A MATRIX STRUCTURAL THEORY OF PIECEWISE LINEAR ELASTOPLASTICITY WITH INTERACTING YIELD PLANES*

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SOMMARIO: Si formulano, con notazione matriciale, generali leggi costitutive linearizzate a tratti e dotate di normalità, e se ne discutono alcune proprietà e la specializzazione ad usuali tipi di incrudimento (in particolare cinematico ed isotropo). Riferendosi a modelli strutturali per elementi finiti, si ottengono i risultati seguenti:

a) *la soluzione olonoma per dati carichi e distorsioni viene caratterizzata da sei proprietà estremali di natura "quadratico-lineare", due di validità generale, quattro di validità condizionata*; b) *corrispondenti teoremi in ambito differenziale vengono proposti per analogia; si danno degli enunciati di confronto tra soluzioni olonome ed anolonome*; c) *si fornisce un teorema sull'assestamento in campo elastico sotto azioni esterne variabili in presenza di forze d'inerzia e resistenze viscose, generalizzando alle strutture incrudenti un teorema di Ceradini, e per specializzazione ai casi quasi-statici quello di Melan*; d) *si propone un metodo per valutare in assenza di scarichi locali, o delimitare superiormente il coefficiente di sicurezza nei confronti di rotture locali dovute a limitata deformabilità plastica.*

SUMMARY: General piecewise linear constitutive laws with associated flow rules are formulated in matrix notation; some properties and specializations (in particular to kinematic and isotropic hardening) are discussed.

With reference to finite element models of structures and, hence, in matrix-vector description, the following results are achieved:

a) *the holonomic solutions to the analysis problem for given loads and dislocations are shown to be characterized by means of six "quadratic-linear" minimum principles, two of general, four of conditioned validity*; b) *the incremental counterparts of the above theorems are indicated by analogy; some comparison properties concerning holonomic and nonholonomic solutions, are pointed out*; c) *a shakedown theorem is established for variable repeated loads and dislocations, with allowance for inertia forces and viscous damping, i. e. a generalization to workhardening structures of Ceradini's and (in quasi-static situations) Melan's theorems;* d) *a method is proposed for evaluating under holonomy hypothesis, or bounding from above, the safety factor with respect to local failure due to limited plastic strain capacity.*

1. Introduction.

A crucial topic of plasticity theory is the choice of the description of the material behavior. A good compromise between generality and fidelity to complex experimental data on one hand, and the simplicity needed for developing useful analysis methods on the other, is an obvious but illdefined and problem-dependent criterion. In fact numerous stress-strain relations have been proposed (surveyed e. g. in [1] [2] [3]) and are still being proposed (cf. e. g. [4]).

For incremental (in rates) problems, the so-called "linear" associated or nonassociated flow-laws, with Koiter-Sanders hypothesis of non-interacting yield modes in singular points, has been regarded by many Authors as a satisfactory basis for a useful and sufficiently general theory. Nevertheless, interesting and meaningful generalizations were proposed, e. g. involving interacting modes [5] [6] [7] and "nonlinearity" [8].

For finite (in total quantities) problems, outside the domain of the perfect plasticity assumption, the picture is less clear and the choice of the hardening rule is the central, controversial point. Piecewise linearization (nonrotating yield planes, linear hardening) has been successfully developed and widely used, particularly by Hodge and his coworkers (see Ref. [9], which surveys previous work). Piecewise linear laws with *non-interacting* yield planes and normality, have led to a "deformation theory" which reduces, in four different ways, the finite boundary value problem to the minimization of a quadratic functional under linear equations and inequalities (briefly "quadratic-linear" minimum principles) [10]. Obviously, such an assumption cannot allow for certain important phenomenological features (e. g. Bauschinger effect). These limitations however are not so serious, as long as radial or quasi radial loading paths are considered, which rule out large regressions of the stress points in the stress-spaces, and often actually give rise to "regularly progressive" yielding. It is not so for shakedown problems in the presence of workhardening: in fact the non-interaction assumption even makes these problems trivial, since, clearly, any system complying with it shakes down under *all* loading programmes. Therefore the interaction among yielding regimes plays an essential role in the theory of workhardening structures.

In this paper, first of all (Sec. 2) the general piecewise linear associated constitutive laws with translating-interacting yield-planes, are given a suitable matrix formulation and some of their basic properties are pointed out. The descriptions of the frequently assumed kinematic and isotropic hardening behaviours are discussed as particular cases. In Sec. 3 the finite boundary value problem in terms of the deformation theory, is discretizised by referring to a finite element model of the system. On this

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basis two pairs of previously known finite minimum theorems [10] are generalized preserving their computationally attractive quadratic-linear nature. Two further quadratic-linear extremum principles are established, which are valid even when lack of symmetry in the interactions invalidates the others.

The analogy between finite holonomic problems and rate problems with general "linear" flow-laws is pointed out, and used for indicating seemingly new extremum properties in the framework of Mandel's generalization [6] of the classical incremental theory. Some results obtained in Sec. 3.4 allow to compare the path-dependent, nonholonomic solution to the holonomic one for the same given external actions.

The static shakedown theorem (Melan's) has been proved by Neal for beams and frames, the moment-curvature law of which exhibit workhardening together with symmetry with respect to the origin of the axes and with a constant elastic range [11] [12]. No further generalization seems to have been carried out sofar, to the author's knowledge, nor does Neal's approach seem to supply a suitable basis for extensions. In ideal elastoplasticity G. Ceradini established recently a dynamic shakedown theorem [13], which reduces to Melan's theorem when inertia forces are negligeable. This result is extended here (Sec. 4) to systems which obey fairly general hardening rules. The present approach differs from Ceradini's, among others, inasmuch it takes the plastic multipliers as principal variables.

By specializing the treatment to quasi-static situations, the basis is obtained for a shakedown theory of hardening structures parallel to that previously expounded in this Journal [14] for perfectly plastic structures (however lack of normality is not dealt with here). The further specialization to a one-parameter load family, leads, in Sec. 5, to a formulation of the limit-analysis problem, which lends itself to allow, under certain restrictions, for limited plastic strain capacity. A method for evaluating under progressive yielding hypothesis, or for bounding from above the safety factor against local failure is proposed. This question is of tecnical importance e. g. in reinforced concrete frames (see Ref. [15] [16] and the abundant literature surveyed in them). For the analysis and design of frames, methods allowing for limited plastic rotation capacity are already in use (A. L. L. Baker's, Macchi's methods). These also are tacitly founded on the progressive yielding (holonomy) hypothesis, but, in contrast to the present one, involve trial-and-error procedures [15] [16]. The method proposed herein requires further research (to be developed elsewhere), particularly from the computational standpoint

2. Piecewise-linear constitutive laws with interacting. yield planes.

2.1. *Analytical formulation.*

In the discrete idealization of the system, element deformation patterns will be prescribed (as in [14]) implying homogeneous strain and, hence, stress fields in each finite element. Therefore the material behaviour is directly reflected by the element behaviour, so that it will be sufficient to describe the latter in the superposed spaces of the "natural" generalized strain m-vector $\mathbf{q}^i$ and the "natural" generalized stress m-vectors $\mathbf{Q}^i$ of the generic element i (for a detailed discussion of these concepts see e. g. [17] [18]; the basis of this approach is clarified in [14]).

The following terminology will be used (¹):

$\mathbf{e}^i$, $\mathbf{p}^i$, $\boldsymbol{\delta}^i$:	elastic, plastic and dislocation (thermal) addends of $\mathbf{q}^i$, respectively.
$\mathbf{S}^i$, $\mathbf{C}^i$,:	element elastic stiffness and, respectively, compliance $m \times m$-matrices (symmetric, positive definite, constant)
$\boldsymbol{\varphi}^i$, $\boldsymbol{\lambda}^i$:	vectors of the y plastic potentials and corresponding plastic multipliers.
$\mathbf{K}^i$:	a positive, constant y-vector.
$\mathbf{N}^i \equiv [\mathbf{N}_1^i \dots \mathbf{N}_y^i]$:	$m \times y$-matrix whose columns are fixed unit vectors in the $\mathbf{Q}^i$-space.
$\mathbf{H}^i$:	$y \times y$-"workhardening matrix".

