INCREMENTAL ELASTOPLASTIC ANALYSIS AND QUADRATIC OPTIMIZATION*

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SOMMARIO: Si discute la soluzione incrementale dei problemi elastoplastici con incrudimento, tenendo conto di distorsioni distribuite. Vengono dimostrati due teoremi di estremo "duali" che trasformano il problema in ottimizzazioni di forme quadratiche convesse vincolate da equazioni e diseguaglianze lineari: il primo teorema concerne gli incrementi degli sforzi e dei moltiplicatori plastici, il secondo gli incrementi degli spostamenti e dei moltiplicatori plastici.

Si specializzano le conclusioni raggiunte al caso dell'elastoplasticità senza incrudimento.

La trattazione viene svolta sia nei termini tradizionali della meccanica dei continui, sia in notazione matriciale in base a discretizzazione per elementi finiti utilizzando i concetti di teoria della programmazione quadratica.

Si effettua infine un confronto con i classici principi incrementali di minimo dell'elastoplasticità (Prager-Hodge, Greenberg), che vengono dedotti dai teoremi qui proposti in una forma generalizzata alle distorsioni diffuse.

SUMMARY: The paper discusses the incremental boundary value problem for elastoplastic workhardening continua, allowing for distributed dislocations. A pair of "dual" extremum theorems reduces the problem to the optimization of convex quadratic forms subject to linear inequalities and equations: the first theorem takes as variables stress and plastic multiplier rates, the latter velocities and plastic multiplier rates.

The conclusions reached are specialized to elastic perfectly plastic (nonhardening) cases.

The problem is discussed both in the traditional terms of continuum mechanics and in matrix notation on the basis of finite element discretization, using some quadratic programming concepts. Finally a comparison is made with the classical incremental minimum principles of plasticity (Prager-Hodge, Greenberg), which are deduced from the present theorems in a form generalized to the distributed dislocations.

1. Introduction.

The boundary value incremental problem is central to flow-law plastic analysis.

For this problem two minimum principles have long been known and are conceptually amenable, respectively, to the complementary and to the potential energy principle of elasticity: the *statical* theorem for stress rates (Prager-Hodge) and the *kinematical* theorem for velocities (Greenberg), see e. g. [1], [2], [3]. However little use has yet been made of either of the above principles.

One reason for this might be the mathematically involved nature of the functionals or functions to be minimized, when stress rates or velocities are assumed as variables.

Minimization of quadratic forms subject to linear inequalities and, possibly, linear equations (here referred to briefly as *quadratic optimization*) is a fairly tractable problem, which becomes quite simple after some discretization, for differential relations are replaced by algebraic linear relations and efficient quadratic programming methods may be applied to numerical solutions.

A theorem which reduces the incremental elastoplastic problem to a quadratic optimization in the plastic multipliers was worked out by Ceradini [4]. The latter writer proved another extremum property of this kind and pointed out the usefulness of quadratic programming concepts in plasticity theory [5], [6].

The former writer formulated the incremental problem as a quadratic optimization in terms of displacement and plastic multiplier rates [7].

In this paper, allowing for prescribed dislocations besides external forces, we reduce the incremental elastoplastic problem to a quadratic optimization in two complementary or *dual* ways: by means of a minimum theorem for stress and plastic multiplier rates, and by means of a minimum theorem for displacement and plastic multiplier rates.

Both theorems are related to the aforementioned pairs of classical extremum principles, as pointed out in subsection 4.2. In the absence of workhardening, they represent the incremental counterparts of the Haar-Kármán theorem generalized to dislocations and of another statement, dual to it, in the deformation theory of ideal plasticity with piecewise linear yield surfaces: these "finite" theorems have been proved in [6] and generalized to the workhardening behavior in [8].

In the absence of dislocations, the latter theorem established herein specializes to the one previously proved in [7].

Compared to the above mentioned extremum theorems in the plastic multiplier rates, [4], [5], the theorems presented here deal with more numerous unknowns ot once, but do not require the preliminary linear calculation of the influence functions or coefficients for stresses due to dislocations.

In Sec. 2 we approach the problem from the familiar angle of continuum mechanics using Cartesian tensor

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notation; in Sec. 3 we reach the same conclusions by the finite element approach in matrix notation, making use of mathematical programming notions.

A comparison between the two treatments is, in our opinion, instructive: it points out in particular some advantages of the latter nontraditional approach, not only as a basis for solving structural problems numerically but also for developing theoretical considerations in a compact and simple manner.

2. The continuous field approach.

2.1. *Formulation of the problem.*

Let σ_{ij} and ε_{ij} represent the stress and strain symmetric tensor in rectangular Cartesian coordinates x_i $(i = 1, 2, 3)$. Strains are conceived as the sum of elastic ε^e_{ij}, plastic ε^p_{ij} and dislocations ε^d_{ij} (1) components; hence strain rates may be expressed as (2)

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij} + \dot{\varepsilon}^d_{ij}. \tag{1}$$

The elastic stress-strain rate relationship reads:

$$\dot{\varepsilon}^e_{ij} = A_{ijhk}\dot{\sigma}_{hk} \tag{2}$$

where the elastic tensor components A_{ijhk} (possibly dependent on the stress state σ_{ij} as in nonlinear elastic cases) are such that

$$A_{ijhk} = A_{jihk} = A_{ijkh} = A_{hkij} \tag{3}$$

$$A_{ijhk}\dot{\sigma}_{ij}\dot{\sigma}_{hk} > 0 \quad \text{for} \quad \dot{\sigma}_{ij} \neq 0\,. \tag{4}$$

We assume the following flow-rules. Let the instantaneous elastic range be defined by the inequalities:

$$f_\alpha(\sigma_{ij}\,,\ \overset{p}{\varepsilon}_{ij}\,,\ k) \leqslant 0 \qquad (\alpha = 1 \dots n) \tag{5}$$

where f_α are (regular) yield functions, k represents the workhardening parameter (which depends on the plastic deformation history). Let the yield functions play the role of plastic potentials: this means that (normality rule):

$$\dot{\varepsilon}^p_{ij} = \sum_1^n{}_\alpha \frac{\partial f_\alpha}{\partial \sigma_{ij}}\,\dot{\lambda}_\alpha\,. \tag{6}$$

The plastic multipliers λ_β conform to the following rules:

$$\dot{\lambda}_\alpha \geqslant 0 \text{ if } f_\alpha = 0 \text{ and } \dot{f}_\alpha = 0, \text{ otherwise } \dot{\lambda}_\alpha = 0 \tag{7}$$

where, if h_α represents the positive, history dependent workhardening coefficient:

$$\dot{f}_\alpha = \frac{\partial f_\alpha}{\partial \sigma_{ij}}\,\dot{\sigma}_{ij} - h_\alpha\dot{\lambda}_\alpha\,.\ (3) \tag{8}$$

The boundary value problem to be discussed can be formulated as follows.

Starting from a known situation Y, consider as given the rates $\dot{X}_i$ of external body forces, $\dot{\varepsilon}^d_{ij}$ of internal dislocations (both defined over the volume V), $\dot{T}_i$ of surface tractions on the part S_T of the boundary, $\dot{u}_i$ of prescribed displacements on the part S_u. The stress and strain response $\dot{\sigma}_{ij}$, $\dot{\varepsilon}_{ij}$ are to be determined in V.

