mathcal{P}} (\mathbb{C}n\cdot W + h\cdot n)\,\mathrm{d}a.$$

Secondly, we complete, and exhaust, the exploitation of the assumed invariance of (2.9) by requiring that (8) be invariant under a change in observer of the form (1), with $q_0 = 0$. Again, in view of various assumptions stated in Section 3, we have

$$\left[\!\!\left[\int_{\mathcal{P}} \rho(\epsilon + \frac{1}{2}IW\cdot W)^{\cdot}\,\mathrm{d}v\right]\!\!\right] = \tag{4.9}$$

$$= \Omega\cdot\left(-\int_{\mathcal{P}} \rho(IW)^{\cdot}\,\mathrm{d}v\right) + \frac{1}{2}\Omega^2\cdot\left(-\int_{\mathcal{P}} \rho I^{\cdot}\,\mathrm{d}v\right)$$

and

$$\left[\!\!\left[\int_{\mathcal{P}} (\rho(B\cdot W + \sigma) + T\cdot V)\,\mathrm{d}v + \int_{\partial\mathcal{P}} (\mathbb{C}n\cdot W + h\cdot n)\,\mathrm{d}a\right]\!\!\right] = \tag{4.10}$$

$$\Omega\cdot\left(\int_{\mathcal{P}} (-\rho B + T - \operatorname{div}\mathbb{C})\,\mathrm{d}v\right) + \frac{1}{2}\Omega^2\cdot\left(-2\int_{\mathcal{P}} \rho IW\,\mathrm{d}v\right).$$

Therefore, (8) is invariant under such a change of observer if and only if

$$A := \rho((IW)^{\cdot} - B) + T - \operatorname{div} \mathfrak{T} = \text{a symmetric tensor} \quad (4.11)$$

and

$$I = IW + W^T I. \text{ (3)} \quad (4.12)$$

We recognize in Equation $(11)_2$ the classical *balance law of angular momentum*, extended as to apply to bodies with affine structure; in Equation (12), the *conservation law of microinertia* (4). The latter law is also obtained through an invariance argument by Kaloni and De Silva [16].

To rephrase our invariance argument, the energy principle is indifferent to changes in observer if and only if each of Equations (5), (7), $(11)_2$ and (12) is satisfied.

We now introduce $(11)_2$ and (12) into the equation of balance of energy (8). Using (12)

$$(IW \cdot W)^{\cdot} = 2((IW)^{\cdot} - W^T IW) \cdot W, \quad (4.13)$$

and thus we find

$$\int_{\mathscr{P}} (\rho\dot{e} + ((IW)^{\cdot} - W^T IW) \cdot W) \, dv = \int_{\mathscr{P}} (\rho B \cdot W + \rho\sigma + T \cdot V + \operatorname{div} \mathfrak{T} \cdot W + \mathfrak{T} \cdot \mathfrak{W} + \operatorname{div} h) \, dv, \quad (4.14)$$

where $\mathfrak{W} := D_x W$. Using (11) and taking the localization, we find

$$\rho\dot{e} = \rho\sigma + \operatorname{div} q + T \cdot V + \mathfrak{T} \cdot \mathfrak{W} + (T + \rho W^T IW - A) \cdot W. \quad (4.15)$$

To obtain a more compact form and to distinguish the several effects more clearly, we replace A by the symmetric tensor

$$Z := \operatorname{sym} T + \rho W^T IW - A, \quad (4.16)$$

in terms of which the energy equation takes the form

$$\rho\dot{e} = \rho\sigma + \operatorname{div} q + T \cdot V + \mathfrak{T} \cdot \mathfrak{W} + \operatorname{skw} T \cdot \operatorname{skw} W + Z \cdot \operatorname{sym} W, \quad (4.17)$$

while the balance of moment of momentum becomes

$$\rho((IW)^{\cdot} - W^T IW) = \rho B + \operatorname{div} \mathfrak{T} - \operatorname{skw} T - Z. \quad (4.18)$$

Thus Z is identified as being the symmetric analogue of $\operatorname{skw} T$, representing the symmetric part of the moment of the interaction of the microstructure and the gross motion, and being therefore the object of a constitutive prescription (in particular, in the case of rigid structure, with W skew, it would appear as a constraint variable, as is pointed out by Allen, de Silva and Kline [17]).

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(3) $(11)_2$ follows from (9) and (10) because of the well-known orthogonality of the spaces of symmetric and skew-symmetric tensors. Less trivially, one obtains (12) because

($\Omega^2 \cdot D = 0$ for any skew-symmetric tensor Ω)

$\Leftrightarrow$ (The symmetric part of D vanishes).

(4) It is easy to give (12) a form analogous to (4). One begins by writing (12) as

$$I^{\cdot} - IG^{-1T} G^{\cdot T} - G^{\cdot} G^{-1} I = 0,$$

or, equivalently,

$$-G^{-1} G^{\cdot} G^{-1} I G^{-1T} + G^{-1} I^{\cdot} G^{-1T} - G^{-1} I G^{-1T} G^{\cdot T} G^{-1T} = 0.$$

But, if a tensor D is invertible,

$$(D^{-1})^{\cdot} = -D^{-1} D^{\cdot} D^{-1}.$$

Thus, the last expression of (12) can be cast into the form

$$(G^{-1} I G^{-1T})^{\cdot} = 0,$$

which parallels (4) very much as (12) does (7).

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