For the sake of brevity we shall omit the element superscript index i throughout the remainder of Sec. 2.

Consider the relations:

$$\dot{\mathbf{Q}} = \mathbf{S}\dot{\mathbf{e}} \tag{2.1}$$

$$\dot{\mathbf{p}} = \mathbf{N}\dot{\boldsymbol{\lambda}} \tag{2.2}$$

$$\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0} \tag{2.3}$$

$$\boldsymbol{\varphi} = \tilde{\mathbf{N}}\mathbf{Q} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{K} \tag{2.4}$$

$$\boldsymbol{\varphi} \leqslant \mathbf{0} \tag{2.5}$$

$$\dot{\boldsymbol{\varphi}} = \tilde{\mathbf{N}}\dot{\mathbf{Q}} - \mathbf{H}\dot{\boldsymbol{\lambda}} \tag{2.6}$$

$$\dot{\boldsymbol{\varphi}} \leqslant \mathbf{0} \tag{2.7}$$

$$\dot{\tilde{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} = 0 \tag{2.8}$$

$$\tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = 0\ . \tag{2.9}$$

Relations (2.4) (2.5) define in the $\mathbf{Q}$-space the elastic polyhedron where $\dot{\mathbf{q}} = \dot{\mathbf{e}}$ according to (2.1). The set of relations (2.2) through (2.9) forms a complete description of the plastic flow rules: (2.2) (2.3) express the outward normality of the plastic strain vector; (2.9) together with (2.3) (2.4) (2.5) rules out the activation of any yield plane which does not contain the current stress point $\mathbf{Q}$; (2.2) (2.6) guarantee linearity (clearly, in a limited sense) to the relation between $\dot{\mathbf{p}}$ and $\dot{\mathbf{Q}}$; (2.8) with (2.3) (2.7) expresses Prager's consistency principle of mutually exclusive yielding and unloading.

Thus the matrix description introduced in [19] acquires

(¹) Matrices and column-vectors are indicated by bold-face letters and sometimes by square brackets enclosing the entries; a tilde means transpose, a dot derivative with respect to time. $\mathbf{0}$ is a matrix or vector whose entries are all zero; vector inequalities concern each pair of corresponding components.

the remarkable novelty that matrix $\mathbf{H}$ is no longer assumed as diagonal: if its element H_{rs} with $r \neq s$ is positive, the activation of the s-th yield plane ($\dot{\lambda}_s > 0$) makes the r-th yield plane translate outward, if $H_{rs} < 0$ inward; therefore the off-diagonal entries of $\mathbf{H}$ might be called "secondary hardening" constants or interaction parameters.

For $\mathbf{H} \equiv \operatorname{diag}[H_j]$ and $\mathbf{H} = \mathbf{0}$, we return to the familiar cases of noninteracting and, respectively, fixed (nonhardening) yield planes. The set of (2.4) (2.5) associated with the further relations

$$\mathbf{p} = \mathbf{N}\boldsymbol{\lambda} \tag{2.10}$$

$$\boldsymbol{\lambda} \geqslant \mathbf{0} \tag{2.11}$$

represents the (indeterminate) link between the "finite" or total variables $\mathbf{p}$ and $\mathbf{Q}$; this can be conceived as obtained by a path-dependent integration of the above flow-laws. With the addition of

$$\tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0 \tag{2.12}$$

the finite relation set becomes a holonomic, reversible $\mathbf{p}(\mathbf{Q})$ law [19]. It is worth stressing that anisotropy, both initial and consequent to yielding (see e. g. [9]), is allowed for by the above formulated laws.

2.2. *Some fundamental properties.*

On the basis of the above remarks the *holonomic* relation between $\mathbf{q}$ (with $\boldsymbol{\delta} = \mathbf{0}$) and $\mathbf{Q}$ can be written as the set of a linear equation

$$\mathbf{q} = \mathbf{C}\mathbf{Q} + \mathbf{N}\boldsymbol{\lambda} \tag{2.13}$$

and of a "linear complementarity problem" [20] in the vectors $\boldsymbol{\varphi}$, $\boldsymbol{\lambda}$

$$\left.\begin{array}{l} -\boldsymbol{\varphi} = \mathbf{H}\boldsymbol{\lambda} + (\mathbf{K} - \tilde{\mathbf{N}}\mathbf{Q}) \\ -\boldsymbol{\varphi} \geqslant \mathbf{0}\,, \quad \boldsymbol{\lambda} \geqslant \mathbf{0}\,, \quad \tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0 \end{array}\right\} . \tag{2.14}$$

The same mathematical structure has been found in [7] for the general "linear" flow-laws of incremental (nonholonomic) plasticity; this remark has led to several statements, which might be immediately transferred from [7] to the present cases by analogy. However only the following minimum properties will be stated here:

(I) If $\mathbf{H}$ is symmetric positive semidefinite, the plastic strains which correspond through the holonomic law to given stresses, are defined by plastic multipliers which minimize the function

$$\omega_1(\boldsymbol{\lambda}) \equiv \frac{1}{2}\,\tilde{\boldsymbol{\lambda}}\mathbf{H}\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}(\tilde{\mathbf{N}}\mathbf{Q} - \mathbf{K}) \tag{2.15}$$

subject to
$$\boldsymbol{\lambda} \geqslant \mathbf{0} \tag{2.16}$$

(II) For generic $\mathbf{H}$, the plastic strains corresponding, through holonomic laws, to given stresses, minimize the function

$$\omega(\boldsymbol{\lambda}) \equiv \tilde{\boldsymbol{\lambda}}\mathbf{H}\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}(\tilde{\mathbf{N}}\mathbf{Q} - \mathbf{K}) \tag{2.17}$$

subject to

$$\boldsymbol{\lambda} \geqslant \mathbf{0}\,, \qquad \mathbf{H}\boldsymbol{\lambda} - \tilde{\mathbf{N}}\mathbf{Q} + \mathbf{K} \geqslant \mathbf{0}\,. \tag{2.18}$$

The clear analogy between on the one hand the relation set (2.14), theorems (I) and (II) and on the other hand the set (3.7) (3.12), theorems (VI) and (VIII), respectively, allows us to give formal proofs of only the latter and more important statements (see Sec. 3). Similarly, another extremum characterization (I′) [dual to (I)] of $\mathbf{p}$ and, hence, of $\mathbf{q}$ for a given $\mathbf{Q}$ can be obtained by analogy from theorem (VII). Moreover the *inverse* law $\mathbf{Q} = \mathbf{Q}(\mathbf{q})$ exhibits again the same mathematical nature as (2.13) (2.14) (2); therefore extremum properties similar to (I) (II) (I′) can be stated for the inverse holonomic laws too (clearly the conditions will concern $\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{N}$ instead of $\mathbf{H}$).

As pointed out in [7] the stress-strain relations can be thoroughly discussed by using some results from operations research concerning problem (2.14). A fairly interesting property, among others, is the following one:

(III) When $\mathbf{H}$ is positive semidefinite (not necessarily symmetric), then for a given stress vector, if there exists any strain vector which complies with the corresponding non holonomic laws (one always exists if $\mathbf{H}$ is definite), there must be at least one strain vector which complies with the corresponding holonomic laws. The statement remains true if the words stress and strain are exchanged and the inverse laws are considered.

Proof. — According to a theorem due to Cottle [21], when $\mathbf{H}$ is as stated, there exists a solution $\boldsymbol{\varphi}$, $\boldsymbol{\lambda}$ to the complementarity problem (2.14) if its linear relations are consistent and these are always so when $\mathbf{H}$ is definite. As observed in Sec. 2.1, holonomic laws (in finite terms) expressed by (2.14) are more restrictive than the corresponding nonholonomic laws, precisely because of the nonlinear orthogonality requirements $\tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0$. So the statement is proved. Its extension to the inverse relation flows from the fact that matrix $\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{N}$, which plays the same role as $\mathbf{H}$ in the direct law, is certainly positive semidefinite if $\mathbf{H}$ is so, since $\mathbf{S}$ is positive definite.