The governing equations express equilibrium:

$$\begin{aligned} &\dot{\sigma}_{ij/i} + \dot{X}_j = 0 && \text{in } V \\ &\dot{\sigma}_{ij}n_i = \dot{T}_j && \text{on } S_T \end{aligned} \tag{9}$$

compatibility:

$$\begin{aligned} &\dot{\varepsilon}_{ij} = \frac{1}{2}(\dot{u}_{i/j} + \dot{u}_{j/i}) && \text{in } V \\ &\dot{u}_i = \bar{\dot{u}}_i && \text{on } S_u\,, \end{aligned} \tag{10}$$

conformity, i. e. fulfilment of the constitutive Eqs. (1) (2) (6) (7) (8); the last two may be conveniently replaced by

$$\dot{f}_\alpha \leqslant 0 \tag{11}$$

$$\dot{\lambda}_\alpha \geqslant 0 \qquad \text{in } V_\alpha \quad (\alpha = 1 \dots n) \tag{12}$$

$$\dot{f}_\alpha\dot{\lambda}_\alpha = 0 \tag{13}$$

$$\dot{\lambda}_\alpha = 0 \qquad \text{in } V - V_\alpha \tag{14}$$

where V_α denotes the region of V in which the α-th inequality (5) is satisfied in Y as an equality.

2.2. *A minimum theorem in the stress and plastic multiplier rates.*

Let us consider the quadratic functional

$$\Omega(\dot{\sigma}_{ij}\,,\ \dot{\lambda}_\alpha) = \frac{1}{2}\int_V A_{ijhk}\dot{\sigma}_{ij}\dot{\sigma}_{hk}dV + \frac{1}{2}\sum{}_\alpha\int_{V_\alpha} h_\alpha\dot{\lambda}_\alpha^2 dV + \int_V \dot{\sigma}_{ij}\dot{\varepsilon}^d_{ij}dV - \int_{S_u}\dot{\sigma}_{ij}n_i\bar{\dot{u}}_j dS \tag{15}$$

(1) E. g.: for thermically isotropic bodies $\dot{\varepsilon}^d_{ij} = \delta_{ij}\gamma\dot{T}$, if δ_{ij} is the Kronecker symbol, γ the thermal coefficient of linear expansion, $\dot{T}$ the temperature rate.

(2) A superposed point indicates derivative with respect to any monotonous function t of time. When increments are referred to, the differential δ will be omitted for brevity.

(3) Relations (7) (8) are equivalent to the traditional ones:

$$\dot{\lambda}_\alpha = \frac{1}{h_\alpha}\frac{\partial f_\alpha}{\partial \sigma_{ij}}\dot{\sigma}_{ij} \qquad \text{if} \quad f_\alpha = 0 \quad \text{and} \quad \frac{\partial f_\alpha}{\partial \sigma_{ij}}\dot{\sigma}_{ij} \geqslant 0$$

otherwise

$$\dot{\lambda}_\alpha = 0\,.$$

and the linear relations

$$\dot{\sigma}_{ij/i} + \dot{X}_j = 0 \qquad \text{in } V \qquad (9)$$

$$\dot{\sigma}_{ij} n_i = \dot{T}_j \qquad \text{on } S_T$$

$$\dot{f}_\alpha = \frac{\partial f_\alpha}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - h_\alpha \dot{\lambda}_\alpha \leqslant 0 \quad \text{in } V_\alpha (\alpha = 1 \,..\, n). \; (11)$$

We shall prove here below that the solution $\dot{\sigma}^0_{ij}$, $\dot{\lambda}_\alpha^0$ is characterized by the absolute minimum of functional Ω within the class of all stress rate and plastic multiplier rate fields, $\dot{\sigma}_{ij}$, $\dot{\lambda}_\alpha$, which satisfy the constraints (9) and (11). Such fields might be called *statically and plastic potential admissible* in view of the meaning of (9) and (11).

Let us express any admissible field as:

$$\dot{\sigma}_{ij} = \dot{\sigma}^0_{ij} + \Delta\dot{\sigma}_{ij}, \qquad \dot{\lambda}_\alpha = \dot{\lambda}^0_\alpha + \Delta\dot{\lambda}_\alpha \qquad (16)$$

and indicate by $\Delta\Omega$ the difference $\Omega(\dot{\sigma}_{ij}, \dot{\lambda}_\alpha) - \Omega(\dot{\sigma}^0_{ij}, \dot{\lambda}_\alpha^0)$. We may write:

$$\Delta\Omega = \int_V \Delta\dot{\sigma}_{ij} A_{ijhk} \dot{\sigma}^0_{hk} dV + \frac{1}{2}\int_V \Delta\dot{\sigma}_{ij} A_{ijhk} \Delta\dot{\sigma}_{hk} dV + \frac{1}{2}\sum_\alpha \int_{V_\alpha} h_\alpha (2\Delta\dot{\lambda}_\alpha \dot{\lambda}^0_\alpha + \Delta\dot{\lambda}^2_\alpha) dV + \int_V \Delta\dot{\sigma}_{ij} \dot{\varepsilon}^d_{ij} dV - \int_{S_u} \Delta\dot{\sigma}_{ij} n_j \dot{\bar{u}}_j dS \qquad (17)$$

whence, through Eqs. (1) (2):

$$\Delta\Omega = \int_V \Delta\dot{\sigma}_{ij}(\dot{\varepsilon}^0_{ij} - \dot{\varepsilon}^{p0}_{ij}) dV + \frac{1}{2}\int_V \Delta\dot{\sigma}_{ij} A_{ijhk} \Delta\dot{\sigma}_{hk} dV + \frac{1}{2}\sum_\alpha \int_V h_\alpha (2\Delta\dot{\lambda}_\alpha \dot{\lambda}^0_\alpha + \Delta\dot{\lambda}^2_\alpha) dV - \int_{S_u} \Delta\dot{\sigma}_{ij} n_i \dot{\bar{u}}_j dS.$$

Taking account of Eqs. (6) (8), after some manipulations, we finally obtain:

$$\Delta\Omega = \int_V \Delta\dot{\sigma}_{ij} \dot{\varepsilon}^0_{ij} dV - \int_{S_u} \Delta\dot{\sigma}_{ij} n_i \dot{\bar{u}}_j dS + \sum_\alpha \int_{V_\alpha} (\dot{f}^0_\alpha - \dot{f}_\alpha) \dot{\lambda}^0_\alpha dV + \frac{1}{2}\int_V \Delta\dot{\sigma}_{ij} A_{ijhk} \Delta\dot{\sigma}_{hk} dV + \frac{1}{2}\sum_\alpha \int_{V_\alpha} h_\alpha \Delta\dot{\lambda}^2_\alpha dV. \qquad (18)$$

By the virtual work principle, the sum of the first two integrals in Eq. (18) vanishes. The third integral is nonnegative because the solution complies with (11) (12), while any admissible field fulfils inequality (11). The sum of the last two integrals is nonnegative and zero if and only

$$\Delta\dot{\sigma}_{ij} = \dot{\sigma}_{ij} - \dot{\sigma}^0_{ij} \equiv 0 \text{ and } \Delta\dot{\lambda}_\alpha = \dot{\lambda}_\alpha - \dot{\lambda}^0_\alpha \equiv 0. \qquad (19)$$

When Eqs. (19) hold, the third integral of Eq. (18) vanishes by virtue of Eq. (8). Therefore we may conclude that

$$\Omega(\dot{\sigma}_{ij}, \dot{\lambda}_\alpha) \geqslant \Omega(\dot{\sigma}^0_{ij}, \dot{\lambda}^0_\alpha) \qquad (20)$$

where the equality sign holds only for the solution. Thus the theorem is proved.