2.3. *Special cases.*

A translatory motion without distorsion of the elastic range in the $\mathbf{Q}$-space by yielding (*kinematic hardening*, fig. 1-a) is a simple assumption, in fairly good agreement with several experimental results, see e. g. [22] [23]. The general piecewise linear stress-strain laws formulated in Sec. 2.1 cover this case, provided that matrix $\mathbf{H}$ (and only matrix $\mathbf{H}$) acquires the special features determined below. The elastic range, after the plastic strains defined

(2) In fact, by solving (2.13) with respect to $\mathbf{Q}$ and substituting in the first (2.14):

$$\mathbf{Q} = \mathbf{S}\mathbf{q} - \mathbf{S}\mathbf{N}\boldsymbol{\lambda} \tag{2.13′}$$

$$\begin{array}{l} -\boldsymbol{\varphi} = (\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{N})\boldsymbol{\lambda} + (\mathbf{K} - \tilde{\mathbf{N}}\mathbf{S}\mathbf{q}) \\ -\boldsymbol{\varphi} \geqslant \mathbf{0}\,, \quad \boldsymbol{\lambda} \geqslant \mathbf{0}\,, \quad \tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0\,. \end{array} \tag{2.14′}$$

by $\boldsymbol{\lambda}$ have developed, is represented by the inequality:

$$\tilde{\mathbf{N}}\mathbf{Q} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{K} \leqslant \mathbf{0} \qquad (2.19)$$

in the original state by the inequality:

$$\tilde{\mathbf{N}}\mathbf{Q} - \mathbf{K} \leqslant \mathbf{0} \qquad (2.20)$$

Shape and size of the original polyhedron are preserved for any $\boldsymbol{\lambda}$, if (2.19) can be given the form

$$\tilde{\mathbf{N}}(\mathbf{Q} - \mathbf{Q}_\lambda) - \mathbf{K} \leqslant \mathbf{0}\,. \qquad (2.21)$$

This condition is fulfilled by assuming $\mathbf{Q}_\lambda$ as linear transform of $\boldsymbol{\lambda}$: $\mathbf{Q}_\lambda = \mathbf{L}\boldsymbol{\lambda}$. Then, by comparing (2.19) to (2.21):

$$\mathbf{H} \equiv \tilde{\mathbf{N}}\mathbf{L} \qquad (2.22)$$

As a more particular case, Prager's hardening rule requires a translation velocity of the yield locus proportional to the plastic strain rate (see [2] [3] and fig. 1-b), i. e.: $\dot{\mathbf{Q}}_\lambda = h\dot{\mathbf{p}}$, with $h > 0$. Therefore it can be expressed by assuming $\mathbf{L} = h\mathbf{N}$ in (2.22) and hence:

$$\mathbf{H} \equiv h\tilde{\mathbf{N}}\mathbf{N} = h[\tilde{\mathbf{N}}_r\mathbf{N}_s]\,. \qquad (2.23)$$

In other terms, Prager's kinematic hardening corresponds to a matrix $\mathbf{H}$ which is (to within the factor h) the Gramian matrix of the outward normal unit vectors to the yield planes, and which is hence symmetric, positive semidefinite (definite if and only if vectors $\mathbf{N}_r$ are linearly independent).

In one-component cases ($y = 2$, $\mathbf{N} \equiv [1 \;\; -1]$), using (2.22) with $\mathbf{L} \equiv [h_1, \; -h_2]$, $h_1 > 0$, $h_2 > 0$, we obtain the analytical description of the behaviour illustrated by fig. 2-a (where tg^{-1} is omitted for brevity and dashed lines define the secondary hardening; $K_2 < 0$):

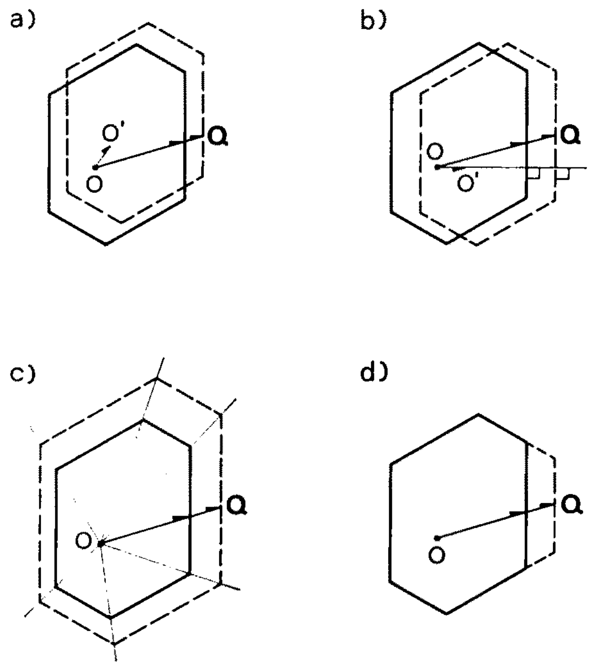

Fig. 1.

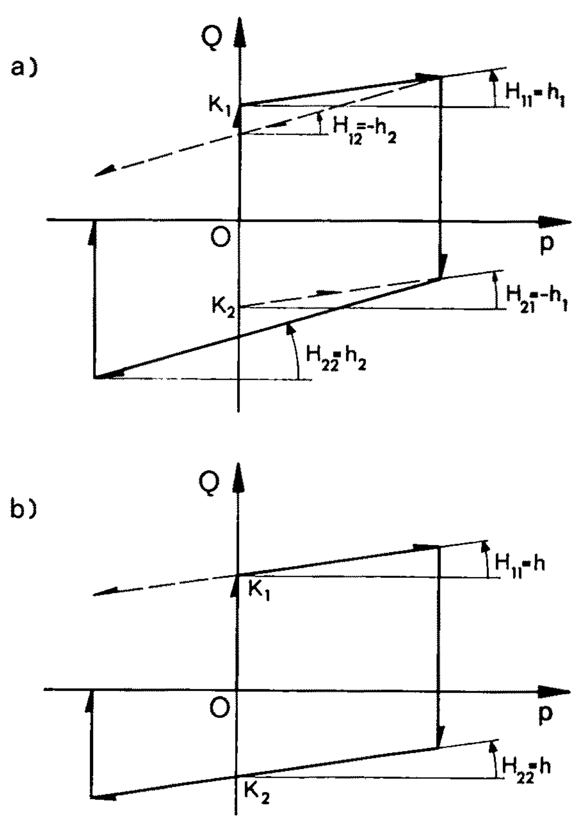

Fig. 2.

$$\left.\begin{aligned} \varphi_1 &= \quad Q - K_1 - h_1\lambda_1 + h_2\lambda_2 \leqslant 0 \\ \varphi_2 &= -Q + K_2 + h_1\lambda_1 - h_2\lambda_2 \leqslant 0 \end{aligned}\right\} \qquad (2.24)$$

$$\dot{\lambda}_r \geqslant 0, \quad \varphi_r\dot{\lambda}_r = 0, \quad \dot{\varphi}_r\dot{\lambda}_r = 0 \quad (r = 1, 2); \; p = \lambda_1 - \lambda_2\,.$$

The familiar picture of fig. 2-b corresponds to the assumption (2.23), i. e. to $h_1 = h_2 = h > 0$.

Although in direct opposition to the concept of Bauschinger effect, the concept of a uniformly expanding yield locus without shape changes (*isotropic hardening*, fig. 1-c) has been frequently accepted and even found to be in general agreement with certain experiments (see e. g. [24]). Inequality (2.19) defines an elastic range homothetic (with respect to the origin of the axes) to that defined by (2.20), when it can be expressed in the form:

$$\tilde{\mathbf{N}}\mathbf{Q} - \alpha\mathbf{K} \leqslant \mathbf{0} \qquad (2.25)$$

where α is a scalar depending on the yielding history. This condition is fulfilled if $\mathbf{H}$ complies with the requirement:

$$\mathbf{H}\boldsymbol{\lambda} = (\alpha - 1)\mathbf{K} \qquad \text{for } any \;\; \boldsymbol{\lambda} \qquad (2.26)$$

which supplies the following $y(y-1)$ equations between the y^2 entries H_{rs} of $\mathbf{H}$:

$$\frac{H_{rs}}{K_r} = \frac{H_{ss}}{K_s} \qquad (2.27)$$

Therefore the constitutive laws of Sec. 2.1 describe isotropic hardening if

$$\mathbf{H} \equiv \left[\frac{K_r}{K_s} H_{ss}\right] = \mathbf{K} \left[\frac{1}{K_1} \cdots \frac{1}{K_y}\right] \operatorname{diag}\,[H_{ss}] \quad (2.28)$$

where the y direct hardening parameters H_{ss} are the only arbitrary material constants after the original yield locus has been fixed. Assuming equal hardening rate for all regimes, $H_{ss} = h$, the expression (2.28) reduces to:

$$\mathbf{H} \equiv h\mathbf{K} \left[\frac{1}{K_1} \cdots \frac{1}{K_y}\right]. \quad (2.29)$$

Matrices (2.28) (2.29) are not symmetric. For $H_{ss} \geqslant 0$ they are not positive semidefinite in general [(2.29) is so only for all K_r equal]; however, as readily seen, they are both positive and strictly copositive, i. e.

$$\mathbf{H} > \mathbf{0}, \quad \tilde{\boldsymbol{\lambda}}\mathbf{H}\boldsymbol{\lambda} > 0 \quad \text{for any} \quad \boldsymbol{\lambda} \geqslant \mathbf{0} \;\; (\neq \mathbf{0}). \quad (2.30)$$

Fig. 3 illustrates the stress-plastic strain relationship with isotropic hardening in one component, according to assumptions (2.28) and (2.29), in (a) and (b) respectively.

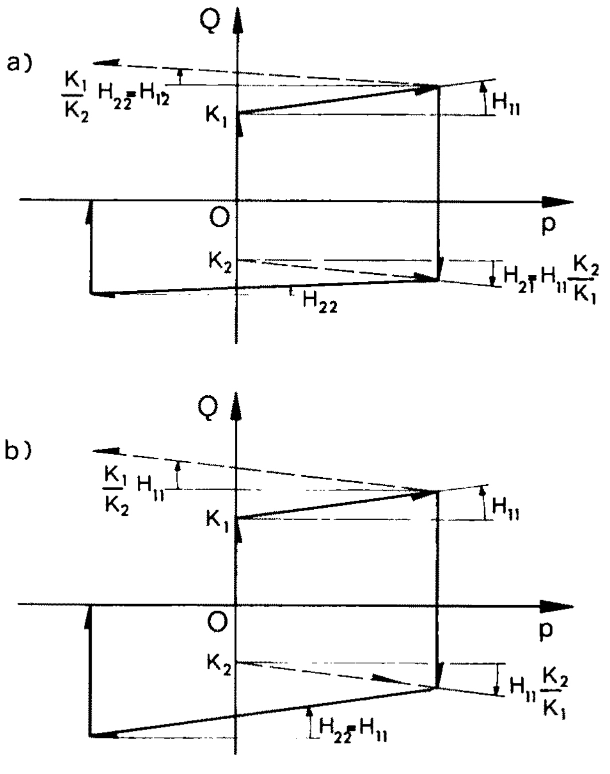

Fig. 3.

A description of expanding yield loci can be obtained by taking all entries of **H** equal to a positive constant h: this might supply only a simple approximation for isotropic hardening ("quasi-isotropic" hardening), because according to such an assumption the elastic range actually changes its shape, unless all K_r coincide.

A diagonal matrix **H**, as already noted, means *independently acting* yield planes (fig. 1-d). Naturally matrix **H** can be adjusted in order to describe many other types of hardening behaviour. Note that strain-softening may be included as well, simply by taking as negative some direct hardening parameters H_{rr}, or the constant h in (2.23) (translation in the inward normal direction) or in (2.29) (uniform homothetic contraction).

3. Finite, quadratic-linear extremum theorems.

3.1. *Basic relations.*

Vectors and matrices without superscript i are understood to assemble as subvectors or, respectively, submatrices in main diagonal location, *all* the analogous vectors and matrices introduced with superscripts i at the beginning of Sec. 2. These are taken in a fixed order $i = 1 \ldots n$, n being the number of the finite elements in which the structure to study has been subdivided. It is essential to keep in mind that, since in the formulae of Sec. 2 the element index i has been dropped to simplify the notation, the same symbols as in Sec. 2 will be used from now on with substantially different meanings: e. g. vectors **q** and **Q** define henceforth the strain and stress state *throughout the structure*: $\tilde{\mathbf{q}} \equiv [\tilde{\mathbf{q}}^1 \ldots \tilde{\mathbf{q}}^n]$. The following additional symbols are introduced:

F: vector of the independent external (nodal) force components;

f: vector of the free nodal displacements;

B: geometrical compatibility matrix.

Suppose the structure be acted upon by a set of loads **F** and dislocations **δ** (proportionally increased, so that deformation theory be practically realiable in many cases). Let vector **δ** include dislocations suitably prescribed in additional stiff elements in order to simulate possible forced nodal displacements. The structural *holonomic* response (**Q**, **q**, **f**) to these straining effects is governed by the following relations:

$$\mathbf{q} = \mathbf{B}\mathbf{f} \quad (3.1)$$

$$\tilde{\mathbf{B}}\mathbf{Q} = \mathbf{F} \quad (3.2)$$

$$\mathbf{q} = \mathbf{p} + \mathbf{e} + \boldsymbol{\delta} \quad (3.3)$$

$$\mathbf{e} = \mathbf{C}\mathbf{Q} \quad (3.4)$$

$$\mathbf{p} = \mathbf{N}\boldsymbol{\lambda} \quad (3.5)$$

$$\boldsymbol{\varphi} = \tilde{\mathbf{N}}\mathbf{Q} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{K} \quad (3.6)$$

$$\boldsymbol{\lambda} \geqslant \mathbf{0} \qquad \tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0 \qquad \boldsymbol{\varphi} \leqslant \mathbf{0}\,. \quad (3.7)$$

Eqs. (3.1) (3.2) express compatibility and equilibrium, respectively; the other relations express the constitutive laws *of all elements simultaneously*, the last equation (3.7) reflects the no-local-unloading or holonomy hypothesis.

By eliminating vectors **q**, **p**, **e**, **Q** from the equation system (3.1) to (3.6), this can be replaced by the pair of equations:

$$\boldsymbol{\varphi} = \tilde{\mathbf{N}}\mathbf{SBf} - (\mathbf{H} + \tilde{\mathbf{N}}\mathbf{SN})\boldsymbol{\lambda} - (\tilde{\mathbf{N}}\mathbf{S}\boldsymbol{\delta} + \mathbf{K}) \qquad (3.8)$$

$$\tilde{\mathbf{B}}\mathbf{SBf} - \tilde{\mathbf{B}}\mathbf{SN}\boldsymbol{\lambda} - \tilde{\mathbf{B}}\mathbf{S}\boldsymbol{\delta} = \mathbf{F} \qquad (3.9)$$

which represent, in association with (3.7), a formulation of the problem in the unknowns $\mathbf{f}$, $\boldsymbol{\lambda}$, $\boldsymbol{\varphi}$ only.