2.3. *A minimum theorem in displacement and plastic multiplier rates.*

Let us now consider the quadratic functional:

$$\Xi(\dot{u}_i, \dot{\lambda}_\alpha) = \frac{1}{2}\int_V a_{ijhk} \dot{\varepsilon}^e_{ij} \dot{\varepsilon}^e_{hk} dV + \frac{1}{2}\sum_\alpha \int h_\alpha \dot{\lambda}^2_\alpha dV - \int_V \dot{X}_i \dot{u}_i dV - \int_{S_T} \dot{T}_i \dot{u}_i dS \qquad (21)$$

where a_{ijhk} is the elastic moduli tensor (the inverse of A_{ijhk}) and

$$\dot{\varepsilon}^e_{ij} = \frac{1}{2}(\dot{u}_{i/j} + \dot{u}_{j/i}) - \sum_\alpha \frac{\partial f_\alpha}{\partial \sigma_{ij}} \dot{\lambda}_\alpha - \dot{\varepsilon}^d_{ij}. \qquad (22)$$

Moreover let us consider the linear relations

$$\dot{\lambda}_\alpha \geqslant 0 \qquad \text{in } V_\alpha \; (\alpha = 1 \dots n) \qquad (23)$$

$$\dot{u}_i = \dot{\bar{u}}_i \qquad \text{on } S_u. \qquad (24)$$

We shall now prove that the solution $\dot{u}_i^0$, $\dot{\lambda}_\alpha^0$ is characterized by the absolute minimum of functional Ξ within the class of all velocity and plastic multiplier rate fields, $\dot{u}_i$, $\dot{\lambda}_\alpha$, which satisfy the constraints (23) (24) (and, hence, can be called *kinematically and direction admissible* (4)).

Let us express any admissible field as:

$$\dot{u}_i = \dot{u}^0_i + \Delta\dot{u}_i, \qquad \dot{\lambda}_\alpha = \dot{\lambda}^0_\alpha + \Delta\dot{\lambda}_\alpha. \qquad (25)$$

The difference

$$\Delta\Xi = \Xi(\dot{u}_i, \dot{\lambda}_\alpha) - \Xi(\dot{u}^0_i, \dot{\lambda}^0_\alpha)$$

can be given the form:

$$\Delta\Xi = \int_V \Delta\dot{\varepsilon}^e_{ij} a_{ijhk} \dot{\varepsilon}^{e0}_{hk} dV + \frac{1}{2}\int_V a_{ijhk} \Delta\dot{\varepsilon}^e_{ij} \Delta\dot{\varepsilon}^e_{hk} dV + \frac{1}{2}\sum_\alpha \int_{V_\alpha} h_\alpha (2\dot{\lambda}^0_\alpha \Delta\dot{\lambda}_\alpha + \Delta\dot{\lambda}^2_\alpha) dV - \int_V \dot{X}_i \Delta\dot{u}_i dV - \int_{S_T} \dot{T}_i \Delta\dot{u}_i dS. \qquad (26)$$

By substituting in the first integral of (26) the elastic rate relationship and Eqs. (1) (22), and rearranging, we obtain:

$$\Delta\Xi = \int_V \dot{\sigma}_{ij} \Delta\dot{\varepsilon}_{ij} dV - \int_V \dot{X}_i \Delta\dot{u}_i dV - \int_{S_T} \dot{T}_i \Delta\dot{u}_i dS - \sum_\alpha \int_{V_\alpha} \left(\frac{\partial f_\alpha}{\partial \sigma_{ij}} \dot{\sigma}^0_{ij} - h_\alpha \dot{\lambda}^0_\alpha\right) \cdot \Delta\dot{\lambda}_\alpha dV + \frac{1}{2}\int_V a_{ijhk} \Delta\dot{\varepsilon}^e_{ij} \Delta\dot{\varepsilon}^e_{hk} dV + \frac{1}{2}\sum_\alpha \int_{V_\alpha} h_\alpha \Delta\dot{\lambda}^2_\alpha dV.$$

(4) "Direction admissible" refers to the outward normality rule expressed by (23) for the vector of the strain rate components.

By means of Eqs. (8) and of the latter Eq. (25), we get:

$$\Delta \Xi = \int_V \overset{\cdot 0}{\sigma}_{ij} \Delta \dot{\varepsilon}_{ij} dV - \int_V \dot{X}_i \Delta \dot{u}_i dV - \int_{S_T} \dot{T}_i \Delta \dot{u}_i dS +$$

$$+ \sum_\alpha \int_{V_\alpha} \overset{\cdot 0}{f}_\alpha (\overset{\cdot 0}{\lambda}_\alpha - \dot{\lambda}) dV + \frac{1}{2} \int_V a_{ijhk} \Delta \dot{\varepsilon}^e_{ij} \Delta \dot{\varepsilon}^e_{hk} dV +$$

$$+ \frac{1}{2} \sum_\alpha \int_{V_\alpha} h_\alpha \Delta \dot{\lambda}^2_\alpha dV. \qquad (27)$$

The principle of virtual work requires the sum of the first three integrals in Eq. (27) to vanish.

The fourth integral is nonnegative since the solution satisfies relations (11) (13), and any admissible field fulfils inequality (12). The sum of the two last terms of (27) is nonnegative and zero if and only if

$$\Delta \dot{\varepsilon}^e_{ij} = \dot{\varepsilon}^e_{ij} - \dot{\varepsilon}^{e0}_{ij} \equiv 0 \qquad (28\text{-}a)$$

and

$$\Delta \dot{\lambda}_\alpha = \dot{\lambda}_\alpha - \overset{\cdot 0}{\lambda}_\alpha \equiv 0 . \qquad (28\text{-}b)$$

Eqs.(28) imply that $\dot{\varepsilon}_{ij} \equiv \dot{\varepsilon}^0_{ij}$, and, consequently, $\dot{u}_i \equiv \dot{u}_i^0$ (possibly to within an immaterial rigid body motion); it follows also, through Eq.(8), that, if Eqs.(28) are satisfied, the fourth integral in Eq. (27) must vanish. In conclusion, we may affirm that

$$\Xi(\dot{u}_i , \dot{\lambda}_\alpha) \geq \Xi(\overset{\cdot 0}{u}_i , \overset{\cdot 0}{\lambda}_\alpha) \qquad (29)$$

the equality sign holding only for the solution.

2.4. *A continued inequality.*

For the solution we may write the following virtual work equation

$$\int_V \overset{\cdot 0}{\sigma}_{ij} \overset{\cdot 0}{\varepsilon}_{ij} \, dV - \int_V \dot{X}_i \overset{\cdot 0}{u}_i dV - \int_{S_T} \dot{T}_i \dot{u}_i dS -$$

$$- \int_{S_u} \overset{\cdot 0}{\sigma}_{ij} n_i \bar{\dot{u}}_j dS = 0 \qquad (30)$$

in which by means of Eqs. (1) and (6), the first integral becomes:

$$\int_V \overset{\cdot 0}{\sigma}_{ij} \dot{\varepsilon}^{e0}_{ij} \, dV + \sum_\alpha \int_{V_\alpha} \overset{\cdot 0}{\sigma}_{ij} \frac{\partial f_\alpha}{\partial \sigma_{ij}} \overset{\cdot 0}{\lambda}_\alpha dV + \int_V \overset{\cdot 0}{\sigma}_{ij} \dot{\varepsilon}^{0d}_{ij} \, dV.$$

Making use of Eqs. (8) and (13) we finally obtain:

$$\int_V \overset{\cdot 0}{\sigma}_{ij} \dot{\varepsilon}^{e0}_{ij} \, dV + \sum_\alpha \int_{V_\alpha} h_\alpha \dot{\lambda}^{02}_\alpha \, dV + \int_V \overset{\cdot 0}{\sigma}_{ij} \dot{\varepsilon}^{d0}_{ij} \, dV +$$

$$- \int_V \dot{X}_i \overset{\cdot 0}{u}_i dV - \int_{S_T} \dot{T}_i \overset{\cdot 0}{u}_i dS - \int_{S_u} \overset{\cdot 0}{\sigma}_{ij} n_i \bar{\dot{u}}_j dS = 0 .$$