Let $\mathbf{Z}$ indicate the (symmetric, negative semidefinite) transformation matrix from dislocations to their corresponding selfstresses in elastic conditions; let superscripts E and P mark the elastic response to external actions $\mathbf{F}$ and $\boldsymbol{\delta}$, and to plastic strains, respectively. With these symbols we may write:

$$\mathbf{Q} = \mathbf{Q}^E + \mathbf{Q}^P = \mathbf{Q}^E + \mathbf{Zp} \qquad (3.10)$$

$$\mathbf{Q}^E = \mathbf{Q}^{EF} + \mathbf{Q}^{E\delta} = \mathbf{Q}^{EF} + \mathbf{Z}\boldsymbol{\delta} . \qquad (3.11)$$

By introducing Eqs. (3.10) (3.5) in Eq. (3.6), this becomes

$$\boldsymbol{\varphi} = -(\mathbf{H} - \tilde{\mathbf{N}}\mathbf{ZN})\boldsymbol{\lambda} + (\tilde{\mathbf{N}}\mathbf{Q}^E - \mathbf{K}) . \qquad (3.12)$$

Eq. (3.12) together with (3.7) supplies a further alternative governing relation set in the unknowns $\boldsymbol{\lambda}$, $\boldsymbol{\varphi}$. Note in passing that the same set, and in addition an explicit expression for matrix $\mathbf{Z}$, can be obtained by eliminating $\mathbf{f}$ from Eq. (3.8) by means of (3.9) (cf. [25]).

3.2. *Theorems for symmetric, positive semidefinite hardenig matrices* $\mathbf{H}$.

When the hardening rules are such that

$$\tilde{\mathbf{H}}^i = \mathbf{H}^i, \quad \tilde{\boldsymbol{\lambda}}^i \mathbf{H}^i \boldsymbol{\lambda}^i \geqslant 0 \quad \text{for any } \boldsymbol{\lambda}^i \ (i = 1 \dots n) \qquad (3.13)$$

the same also holds for matrix $\mathbf{H} \equiv \mathrm{diag}\,[\mathbf{H}^1 \dots \mathbf{H}^n]$ concerning the whole structure. Under this hypothesis: *the/any solution to the holonomic problem formulated in Sec.* 3.1 *for given* $\mathbf{F}$, $\boldsymbol{\delta}$, *minimizes*:

$$\text{(IV)} \quad \Psi_1(\mathbf{f}, \boldsymbol{\lambda}) \equiv \frac{1}{2}\tilde{\mathbf{e}}\mathbf{Se} + \frac{1}{2}\tilde{\boldsymbol{\lambda}}\mathbf{H}\boldsymbol{\lambda} + \tilde{\mathbf{K}}\boldsymbol{\lambda} - \tilde{\mathbf{F}}\mathbf{f}\,(^3) \qquad (3.14)$$

subject to:

$$\mathbf{e} = \mathbf{Bf} - \mathbf{N}\boldsymbol{\lambda} - \boldsymbol{\delta} , \qquad \boldsymbol{\lambda} \geqslant \mathbf{0} \qquad (3.15)$$

$$\text{(V)} \quad \Psi_2(\mathbf{Q}, \boldsymbol{\lambda}) \equiv \frac{1}{2}\tilde{\mathbf{Q}}\mathbf{CQ} + \frac{1}{2}\tilde{\boldsymbol{\lambda}}\mathbf{H}\boldsymbol{\lambda} + \tilde{\mathbf{Q}}\boldsymbol{\delta} \qquad (3.16)$$

subject to:

$$\tilde{\mathbf{B}}\mathbf{Q} = \mathbf{F} , \qquad \tilde{\mathbf{N}}\mathbf{Q} - \mathbf{H}\boldsymbol{\lambda} \leqslant \mathbf{K} \qquad (3.17)$$

$$\text{(VI)}\,(^4) \quad \Omega_1(\boldsymbol{\lambda}) \equiv \frac{1}{2}\tilde{\boldsymbol{\lambda}}(\mathbf{H} - \tilde{\mathbf{N}}\mathbf{ZN})\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}(\tilde{\mathbf{N}}\mathbf{Q}^E - \mathbf{K}) \qquad (3.18)$$

subject to:

$$\boldsymbol{\lambda} \geqslant \mathbf{0} \qquad (3.19)$$

$$\text{(VII)}\,(^4) \quad \Omega_2(\boldsymbol{\lambda}) \equiv \frac{1}{2}\tilde{\boldsymbol{\lambda}}(\mathbf{H} - \tilde{\mathbf{N}}\mathbf{ZN})\boldsymbol{\lambda} \qquad (3.20)$$

subject to:

$$(\mathbf{H} - \tilde{\mathbf{N}}\mathbf{ZN})\boldsymbol{\lambda} \geqslant \tilde{\mathbf{N}}\mathbf{Q}^E - \mathbf{K} . \qquad (3.21)$$

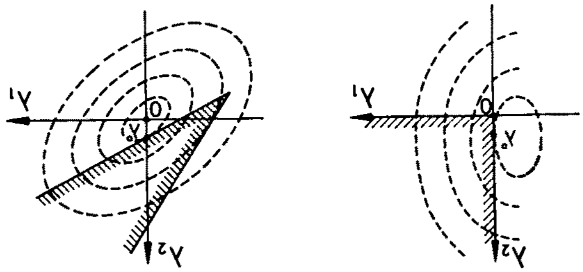

Fig. 4.

Proof. — Only an outline is given below, since the underlying mathematical concepts are available in standard books, e. g. [26], the algebraic manipulations are elementary and demonstrations of this kind have been already developed in related papers, e. g. [19] [25]. If $\mathbf{E}$ is a symmetric positive semidefinite matrix and $\mathbf{a}$, $\boldsymbol{\xi}$ are vectors, the convex quadratic programming problem

$$\text{minimize} \quad \frac{1}{2}\tilde{\boldsymbol{\xi}}\mathbf{E}\boldsymbol{\xi} + \tilde{\mathbf{a}}\boldsymbol{\xi} \quad \text{subject to} \quad \boldsymbol{\xi} \geqslant 0 \qquad (3.22)$$

is equivalent to the set of the Kuhn-Tucker local optimality conditions [26], which can be expressed in the form:

$$\boldsymbol{\eta} = \mathbf{E}\boldsymbol{\xi} + \mathbf{a} , \qquad \boldsymbol{\eta} \geqslant \mathbf{0} , \qquad \boldsymbol{\xi} \geqslant \mathbf{0} , \qquad \tilde{\boldsymbol{\eta}}\boldsymbol{\xi} = 0 \qquad (3.23)$$

The "dual" problem (such that it is solved by any solution of the former, "primal" problem, with equal optimal value for the objective function) has the form:

$$\text{maximize} \quad -\frac{1}{2}\tilde{\boldsymbol{\xi}}\mathbf{E}\boldsymbol{\xi} \quad \text{subject to} \quad \mathbf{a} + \mathbf{E}\boldsymbol{\xi} \geqslant \mathbf{0} \qquad (3.24)$$

The optimizations postulated by (VI) and (VII) have, clearly, the forms (3.22) and (3.24) respectively; as readily seen through the same identifications, the necessary and sufficient conditions (3.23) coincide with the formulation (3.7) (3.12) of the mechanical problem in hand, and therefore the statements (VI) and (VII) are justified.

Also the minimization (IV) can be reduced to the form (3.22), by substituting the expression (3.15) of $\mathbf{e}$ in (3.14), and putting:

$$\mathbf{f} = -\mathbf{f}^- + \mathbf{f}^+ \qquad \mathbf{f}^- \geqslant \mathbf{0} \qquad \mathbf{f}^+ \geqslant \mathbf{0} \qquad (3.25)$$

---

(3) The only variables are $\mathbf{f}$, $\boldsymbol{\lambda}$, since $\mathbf{e}$ can be eliminated by means of the compatibility equation included in the constraint set (3.15).

(4) Fig. 4 visualizes geometrically, for a case of strict convexity and of two variables, these dual optimization problems and their mutual connexions (feasible domains are shaded, lines of constant objective dashed, $\boldsymbol{\lambda}^\circ$ indicates the optimal vector).

$$\xi \equiv \begin{bmatrix} \mathbf{f}^- \\ \cdots \\ \mathbf{f}^+ \\ \cdots \\ \boldsymbol{\lambda} \end{bmatrix}; \quad \mathbf{A} \equiv \begin{bmatrix} \tilde{\mathbf{B}} \\ \cdots \\ -\tilde{\mathbf{B}} \\ \cdots \\ \tilde{\mathbf{N}} \end{bmatrix}; \quad \mathbf{a} \equiv \begin{bmatrix} \mathbf{F} \\ \cdots \\ -\mathbf{F} \\ \cdots \\ \mathbf{K} \end{bmatrix} + \mathbf{AS}\boldsymbol{\delta} \quad (3.26)$$

$$\mathbf{E} \equiv \mathbf{AS}\tilde{\mathbf{A}} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix}. \quad (3.27)$$

It is easy to check, by means of the above identifications, the coincidence of (3.23) with the governing relation set (3.7) (3.8) (3.9) and of (3.24) with (V); thus also the assertions (IV) and (V) are proved.