Since:

$$\frac{1}{2} \int_V A_{ijhk} \overset{\cdot 0}{\sigma}_{ij} \overset{\cdot 0}{\sigma}_{hk} dV + \frac{1}{2} \int_V a_{ijhk} \dot{\varepsilon}^{e0}_{ij} \dot{\varepsilon}^{e0}_{hk} dV =$$

$$= \int_V \overset{\cdot 0}{\sigma}_{ij} \dot{\varepsilon}^{e0}_{ij} dV$$

it readily follows, through Eqs. (15) and (21), that:

$$- \Omega(\overset{\cdot 0}{\sigma}_{ij} , \overset{\cdot 0}{\lambda}_\alpha) = \Xi(\overset{\cdot 0}{u}_i , \overset{\cdot 0}{\lambda}_\alpha) . \qquad (31)$$

Therefore, the minimum properties (20) and (29) of the solution may be compressed into the continued inequality:

$$- \Omega(\dot{\sigma}_{ij} , \dot{\lambda}_\alpha) \leq - \Omega(\overset{\cdot 0}{\sigma}_{ij} , \overset{\cdot 0}{\lambda}_\alpha) = \Xi(\overset{\cdot 0}{u}_i , \overset{\cdot 0}{\lambda}_\alpha) \leq \Xi(\dot{u}_i , \dot{\lambda}_\alpha) . \qquad (32)$$

2.5. *Specialization to nonhardening behaviour.*

In ideal plasticity the plastic potentials are independent from the yielding history; this behaviour is described simply by assuming throughout:

$$h_\alpha = 0 \qquad (\alpha = 1 \ldots n). \qquad (33)$$

The former of the above extremum properties reduces to:

$$\Omega(\overset{\cdot 0}{\sigma}_{ij}) = \frac{1}{2} \int_V A_{ijhk} \overset{\cdot 0}{\sigma}_{ij} \overset{\cdot 0}{\sigma}_{hk} dV + \int_V \overset{\cdot 0}{\sigma}_{ij} \dot{\varepsilon}^d_{ij} dV -$$

$$- \int_{S_u} \overset{\cdot 0}{\sigma}_{ij} n_i \bar{\dot{u}}_j dS = \min \qquad (34)$$

within the class defined by the relations:

$$\begin{aligned} &\dot{\sigma}_{ij/i} + \dot{X}_j = 0 && \text{in } V \\ &\dot{\sigma}_{ij} n_i = \dot{T}_j && \text{on } S_T \qquad (35) \\ &\dot{f}_\alpha = \frac{\partial f_\alpha}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \leq 0 && \text{in } V_\alpha \, (\alpha = 1 \ldots n) . \end{aligned}$$

In fact the proof given in subsec. 2.2 still holds even on the assumption (33), which rules out the plastic multiplier rates from the quadratic optimization. Clearly constraints (35) may now not be compatible and, hence, no solution exists. This means that the starting situation Y represents a plastic collapse state and the external force increments $\dot{X}_i$, $\dot{T}_i$ are inadmissible with such state.

Analogously, the latter minimum property specializes, for Eq. (33), to the form:

$$\Xi(\overset{\cdot 0}{u}_i , \overset{\cdot 0}{\lambda}_\alpha) = \frac{1}{2} \int a_{ijhk} \dot{\varepsilon}^{e0}_{ij} \dot{\varepsilon}^{e0}_{hk} dV - \int_V \dot{X}_i \overset{\cdot 0}{u}_i dV -$$

$$- \int_{S_T} \dot{T}_i \overset{\cdot 0}{u}_i dS = \min \qquad (36)$$

within the class defined by the relations:

$$\dot{\lambda}_\alpha \geqslant 0 \qquad \text{in } V_\alpha \ (\alpha = 1 \dots n)$$
$$\dot{u}_i = \dot{\bar{u}}_i \qquad \text{on } S_u \tag{37}$$

if $\dot{\varepsilon}^e_{ij}$ is still expressed in the form (22).

We note that, since the last integral in Eq. (27) vanishes, the latter of Eq. (28) is no longer required in order that (29) be satified as an equality. However, the nonegativity of the fourth integral in Eq. (27) implies that:

$$\dot{\lambda}_\alpha = 0 \text{ if } \dot{f}^0_\alpha < 0\,, \qquad \dot{\lambda}_\alpha \geqslant 0 \text{ if } \dot{f}^0_\alpha = 0\,. \tag{38}$$

The former of (28) combined with (38) shows that the equality sign in Eq. (29) holds also for fields $\dot{u}_i$, $\dot{\lambda}_\alpha$ different from the hypothesized comparison solution $\dot{u}_i^0$, $\dot{\lambda}_\alpha^0$, provided that the difference consists of a compatible plastic strain distribution. Such fields too represent solutions of the problem which are thus all characterized by the minimum properties. Clearly functional (36) may turn out to be unbounded from below in the feasible class defined by (37). This means again that situation Y represents a collapse state and that the given external force increments exceed the carrying capacity of the continuum.

3. The finite element, matrix approach.

3.1. *Formulation of the problem.*

Let us now consider an assemblage of a discrete number N of structural elements interconnected in a finite number of points (nodes). Let the actual or the admitted deformation patterns and stress fields in any element (say in the i-th element) be governed by the generalized strain vector $\mathbf{q}^i$ and by the corresponding generalized stress vector $\mathbf{Q}^i$: the element mechanical behaviour is pictured by a relation between $\mathbf{q}$ and $\mathbf{Q}$ (*constitutive law*) to be formulated later. As a consequence, the strain and stress state throughout the assemblage is defined by the vectors:

$$\tilde{\mathbf{q}} \equiv [\tilde{\mathbf{q}}^1 \dots \tilde{\mathbf{q}}^N]\,; \qquad \tilde{\mathbf{Q}} \equiv [\tilde{\mathbf{Q}}^1 \dots \tilde{\mathbf{Q}}^N]\,. \quad (^5)$$

A distribution of imposed dislocations (e. g. thermal strains) shall be described by the vector: $\tilde{\boldsymbol{\delta}} \equiv [\tilde{\boldsymbol{\delta}}^1 \dots \tilde{\boldsymbol{\delta}}^N]$.

The homologous vectors $\tilde{\mathbf{F}} \equiv [F_1 \dots F_m]$, $\tilde{\mathbf{f}} \equiv [f_1 \dots f_m]$ will define any set of external (nodal) forces and any set of kinematically free nodal displacements (both referred to the coordinate system chosen for the whole structure).

The matrix theory of structures, see e. g. [9], [10], shows that any continuum may be discretized as an approximation, into the above assemblage of finite segments: provided that appropriate criteria are followed in evaluating element properties and in reducing volume and surface forces to nodal loads, the finite element idealization supplies the basis for an algebraic description of the structural behaviour, which is fully analogous to the traditional continuous field description.

For the sake of simplicity but without loss of generality, we shall refer here to homogeneous stress and deformation patterns in all elements and simulate the prescribed displacements by means of some components of $\boldsymbol{\delta}$ attributed to suitable additional boundary elements [10]. The former assumption ensures full compatibility at the elements boundaries and implies an obvious strict correspondence between the mechanical properties of the material and the element. The latter assumption avoids introducing into the analysis a separate term for explicit reference to the work performed by reactions at movable supports or constraints.

On the basis of the above structural idealization, let us now write the basic relationships governing the elastic-plastic response to given external action rates $\dot{\mathbf{F}}$, $\dot{\boldsymbol{\delta}}$ (or to within a common factor δt, increments), added to a known situation Y.

Compatibility and equilibrium can be expressed by the matrix equations:

$$\dot{\mathbf{q}} = \mathbf{B}\dot{\mathbf{f}} \tag{1}$$

$$\tilde{\mathbf{B}}\dot{\mathbf{Q}} = \dot{\mathbf{F}} \tag{2}$$

where $\mathbf{B}$ is a matrix built up only from the geometrical properties of the undeformed structure.