3.3 *Theorems for generic hardening matrices* H.

When the interaction matrices $\mathbf{H}^i$ of some or all elements do not comply with either or both conditions (3.13), the holonomic analysis problem can still be solved by a quadratic-linear optimization, according to either of the following statements: *the/any holonomic solution of the analysis problem for given* **F**, **δ** *minimizes*:

$$\text{(VIII)} \quad \Psi(\mathbf{f}, \boldsymbol{\lambda}) \equiv \tilde{\mathbf{e}}\mathbf{Se} + \tilde{\boldsymbol{\lambda}}\mathbf{H}\boldsymbol{\lambda} + \tilde{\boldsymbol{\delta}}\mathbf{Se} + \tilde{\mathbf{K}}\boldsymbol{\lambda} - \tilde{\mathbf{F}}\mathbf{f}\,(^5) \quad (3.28)$$

subject to:

$$\mathbf{e} = \mathbf{Bf} - \mathbf{N}\boldsymbol{\lambda} - \boldsymbol{\delta}, \qquad \boldsymbol{\lambda} \geqslant \mathbf{0} \quad (3.29)$$

$$\tilde{\mathbf{B}}\mathbf{Se} = \mathbf{F}, \quad \tilde{\mathbf{N}}\mathbf{Se} - \mathbf{H}\boldsymbol{\lambda} \leqslant \mathbf{K} \quad (3.30)$$

$$\text{(IX)} \quad \Omega(\boldsymbol{\lambda}) \equiv \tilde{\boldsymbol{\lambda}}(\mathbf{H} - \tilde{\mathbf{N}}\mathbf{ZN})\boldsymbol{\lambda} - \tilde{\boldsymbol{\lambda}}(\tilde{\mathbf{N}}\mathbf{Q}^E - \mathbf{K}) \quad (3.31)$$

subject to:

$$\boldsymbol{\lambda} \geqslant \mathbf{0}, \quad (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{ZN})\boldsymbol{\lambda} \geqslant \tilde{\mathbf{N}}\mathbf{Q}^E - \mathbf{K}. \quad (3.32)$$

If the minimum is not zero, no solution exists.

Proof. — For whatever matrix **E**, the set of relations (3.23) in the vectors **ξ**, **η** (a linear complementarity problem) is fully equivalent to the (generally nonconvex) quadratic programming problem

minimize:

$$\tilde{\xi}\mathbf{E}\xi + \tilde{\mathbf{a}}\xi \quad (3.33)$$

subject to:

$$\xi \geqslant \mathbf{0}, \qquad \mathbf{E}\xi + \mathbf{a} \geqslant \mathbf{0} \quad (3.34)$$

supplemented by the statement that, if the minimum is not zero, (3.23) is not consistent. This equivalence is readily proved by noting that (3.33) is the inner product $\tilde{\eta}\xi$ and, hence, cannot be negative. It has been observed in the preceding proof that both the **λ**, **φ** - formulation and the **f**, **λ**, **φ** - formulation of the structural problem considered (Sec. 3.1) can be reduced to the form (3.23) through suitable identifications, which are precisely those which transform (IV) and (VI) to (3.22) and (V) and (VII) to (3.24). Through the same identifications, the minimum seeking problems (VIII) and (IX) are reduced to the form (3.33) and (3.34) and hence, via (3.23), turn out to be equivalent to the original structural problem.

(5) See footnote (3).

3.4. *Remarks.*

3.4.1. From the mechanical standpoint, it is worth noting that the optimization (IV) yields equilibrium besides the constitutive rules $\boldsymbol{\lambda} \geqslant \mathbf{0}$, $\tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0$; (V) supplies compatibility besides the constitutive rules $\boldsymbol{\varphi} \leqslant \mathbf{0}$, $\tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0$. Therefore these theorems are conceptually amenable to the principles of potential and complementary energy respectively. The other four theorems are extremal formulations of constitutive requirements only.

From the operative standpoint, the optimizations (IV) to (VII) can be achieved by any quadratic programming algorithm (e. g. Wolfe's, Beale's) [26]; (VIII) and (IX) can be achieved by Ritter's method [27] in general, by the simpler methods of Lemke or Cottle [20] if all $\mathbf{H}^i$, though violating condition (3.13), are strictly copositive, as for isotropic and quasi-isotropic hardening (Sec. 2.3) or positive semidefinite matrices, respectively.

For the validity of theorems (VI) and (VII) condition (3.13) can be relaxed to the analogous condition concerning matrix $\mathbf{H} - \tilde{\mathbf{N}}\mathbf{ZN}$.

The number of solutions, in particular uniqueness, and the solvability can be discussed and checked in any case on the basis of recent mathematical results (cf. [20] [25] and ref. thereof).

We only note here that in cases of lack of uniqueness the dual properties (V) and (VII) are not only necessary but also sufficient for the solutions, provided that they be further constrained by $\boldsymbol{\lambda} \geqslant \mathbf{0}$; a formal proof of this assertion can be deduced via specialization and analogy, from [25], Sec. 6.3.

3.4.2. The results obtained for discrete models of continua can be straightforwardly translated in the traditional tensor field description. However, this does not rule out the interest of a separate discussion. Simply on the basis of the mechanical meaning of each term, statement (VIII), e. g., can be expressed in the form:

minimize

$$\Psi'(u, \lambda) \equiv \int_V \overset{e}{\varepsilon}_{ij} S_{ijhk} \overset{e}{\varepsilon}_{hk} dV + \int_V H_{rs} \lambda_r \lambda_s dV + \int_V K_r \lambda_r dV - \int_V F_i u_i dV - \int_{B_T} T_i u_i dB + \int_V \overset{d}{\varepsilon}_{ij} S_{ijhk} \overset{e}{\varepsilon}_{hk} dV - \int_{B_u} n_i \bar{u}_j S_{ijhk} \overset{e}{\varepsilon}_{hk} dB \quad (3.35)$$

subject to:

$$\overset{e}{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - N_{ijr}\lambda_r - \overset{d}{\varepsilon}_{ij}, \quad \lambda_r \geqslant 0 \text{ in } V \quad (3.36)$$

$$u_i = \bar{u}_i \text{ on } B_u$$

$$\frac{\partial}{\partial x_i}(S_{ijrs}\overset{e}{\varepsilon}_{rs}) + F_j = 0 \text{ in } V, \; n_i S_{ijhk}\overset{e}{\varepsilon}_{hk} = T_j \text{ on } B_T \tag{3.37}$$

$$N_{ijr}S_{ijhk}\overset{e}{\varepsilon}_{hk} - K_r - H_{rs}\lambda_s \leqslant 0 \text{ in } V$$

where: index summation convention is assumed; u_i = displacements; σ_{ij}, ε_{ij}^c, ε_{ij}^d = stress, elastic strain and thermal strain Cartesian tensors; S_{ijhk} = elastic moduli tensor; V = volume of the body; B_u, B_T = boundary parts where displacements $\bar{u}_i$ and traction T_i are prescribed; F_i = volume forces; n_i = normal unit vector to B_u; x_i = space coordinates. The constraint set (3.36) and (3.37) correspond to (3.29) and (3.30), respectively.