The strain rates are considered as a superposition of elastic $\dot{\mathbf{e}}$, plastic $\dot{\mathbf{p}}$ and dislocation components:

$$\dot{\mathbf{q}} = \dot{\mathbf{e}} + \dot{\mathbf{p}} + \dot{\boldsymbol{\delta}}\,. \tag{3}$$

By assembling all the element elastic stiffness matrices $\mathbf{S}^i$ (symmetric positive definite) in a matrix $\mathbf{D}^s = \text{diag.}\ [\mathbf{S}^1 \dots \mathbf{S}^N]$, we may write:

$$\dot{\mathbf{Q}} = \mathbf{D}^s\dot{\mathbf{e}} \quad \text{or} \quad \dot{\mathbf{e}} = \mathbf{D}^c\dot{\mathbf{Q}} \tag{4}$$

where $\mathbf{D}^c = (\mathbf{D}^s)^{-1}$ is the *internal* elastic flexibility matrix of the structure (as in Sec. 2, $\mathbf{D}^c$ and $\mathbf{D}^s$ might vary with the stress state, i. e. the elastic behavior is not required to be linear).

The plastic flow rules according to Koiter's generalized plastic potential theory [1], which were adopted for the material behavior in the preceding Section, will hold again for the element behavior. It is useful for the sequel to formulate these laws in a fully formal way and for the whole structure simultaneously (see also [5]). In the given starting situation let n^Y elements be at the yield limit: namely the corresponding stress point ${}^Y\mathbf{Q}^i$, for $i = 1 \dots n^Y$, lies on the intersection of $s_i{}^Y$ instantaneous regular yield surfaces $\varphi_{ij}(\mathbf{Q}^i) = 0$ $(j = 1 \dots s_i{}^Y)$ defined in the $\mathbf{Q}^i$-space. We shall indicate by:

$$\mathbf{N}^Y_{ij} = (\text{grad}_{Q^i}\varphi_{ij})_{{}^Y Q^i} \text{ and } \dot{\lambda}_{ij}$$

the outward normal vectors and the plastic multipliers which define the plastic strain rates $\dot{\mathbf{p}}^i$ generated in any possible yielding process starting from ${}^Y\mathbf{Q}^i$.

(5) Notation: Bold face letters indicate matrices or column-vectors; a superposed tilde means transpose; vector inequalities apply to each component separately.

Let us assemble all vectors $\mathbf{N}_{ij}$ for $j = 1 \ldots s_i^Y$ as columns in a matrix $\mathbf{N}_i$, all matrices $\mathbf{N}_i$ for $i = 1 \ldots n^Y$ as diagonal elements in a matrix ${}^Y\mathbf{D}^N \equiv \text{diag}\,[\mathbf{N}_1{}^Y \ldots \mathbf{N}^Y{}_{n^Y}]$, all scalars $\dot{\lambda}_{ij}$ for $i = 1 \ldots s_i^Y$, $i = 1 \ldots n^Y$, as components of a vector $\dot{\boldsymbol{\lambda}}$. By means of these symbols we may express the plastic strain rate vector according to the outward normality rule as follows:

$$\dot{\mathbf{p}} = \mathbf{N}\dot{\boldsymbol{\lambda}} \qquad (5), \qquad \text{where } \mathbf{N} \equiv \begin{bmatrix} {}^Y\mathbf{D}^N \\ \hline \mathbf{0} \end{bmatrix} \qquad (5')$$

$$\dot{\boldsymbol{\lambda}} \geqslant 0 \qquad (6)$$

Finally let us assemble for $j = 1 \ldots s_i^Y$, $i = 1 \ldots n^Y$, all plastic potential rates $\dot{\varphi}_{ij}$ in a vector $\dot{\boldsymbol{\varphi}}$, all strainhardening coefficients $H_{ij}{}^Y$ (history dependent, calculated at Y) in a diagonal matrix $\mathbf{D}^H$. With these new symbols we can complete the description of the plastic flow rules as follows:

$$\dot{\boldsymbol{\varphi}} = \tilde{\mathbf{N}}\dot{\mathbf{Q}} - \mathbf{D}^H\dot{\boldsymbol{\lambda}} \tag{7}$$

$$\dot{\boldsymbol{\varphi}} \leqslant \mathbf{0} \tag{8}$$

$$\tilde{\dot{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} = 0\,. \tag{9}$$

The set of equations and inequalities (1) to (9), all linear except the last one, represents a first mathematical model for the mechanical phenomenon under consideration.

We shall now construct alternative formulations in terms of quadratic programs, which represent the algebraic counterparts of the extremum properties established in Sec. 2 in the continuous description. In order to emphasize the constructive character of the elastoplastic matrix theory founded on mathematical programming, instead of stating first and then proving the desired minimum properties, we shall follow a path which automatically leads to them in the end.

3.2. *A quadratic program in the stress and plastic multiplier rates, the relevant minimum principle.*

Let us combine the unknown vectors $\dot{\mathbf{Q}}, \dot{\boldsymbol{\lambda}}$ in a single vector $[\tilde{\dot{\mathbf{Q}}}|\tilde{\dot{\boldsymbol{\lambda}}}] \equiv \tilde{\mathbf{x}}$. Making use of nonnegative slack variable vectors $\mathbf{y}^+$, $\mathbf{y}^-$, we may express the equilibrium Eq. (2) in the equivalent form:

$$\begin{aligned} \tilde{\mathbf{B}}\dot{\mathbf{Q}} + \mathbf{y}^+ &= \dot{\mathbf{F}} \\ -\tilde{\mathbf{B}}\dot{\mathbf{Q}} + \mathbf{y}^- &= \dot{\mathbf{F}} \end{aligned} \qquad (10) \qquad\qquad \begin{aligned} \mathbf{y}^+ &\geqslant 0 \\ \mathbf{y}^- &\geqslant 0\,. \end{aligned} \qquad (11)$$

In this way the three basic relations (2), (7) and (8) (equilibrium and conformity) can be condensed into the following two matrix relations involving vector $\mathbf{x}$:

$$\begin{bmatrix} \tilde{\mathbf{B}} & \mathbf{O} \\ -\tilde{\mathbf{B}} & \mathbf{O} \\ \tilde{\mathbf{N}} & -\mathbf{D}^H \end{bmatrix} \begin{bmatrix} \dot{\mathbf{Q}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} + \begin{bmatrix} \mathbf{y}^+ \\ \mathbf{y}^- \\ -\dot{\boldsymbol{\varphi}} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{F}} \\ -\dot{\mathbf{F}} \\ \mathbf{O} \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} \mathbf{y}^+ \\ \mathbf{y}^- \\ -\dot{\boldsymbol{\varphi}} \end{bmatrix} \geqslant \mathbf{O}\,. \tag{13}$$

In more compact notation (the meaning of the new symbols is defined by comparison):

$$\mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{a} \qquad (12') \qquad\qquad \mathbf{y} \geqslant \mathbf{0}\,. \qquad (13')$$

The basic Eqs. (1) (4) (5) and (3) (compatibility and conformity), the first three being introduced into the last one, allow us to eliminate vectors $\dot{\mathbf{q}}$, $\dot{\mathbf{e}}$, $\dot{\mathbf{p}}$, and reduce to the equation:

$$-\mathbf{B}\dot{\mathbf{f}} + \mathbf{D}^c\dot{\mathbf{Q}} + \mathbf{N}\dot{\boldsymbol{\lambda}} = -\dot{\boldsymbol{\delta}}\,. \tag{14}$$

Let us express the free variables $\dot{\mathbf{f}}$ as the difference of two sign-restricted variables:

$$\dot{\mathbf{f}} = -\dot{\mathbf{f}}^- + \dot{\mathbf{f}}^+ \quad (15); \qquad \dot{\mathbf{f}}^- \geqslant \mathbf{0}, \quad \dot{\mathbf{f}}^+ \geqslant \mathbf{0} \quad (16)$$

and associate to Eq. (14) the identity

$$-\mathbf{D}^H\dot{\boldsymbol{\lambda}} + \mathbf{D}^H\dot{\boldsymbol{\lambda}} = \mathbf{0}\,. \tag{17}$$