4. Incremental versus deformation theory.

4.1. *Analogous rate statements.*

When the deformation theory cannot be accepted (complex loading paths) or relied upon, the loading process must be followed step by step in the spirit of the flow-law plasticity theory. It is important to observe that all the preceding extremum characterizations of holonomic finite solutions can be transferred by analogy to the solutions of the incremental (step) problems. In fact, consider the element (or material) flow-laws (2.1) (2.2) (2.3) (2.6) (2.7) (2.8) (2.9). The last condition (2.9) can be disregarded, as implicitly satisfied, if we assemble in matrix $\mathbf{N}^i$ only the normal vectors $\mathbf{N}_j^i$ of those yield planes which contain the stress point at the considered situation, and if we dimension $\mathbf{H}^i$, $\dot{\boldsymbol{\lambda}}^i$ and $\dot{\boldsymbol{\varphi}}^i$ accordingly. Let $\mathbf{N}$ indicate again the supermatrix containing the n above defined matrices $\mathbf{N}^i$ in diagonal positions, and zero elsewhere, so that $\dot{\mathbf{p}} = \mathbf{N}\dot{\boldsymbol{\lambda}}$ with $\dot{\tilde{\boldsymbol{\lambda}}} \equiv [\dot{\tilde{\boldsymbol{\lambda}}}^1 \dots \dot{\tilde{\boldsymbol{\lambda}}}^n]$, $\dot{\tilde{\mathbf{p}}} \equiv [\dot{\tilde{\mathbf{p}}}^1 \dots \dot{\tilde{\mathbf{p}}}^n]$, ($n$ is the number of finite elements). By dotting the symbols of all variable vectors, giving to $\mathbf{N}$, $\mathbf{H}$, $\dot{\boldsymbol{\lambda}}$, $\dot{\boldsymbol{\varphi}}$ the meanings specified above and dropping the constant vector $\mathbf{K}$, the set of relations (3.1) to (3.7) is transformed into that which governs the incremental problem. The clear analytical analogy between the formulations extends to the developments and conclusions. Therefore we may state e. g., as a counterpart of theorem (VIII) but eliminating $\dot{\mathbf{e}}$, that:

(VIII*) whatever matrix $\mathbf{H}$ may be, the/any solution to the rate problem for given $\dot{\mathbf{F}}$, $\dot{\boldsymbol{\delta}}$ is characterized by the minimum (if zero) of the quadratic function:

$$\Psi^*(\dot{\mathbf{f}}, \dot{\boldsymbol{\lambda}}) \equiv (\dot{\tilde{\mathbf{f}}}\tilde{\mathbf{B}} - \dot{\tilde{\boldsymbol{\lambda}}}\tilde{\mathbf{N}})\mathbf{S}(\mathbf{B}\dot{\mathbf{f}} - \mathbf{N}\dot{\boldsymbol{\lambda}}) + \dot{\tilde{\boldsymbol{\lambda}}}\mathbf{H}\dot{\boldsymbol{\lambda}} - \dot{\tilde{\mathbf{F}}}\dot{\mathbf{f}} - \dot{\tilde{\boldsymbol{\delta}}}\mathbf{S}(\mathbf{B}\dot{\mathbf{f}} - \mathbf{N}\dot{\boldsymbol{\lambda}}) \tag{4.1}$$

subject to:

$$\dot{\boldsymbol{\lambda}} \geqslant 0, \quad \tilde{\mathbf{B}}\mathbf{S}\mathbf{B}\dot{\mathbf{f}} - \tilde{\mathbf{B}}\mathbf{S}\mathbf{N}\dot{\boldsymbol{\lambda}} = \dot{\mathbf{F}} + \tilde{\mathbf{B}}\mathbf{S}\dot{\boldsymbol{\delta}} \tag{4.2}$$

$$\tilde{\mathbf{N}}\mathbf{S}\mathbf{B}\dot{\mathbf{f}} - (\mathbf{H} + \tilde{\mathbf{N}}\mathbf{S}\mathbf{N})\dot{\boldsymbol{\lambda}} \leqslant \tilde{\mathbf{N}}\mathbf{S}\dot{\boldsymbol{\delta}} \; . \tag{4.3}$$

All the six finite theorems can be similarly translated in incremental terms, without needing separate treatment.

In the rate problem the piecewise linearization is an immaterial restriction; vectors $\mathbf{N}_j^i$ may be thought of as gradients of *regular, but not necessarily linear*, yield functions in the current stress point. With this interpretation, the incremental counterparts of the six minimum principles of Sec. 3.2 and 3.3, include the generalizations (to interacting yield regimes) of the theorems proved in Ref. [28] and are specializations (to small displacements and associated flow-laws) of some results obtained in [25].

4.2. *Comparison properties.*

Given an external action set $\mathbf{F}$, $\boldsymbol{\delta}$, the relevant nonholonomic, history-dependent solutions involve vectors which clearly satisfy the constraints of all the optimizations (IV) to (IX). In fact the nonlinear holonomy condition $\tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0$ never appears as a constraint and is always a consequence of the optimization process. Therefore, six bounding properties follow and can be expressed by the statement:

(X) *The energy functions Ψ_1, Ψ_2, Ω_1, Ω_2, Ψ and Ω attain for the holonomic solution, values which bound from below the values they attain for the nonholonomic solutions relative to any path leading to the same final loading condition* $\mathbf{F}$, $\boldsymbol{\delta}$.

It has been seen above that the stress and strain state in a structure subject to $\mathbf{F}$ and $\boldsymbol{\delta}$, must obey, under the holonomy assumption, a further requirement ($\tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0$) than for nonholonomic laws. The following theorem is of interest in this regard:

(XI) *When matrix* $\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N}$ *is positive semidefinite (not necessarily symmetric), then for given external actions* $\mathbf{F}$, $\boldsymbol{\delta}$, *if there exists any nonholonomic solution (one always exists if the matrix is definite), there must be an holonomic solution.*

Proof. — The multiplier strain vector $\boldsymbol{\lambda}$ defines completely a solution, through Eqs. (3.5)(3.10); the linear relations contained in (3.7) (3.12) govern all the nonholonomic solution. The same mathematical theorem (Cottle [21]) used in Sec. 2.2 for statement (III), leads directly to the above conclusion, if it is applied to the complementarity problem (3.7) (3.12).

5. Shakedown analysis in dynamic and quasi-static conditions.

5.1. *Formulation of the problem.*

Let the structure be subject to external actions $\mathbf{F}(t)$, $\boldsymbol{\delta}(t)$ varying in time so that dynamic effects cannot be neglected, but the influence of geometric changes on the equilibrium relations can ("small deformation" hypothesis).

First suppose that the material be *linear elastic*: then if damping of only a viscous nature is assumed, the displacement response $\mathbf{f}(t)$ of a finite element model of the system is governed by the following set of linear ordinary differential equations:

$$\mathscr{L}[\mathbf{f}(t)] \equiv \mathbf{m}\ddot{\mathbf{f}} + \mathbf{D}\dot{\mathbf{f}} + \tilde{\mathbf{B}}\mathbf{S}\mathbf{B}\mathbf{f} = \mathbf{F}(t) + \tilde{\mathbf{B}}\mathbf{S}\boldsymbol{\delta}(t) \tag{5.1}$$

associated with the initial conditions:

$$\mathbf{f}(0) = \mathbf{f}_0, \qquad \dot{\mathbf{f}}(0) = \dot{\mathbf{f}}_0 \; . \tag{5.2}$$

In Eq. (5.1), $\mathbf{m}$ is the equivalent mass matrix for the assembled structure (diagonal in lumped-mass models), $\tilde{\mathbf{B}}\mathbf{S}\mathbf{B}$ the (external) stiffness matrix, $\mathbf{D}$ the damping matrix (often assumed proportional to either of the preceding ones). For details on these notions see e. g. [18]. Eq. (5.1) readily derives from Eqs. (3.1) to (3.4) provided that inertia and damping forces are considered in the equilibrium Eq. (3.2).

Studying the motion $\mathbf{f}(t)$ of the same system on the basis of elastoplastic constitutive laws, is a formidable task in most cases, far more difficult than solving problem (5.1) (5.2). An important question on the dynamic evolution of an elastoplastic system is whether it shakes down to purely elastic behaviour or undergoes unlimited plastic yielding and, hence, becomes eventually inserviceable.