Through the above artifices we may write, in lieu of Eqs. (14) (15) (17):

$$\begin{bmatrix} \mathbf{B} & -\mathbf{B} & \mathbf{N} \\ \mathbf{O} & \mathbf{O} & -\mathbf{D}^H \end{bmatrix} \begin{bmatrix} \dot{\mathbf{f}}^- \\ \dot{\mathbf{f}}^+ \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} + \begin{bmatrix} \mathbf{D}^c & \mathbf{O} \\ \mathbf{O} & \mathbf{D}^H \end{bmatrix} \begin{bmatrix} \dot{\mathbf{Q}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} -\dot{\boldsymbol{\delta}} \\ \mathbf{O} \end{bmatrix} \tag{18}$$

or with more compact symbols, defined by comparison with Eq. (18):

$$\tilde{\mathbf{A}}\mathbf{v} + \mathbf{S}\mathbf{x} = \mathbf{b}\,. \tag{18'}$$

The two remaining basic relationships (6) and (9) may be incorporated in the following one:

$$\begin{bmatrix} \dot{\mathbf{f}}^- \\ \dot{\mathbf{f}}^+ \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} \geqslant \mathbf{0} \quad (19) \qquad [\tilde{\dot{\mathbf{f}}}^- \,|\, \tilde{\dot{\mathbf{f}}}^+ \,|\, \tilde{\dot{\boldsymbol{\lambda}}}] \begin{bmatrix} \mathbf{y}^+ \\ \mathbf{y}^- \\ -\dot{\boldsymbol{\varphi}} \end{bmatrix} = 0 \quad (20)$$

or,

$$\mathbf{v} \geqslant \mathbf{0} \quad (19') \qquad\qquad \tilde{\mathbf{v}}\mathbf{y} = 0\,. \quad (20')$$

In fact, the former, (19), includes (16) and (6); the latter, (20), covers (9) and does not further restrict vector $\dot{\mathbf{f}}$, since both $\mathbf{y}^-$ and $\mathbf{y}^+$ must vanish because of (10) (11).

The set of five relations (12) (13) (18) (19) (20) is fully equivalent to the original set of basic relations (1) to (9).

On the other hand, as matrix

$$\mathbf{S} \equiv \left[\begin{array}{c|c} \mathbf{D}^c & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{D}^H \end{array}\right]$$

is symmetric positive definite, the former set exhibits the typical structure of the Kuhn-Tucker conditions of a quadratic program in the vector $[\dot{\tilde{\mathbf{Q}}}|\dot{\tilde{\boldsymbol{\lambda}}}]$. That is to say, a vector $[\dot{\tilde{\mathbf{Q}}}|\dot{\boldsymbol{\lambda}}]$ belongs to the solution of either above relation sets, and, hence, to the solution of the problem in hand, if and only if, it solves the following quadratic programming problem:

$$\text{minimize} \qquad \Omega \equiv \frac{1}{2}\,\tilde{\mathbf{x}}\mathbf{S}\mathbf{x} - \tilde{\mathbf{b}}\mathbf{x} \tag{21'}$$

$$\text{subject to} \qquad \mathbf{A}\mathbf{x} \leqslant \mathbf{a}\,. \tag{22'}$$

In order to prove the above assertions, let us consider the generalized Lagrange function of this inequality-constrained, convex optimization:

$$\Lambda(\mathbf{x}, \mathbf{w}) \equiv \frac{1}{2}\,\tilde{\mathbf{x}}\mathbf{S}\mathbf{x} - \tilde{\mathbf{b}}\mathbf{x} + \tilde{\mathbf{w}}(\mathbf{A}\mathbf{x} - \mathbf{a}) \tag{23}$$

where $\mathbf{w}$ is the vector of Lagrange multipliers. The Kuhn-Tucker theorem (12) (13) ensures the equivalence between the minimization (21′) (22′) and the following minimax problem:

find vectors $\mathbf{x}^0\mathbf{w}^0$ such that

$$\Lambda(\mathbf{x}^0, \mathbf{w}) \leqslant \Lambda(\mathbf{x}^0, \mathbf{w}^0) \leqslant \Lambda(\mathbf{x}, \mathbf{w}^0) \tag{24}$$

subject to $\mathbf{w} \geqslant \mathbf{0}$.

The necessary and sufficient (local) conditions for the solution $(\mathbf{x}^0, \mathbf{w}^0)$ of problem (24) are [12]:

$$\frac{\partial \Lambda}{\partial \mathbf{x}} = \mathbf{0}; \quad \frac{\partial \Lambda}{\partial \mathbf{w}} \leqslant \mathbf{0}; \quad \mathbf{w} \geqslant \mathbf{0}; \quad \tilde{\mathbf{w}}\,\frac{\partial \Lambda}{\partial \mathbf{w}} = 0\,. \tag{25}$$

After performing the derivatives, we note that, by identifying $\mathbf{w} \equiv \mathbf{v}$, these conditions coincide precisely with relations (12) (13) (18) (19) (20).

In the mechanical interpretation of this result, let us revert to the previous less compact but more transparent symbols and carry out the matrix multiplications. Thus we come to the following form:

$$\text{minimize} \qquad \Omega \equiv \frac{1}{2}\,\dot{\tilde{\mathbf{Q}}}\mathbf{D}^c\dot{\mathbf{Q}} + \frac{1}{2}\,\dot{\tilde{\boldsymbol{\lambda}}}\mathbf{D}^H\dot{\boldsymbol{\lambda}} + \dot{\tilde{\mathbf{Q}}}\dot{\delta} \tag{21}$$

subject to $\tilde{\mathbf{B}}\dot{\mathbf{Q}} = \dot{\mathbf{F}}$ (22*a*), $\qquad \tilde{\mathbf{N}}\dot{\mathbf{Q}} - \mathbf{D}^H\dot{\boldsymbol{\lambda}} \leqslant \mathbf{0}$ (22*b*)

or, with a customary brief notation

$$\min\,\{\Omega(\dot{\mathbf{Q}}, \dot{\boldsymbol{\lambda}}) \mid \tilde{\mathbf{B}}\dot{\mathbf{Q}} = \dot{\mathbf{F}}; \qquad \tilde{\mathbf{N}}\dot{\mathbf{Q}} - \mathbf{D}^H\dot{\boldsymbol{\lambda}} \leqslant \mathbf{0}\}. \tag{26}$$

To within an immaterial additive constant, we may also take as energy function the expression:

$$\Phi \equiv \frac{1}{2}\,(\dot{\tilde{\mathbf{e}}} + \dot{\tilde{\boldsymbol{\delta}}})\mathbf{D}^S(\dot{\mathbf{e}} + \dot{\delta}) + \frac{1}{2}\,\dot{\tilde{\boldsymbol{\lambda}}}\mathbf{D}^H\dot{\boldsymbol{\lambda}}\,. \tag{27}$$

3.3. *A quadratic program in the displacement and plastic multiplier rates; the relevant maximum principle.*

We shall now dualize the quadratic program (21′) (22′) according to a well-known procedure of programming theory, see [13] p. 106. Referring to Lagrange function (23) with $\mathbf{w} = \mathbf{v}$ it can be proved [11][12] that the following problem is dual to (21′) (22′) in the sense clarified later:

$$\begin{array}{lll} \text{maximize} & \Lambda(\mathbf{x}, \mathbf{v}) & \\ \text{subject to:} & \mathbf{v} \geqslant \mathbf{0} & \\ \text{and to} & \dfrac{\partial \Lambda}{\partial \mathbf{x}} = \mathbf{S}\mathbf{x} - \mathbf{b} + \tilde{\mathbf{A}}\mathbf{v} = \mathbf{0}\,. & \end{array} \tag{28}$$

Since matrix $\mathbf{S}$ is not singular, we can make use of the equality constraints (28) and eliminate vector $\mathbf{x}$ from the above problem, which thus becomes (formulated as a minimum problem):

$$\min\left\{\frac{1}{2}\,(\tilde{\mathbf{v}}\mathbf{A} - \tilde{\mathbf{b}})\mathbf{S}^{-1}(\tilde{\mathbf{A}}\mathbf{v} - \mathbf{b}) + \tilde{\mathbf{v}}\mathbf{a} \mid \mathbf{v} \geqslant \mathbf{0}\right\}. \tag{29}$$

Taking account of Eqs. (18′) and (4), we may write

$$\tilde{\mathbf{A}}\mathbf{v} - \mathbf{b} = -\left[\begin{array}{c} \dot{\mathbf{e}} \\ \hline \mathbf{D}^H\dot{\boldsymbol{\lambda}} \end{array}\right] \tag{30}$$

which implies the compatibility requirement (14). Hence, the objective function of (27) becomes:

$$\Xi(\dot{\mathbf{f}}, \dot{\boldsymbol{\lambda}}) \equiv \frac{1}{2}\,[\dot{\tilde{\mathbf{e}}} \mid \dot{\tilde{\boldsymbol{\lambda}}}\mathbf{D}^H]\left[\begin{array}{c|c} \mathbf{D}^S & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{D}^H \end{array}\right]^{-1}\left[\begin{array}{c} \dot{\mathbf{e}} \\ \hline \mathbf{D}^H\dot{\boldsymbol{\lambda}} \end{array}\right] + \\ + [\tilde{\mathbf{F}} \mid -\tilde{\mathbf{F}} \mid \tilde{\mathbf{O}}]\left[\begin{array}{c} \dot{\mathbf{f}}^- \\ \hline \dot{\mathbf{f}}^+ \\ \hline \dot{\boldsymbol{\lambda}} \end{array}\right]$$

and, after the matrix multiplications, problem (29) can be reformulated as:

$$\min\left\{\frac{1}{2}\,\dot{\tilde{\mathbf{e}}}\mathbf{D}^S\dot{\mathbf{e}} + \frac{1}{2}\,\dot{\tilde{\boldsymbol{\lambda}}}\mathbf{D}^H\dot{\boldsymbol{\lambda}} - \tilde{\mathbf{F}}\dot{\mathbf{f}} \mid \dot{\mathbf{e}} = \mathbf{B}\dot{\mathbf{f}} - \mathbf{N}\dot{\boldsymbol{\lambda}} - \dot{\boldsymbol{\delta}}\,; \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}\right\}. \tag{31}$$

From one of the duality theorems of mathematical programming [12] it follows that, for the solution (indicated by ^{0}):

$$\Omega(\dot{\mathbf{Q}}^0, \dot{\boldsymbol{\lambda}}^0) + \Xi(\dot{\mathbf{f}}^0, \dot{\boldsymbol{\lambda}}^0) = 0\,. \tag{32}$$

3.4. *Specialization to nonhardening behavior.*

In ideal elastoplasticity the plastic potentials are assumed to be independent of the yielding history: therefore the plastic multiplier rates do not influence the plastic potential rates. In this, nonhardening, case the incremental problem is still governed by the set of relations (1) to (9), Sec. 3.1, provided that all strainhardening coefficients H_{ij} are taken as zero. Thus, matrix $\mathbf{D}^H$ becomes the zero matrix, $\mathbf{D}^H = \mathbf{0}$; hence vector $\dot{\lambda}$ disappears from Eq. (7), and, as a consequence, from the programming problem (32), which reduces to:

$$\min \left\{ \frac{1}{2} \tilde{\dot{\mathbf{Q}}} \mathbf{D}^c \dot{\mathbf{Q}} + \tilde{\dot{\mathbf{Q}}} \dot{\delta} \mid \tilde{\mathbf{B}} \dot{\mathbf{Q}} = \dot{\mathbf{F}} ; \tilde{\mathbf{N}} \dot{\mathbf{Q}} \leqslant 0 \right\} . \quad (33)$$

In fact, we may follow the same path of reasoning as in Sec. 3.2 and give to the governing relations the form (12') (13') (18') (19') (20'); where, however, the following simplifications hold:

$$\tilde{\mathbf{A}} \equiv [\mathbf{B} \mid -\mathbf{B} \mid \mathbf{N}] ; \qquad \mathbf{x} \equiv \dot{\mathbf{Q}} ;$$
$$\mathbf{S} \equiv \mathbf{D}^c ; \qquad \mathbf{b} \equiv -\dot{\boldsymbol{\delta}} . \quad (34)$$

Since matrix $\mathbf{S}$ is still positive definite, the equivalence to a quadratic program, namely to (33), readily follows as before. Moreover, taking account of the new meaning (34) of the symbols, the same dualization process as in Sec. 3.3 leads to the quadratic program:

$$\min \left\{ \frac{1}{2} \tilde{\dot{\mathbf{e}}} \mathbf{D}^S \dot{\mathbf{e}} - \tilde{\dot{\mathbf{F}}} \dot{\mathbf{f}} \mid \dot{\mathbf{e}} = \mathbf{B} \dot{\mathbf{f}} - \mathbf{N} \dot{\boldsymbol{\lambda}} - \dot{\boldsymbol{\delta}} ; \dot{\boldsymbol{\lambda}} \geqslant 0 \right\} . \quad (35)$$

This might be obtained also from (31), simply by assuming $\mathbf{D}^H = \mathbf{0}$.

3.5. *Existence and uniqueness of solution.*

3.5.1. *Strainhardening behavior.*

Quadratic programming theory, [11], [12], ensures that: (α) the boundedness from below of the function to be minimized (objective function) ensures the existence of a (bounded) solution provided that the set of vectors fulfilling the constraints is not empty; (β) the dual problem has a solution, if and only if the primal has a solution; (γ) strict convexity of the objective function guarantees uniqueness of solution (if there is one).

The objective of (26), if it is written in form(27), appears immediately to be bounded from below; the constraints of (26) can always be satisfied by some vectors $\dot{\mathbf{Q}}$, $\dot{\boldsymbol{\lambda}}$ (in fact any solution $\dot{\mathbf{Q}}$ of the equality constraints, can comply with the inequalities by a suitable choice of $\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}$).

Therefore, by virtue of theorems (α) and (β), both problems are always solvable, as physically evident. In problem (26) the function to be minimized is strictly convex, hence through theorem (γ) the stress rate response is unique.

Let $\mathbf{v}^0$ be a solution of problem (29): the totality of solutions can be proved in general [12] to be the set of vectors defined as follows:

$$\mathbf{v} = \mathbf{v}^0 + \mathbf{v}' \geqslant \mathbf{0} \quad (36a)$$

$$\mathbf{A} \mathbf{S}^{-1} \tilde{\mathbf{A}} \mathbf{v}' = \mathbf{0} \quad (36b)$$

$$(\tilde{\mathbf{b}} \mathbf{S}^{-1} \tilde{\mathbf{A}} - \tilde{\mathbf{a}}) \mathbf{v}' = 0 . \quad (36c)$$

Matrix $\mathbf{S}^{-1}$ is positive definite, $\mathbf{A}\mathbf{S}^{-1}\tilde{\mathbf{A}}$ positive semidefinite. Therefore both Eq. (36*b*) and

$$\tilde{\mathbf{A}} \mathbf{v} = \mathbf{0} \quad (37)$$

are necessary and sufficient conditions for

$$\tilde{\mathbf{v}}' \mathbf{A} \mathbf{S}^{-1} \tilde{\mathbf{A}} \mathbf{v}' = 0$$

therefore (36*b*) holds if and only if (37) holds.

In the hardening case Eq. (37) means (see Eqs. (18) (18')):

$$-\mathbf{B} \dot{\mathbf{f}}' + \mathbf{N} \dot{\boldsymbol{\lambda}}' = \mathbf{0} \quad (38a)$$

$$\mathbf{D}^H \dot{\boldsymbol{\lambda}}' = \mathbf{0} . \quad (38b)$$

It follows from (38*b*) that $\dot{\boldsymbol{\lambda}}' = \mathbf{0}$, whence, since matrix $\mathbf{B}$ has full column-rank, $\dot{\mathbf{f}}' = \mathbf{0}$.

Eq. (36*c*) remains satisfied as a consequence, because of Eq. (37) and since [see Eqs. (12) (12') and (18) (18')]:

$$\tilde{\mathbf{a}} \mathbf{v}' = \tilde{\dot{\mathbf{F}}} \dot{\mathbf{f}}' . \quad (39)$$

Thus we have come back to the known conclusion that in this case the structural response in terms of $\dot{\mathbf{f}}$ and $\dot{\boldsymbol{\lambda}}$ is always unique.

3.5.2. *Nonhardening behavior.*

By virtue of theorem (γ), the minimization problem (33) admits unique solution $\dot{\mathbf{Q}}^0$, if any. However its constraints need not be compatible, i. e. the problem may not be solvable: this means that situation Y represents a plastic collapse state.

In this case also problem (35) will not be solvable, by theorem (β). The entire set of solutions to the latter problem can be described by means of relations (36). Eq. (36*b*) implies again Eq. (37), which now simply reads:

$$\mathbf{N} \dot{\boldsymbol{\lambda}}' = \mathbf{B} \dot{\mathbf{f}}' . \quad (40)$$

Eq. (36*c*), through Eqs. (37) (39), becomes

$$\tilde{\dot{\mathbf{F}}} \dot{\mathbf{f}}' = 0 . \quad (41)$$

Eqs. (40) and (41) show that, whenever more than one solution exists, they differ by a compatible stressless plastic strain rate set and by displacement rate vector orthogonal to the given load increment vector.

If Eqs. (36*b*) (36*c*) can be satisfied with $\mathbf{v}' \geqslant \mathbf{0}$, i. e. with $\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}$, then the set of solutions turns out to be either empty (when there is no $\mathbf{v}^0$) or unbounded: the former case implies that the objective is unbounded from below over the feasible domain; both cases mean that the structure is at plastic collapse in the starting situation Y.

4. Concluding remarks.

4.1. *Statements in mechanical terms.*

It seems worthwhile to formulate the minimum properties established analytically in the preceding sections by statements in mechanical terms.

The former quadratic optimization problem, Sec. 2.2 and 3.2 admits the following interpretation:

Theorem I: Within the class of all stress and plastic multiplier rates which are statically and plastic potential admissible (i.e. which obey equilibrium and make the plastic potentials non positive), the actual one is characterized by the fact that it minimizes the sum of the second order elastic stress energy, dissipated work and work connected with the internal and external dislocations.

Correspondingly the latter optimization, Sec. 2.3 and 3.3, can be expressed by the statement:

Theorem II: Within the class of all velocities and plastic multiplier rates which are kinematically and direction admissible (i. e. which comply with compatibility and outward normal rule), the actual one minimizes the sum of the second order elastic strain energy and dissipated work and the opposite of the (second order) work performed by the body and surface forces.

Clearly the above mechanical interpretations of the objective functionals are rigorously valid only for the solution. The specialization of the above statements to nonhardening behavior (see Sec. 2.3, 3.4) is quite obvious.

4.2. *Comparison with the classical rate principles.*

It might be interesting to deduce the classical minimum principles of flow-law plasticity from the theorems discussed in this paper. For this purpose note simply that full conformity is guaranteed by any solution, although not all of the flow-rules are imposed through the constraints.

In the former optimization problem (subsecs. 2.2 and 3.2) the constraints involve only plastic potential admissibility, see inequalities (11) Sec. 2, (8) Sec. 3. Since the remaining rules, (12) (13) Sec. 2 or (6) (9) Sec. 3, are certainly fulfilled in the solution, we may introduce these rules as additional constraints in the optimization without altering the results. For the so restricted feasible classes, we may write in tensor and matrix notation, respectively:

$$\textstyle\sum_\alpha h_\alpha \dot{\lambda}_\alpha^2 = \dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^p \qquad (1)$$

$$\tilde{\dot{\boldsymbol{\lambda}}} \mathbf{D}^H \dot{\boldsymbol{\lambda}} = \tilde{\dot{\mathbf{Q}}} \dot{\mathbf{p}} \, . \qquad (1')$$

Taking into account these relations, and Eqs. (1) (2) Sec. 2 and Eqs. (3) (4) Sec. 3, the former optimization problem can be transformed into the following one:

$$\min \left\{ \frac{1}{2} \int_V \dot{\sigma}_{ij}(\dot{\varepsilon}_{ij} + \dot{\varepsilon}_{ij}^d) dV - \int_{S_u} \dot{\sigma}_{ij} n_i \dot{\bar{u}}_j dS \;\middle|\; \dot{\sigma}_{ij/i} + \dot{X}_j = 0 \text{ in } V; \quad \dot{\sigma}_{ij} n_i = \dot{T}_j \text{ on } S_T \right\} \qquad (2)$$

$$\min \left\{ \frac{1}{2} \tilde{\dot{\mathbf{Q}}}(\dot{\mathbf{q}} + \dot{\boldsymbol{\delta}}) \;\middle|\; \tilde{\mathbf{B}}\dot{\mathbf{Q}} = \dot{\mathbf{F}} \right\} . \qquad (2')$$

It is essential to note that strain rates are understood to correspond to any trial stress rates through the constitutive laws.

The above non quadratic minimization problems express the Prager-Hodge theorem [1] [2] [3] generalized to imposed internal dislocations.

Correspondingly, in the optimization problem formulated herein in subsecs. 2.3 and 3.3, the constraints do not involve the flow-rules (11) (13) Sec. 2, (8) (9) Sec. 3. However these are certainly obeyed in the solution, so that they can be added without influencing the results, and once again reference can be made to a restricted class where Eqs. (1) (1′) hold. Obvious substitutions, as before, lead the the following non quadratic optimization problems:

$$\min \left\{ \frac{1}{2} \int_V \dot{\sigma}_{ij}(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^d) dV - \int_V \dot{X}_i \dot{u}_i dV - \int_{S_T} \dot{\bar{T}}_i \dot{u}_i dS \;\middle|\; \dot{\varepsilon}_{ij} = \frac{1}{2}(\dot{u}_{i/j} + \dot{u}_{j/i}) \text{ in } V; \; \dot{u}_i = \dot{\bar{u}}_i \text{ on } S_u \right\} \qquad (3)$$

$$\min \left\{ \frac{1}{2} \tilde{\dot{\mathbf{Q}}}(\dot{\mathbf{q}} - \dot{\boldsymbol{\delta}}) - \tilde{\dot{\mathbf{F}}}\dot{\mathbf{f}} \;\middle|\; \dot{\mathbf{q}} = \mathbf{B}\dot{\mathbf{f}} \right\} . \qquad (3')$$

Note that the stress rates in (3) (3′) are meant to be related to the strain rates through the constitutive laws.

The preceding minimization problems generalize the Greenberg theorem [1] [2] [3] to cases where internal dislocations are prescribed.

It is worth noting that the above generalized classical theorems implicitly involve full conformity; hence, if they are transformed, using the constitutive laws and following in the reverse direction the above path of reasoning, they give rise to the quadratic optimization problems discussed in this paper, but with all constraints which reflect full conformity. However the sets of these constraints can be proved to be redundant; this would be an alternative demonstration, founded on the classical rate theorems, of the conclusions reached in the paper.

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