It will be shown in the next sections that, for a large class of elastic-workhardening structures (including those which obey Prager's kinematic hardening rule), this question can be answered only on the basis of the *elastic* dynamic response, i. e. of (5.1) (5.2), and of certain *static* characteristics of the actual system (matrix $\mathbf{Z}$). For perfectly plastic continua a similar result has been previously established by Ceradini [13]. The path of reasoning will be partially patterned on his and on Koiter's demonstration [29] of Melan's theorem.

5.2. *Basic dynamic shakedown theorems.*

(XII) *When the workhardening matrix* $\mathbf{H}$ *is symmetric positive semidefinite, shakedown will occur if there exist a plastic multiplier set* $\bar{\boldsymbol{\lambda}}$ *and a displacement and velocity set* $\bar{\mathbf{f}}_0$, $\bar{\dot{\mathbf{f}}}_0$, *such that, should they be imposed on the structure at* $t = 0$, *the whole dynamic evolution under the given loading programme* $\mathbf{F}(t)$, $\boldsymbol{\delta}(t)$ *would lead to stresses below the yield limit at all elements and instants.*

Proof. — At any instant t of the actual process, the plastic potentials throughout the structure can be expressed, via (3.6), (3.10), (3.5), in the form:

$$\boldsymbol{\varphi}(t) = \tilde{\mathbf{N}}\mathbf{Q}^{E}(t) + \tilde{\mathbf{N}}\mathbf{Q}^{P}(t) - \mathbf{H}\boldsymbol{\lambda}(t) - \mathbf{K} \leqslant \mathbf{0} \tag{5.3}$$

where:

$$\mathbf{Q}^{P}(t) = \mathbf{Z}\mathbf{N}\boldsymbol{\lambda}(t) \tag{5.4}$$

$$\mathbf{Q}^{E}(t) = \mathbf{Q}^{EF} + \mathbf{Q}^{E\delta} + \mathbf{Q}^{EI} + \mathbf{Q}^{ED}. \tag{5.5}$$

The expression (5.5) of the elastic stress response, compared to (3.11) is implemented by the addends due to inertia and damping forces.

For the fictitious elastic process postulated by the hypothesis, we may write (symbols relevant to this process are overlined):

$$\bar{\boldsymbol{\varphi}}(t) = \tilde{\mathbf{N}}\bar{\mathbf{Q}}^{E}(t) + \tilde{\mathbf{N}}\bar{\mathbf{Q}}^{P} - \mathbf{H}\bar{\boldsymbol{\lambda}} - \mathbf{K} < \mathbf{0} \tag{5.6}$$

where

$$\bar{\mathbf{Q}}^{P} = \mathbf{Z}\mathbf{N}\bar{\boldsymbol{\lambda}} \tag{5.7}$$

$$\bar{\mathbf{Q}}^{E}(t) = \mathbf{Q}^{EF} + \mathbf{Q}^{E\delta} + \bar{\mathbf{Q}}^{EI} + \bar{\mathbf{Q}}^{ED}. \tag{5.8}$$

$\bar{\mathbf{Q}}^{E}(t)$ generally differs from $\mathbf{Q}^{E}(t)$ both because of the different initial conditions and because of the absence of plastic yielding.

Henceforth, dependence on time will not be explicitly marked by (t), and Δ will indicate the difference between the values that the argument assumes, at the same instant, in the actual and in the fictitious process.

Consider the never negative function χ:

$$2\chi \equiv \Delta\tilde{\mathbf{Q}}^{P}\mathbf{C}\Delta\mathbf{Q}^{P} + \Delta\tilde{\boldsymbol{\lambda}}\mathbf{H}\Delta\boldsymbol{\lambda} + \Delta\tilde{\mathbf{Q}}^{E}\mathbf{C}\Delta\mathbf{Q}^{E} + \Delta\tilde{\dot{\mathbf{f}}}\mathbf{m}\Delta\dot{\mathbf{f}} \tag{5.9}$$

and the derivatives with respect to the time of each of its addends, separately.

$$\dot{\chi}_1 \equiv \Delta\tilde{\mathbf{Q}}^{P}\mathbf{C}\Delta\dot{\mathbf{Q}}^{P} \tag{5.10}$$

which, by means of the virtual work principle and of Eqs. (3.5) (5.3) (5.6) successively, becomes:

$$\dot{\chi}_1 = -\Delta\tilde{\mathbf{Q}}^{P}\dot{\mathbf{p}} = -\tilde{\dot{\boldsymbol{\lambda}}}\tilde{\mathbf{N}}\Delta\mathbf{Q}^{P} = -\tilde{\dot{\boldsymbol{\lambda}}}(\Delta\boldsymbol{\varphi} + \mathbf{H}\Delta\boldsymbol{\lambda} - \tilde{\mathbf{N}}\Delta\mathbf{Q}^{E});$$

because $\tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = 0$, $\dot{\mathbf{p}} = \Delta\dot{\mathbf{p}}$, and via (5.5) and the analogous expression (5.8) for $\bar{\mathbf{Q}}^{E}$, $\dot{\chi}_1$ can be expressed as:

$$\dot{\chi}_1 = \tilde{\bar{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} - \Delta\tilde{\boldsymbol{\lambda}}\mathbf{H}\dot{\boldsymbol{\lambda}} + \Delta\tilde{\dot{\mathbf{p}}}(\Delta\mathbf{Q}^{EI} + \Delta\mathbf{Q}^{ED}) \,. \tag{5.11}$$

$$\dot{\chi}_2 \equiv \Delta\tilde{\boldsymbol{\lambda}}\mathbf{H}\Delta\dot{\boldsymbol{\lambda}} \,. \tag{5.12}$$

$$\dot{\chi}_3 \equiv \Delta\tilde{\mathbf{Q}}^{E}\mathbf{C}\Delta\dot{\mathbf{Q}}^{E}$$

through Eq. (3.4) and (5.5), $\dot{\chi}_3$ can be written as:

$$\dot{\chi}_3 = \Delta\tilde{\dot{\mathbf{e}}}(\Delta\mathbf{Q}^{EI} + \Delta\mathbf{Q}^{ED}) \,. \tag{5.13}$$

$$\dot{\chi}_4 \equiv \Delta\tilde{\dot{\mathbf{f}}}\mathbf{m}\Delta\ddot{\mathbf{f}}$$

by virtue of the virtual work principle becomes

$$\dot{\chi}_4 = -(\Delta\tilde{\dot{\mathbf{e}}} + \Delta\tilde{\dot{\mathbf{p}}})\Delta\mathbf{Q}^{EI} \,. \tag{5.14}$$

By summing up Eqs. (5.11) (5.12) (5.13) (5.14) and taking account of the further virtual work equation:

$$\Delta\tilde{\mathbf{Q}}^{ED}\Delta\dot{\mathbf{q}} = -\Delta\tilde{\dot{\mathbf{f}}}\mathbf{D}\Delta\dot{\mathbf{f}}$$

we finally obtain:

$$\dot{\chi} = \tilde{\bar{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} - \Delta\tilde{\dot{\mathbf{f}}}\mathbf{D}\Delta\dot{\mathbf{f}} \,. \tag{5.15}$$

Because of the hypothesis (5.6) and the positive definiteness of $\mathbf{D}$, $\dot{\chi}$ is never positive and becomes zero if and only if both $\dot{\mathbf{f}} = \bar{\dot{\mathbf{f}}}$ and $\dot{\boldsymbol{\lambda}} = \mathbf{0}$. Therefore, since $\chi \geqslant 0$, the laws of the two motions must tend to coincide [6]

[6] This collateral conclusion clearly rests on the presence of the viscous damping dissipation ($\mathbf{D} \neq \mathbf{0}$), which on the other hand makes the motion of the fictitious system asymptotically independent of the initial conditions.

and plastic yielding must vanish at least asymptotically in time (q. e. d.).

(XII*) *When the external actions are periodic, if shakedown occurs, then there exist distributions of plastic multipliers, displacements and velocities* $\bar{\lambda}$, $\bar{\mathbf{f}}_0$, $\dot{\bar{\mathbf{f}}}_0$, *such that, should they be imposed on the structure at* $t = 0$, *the whole dynamic process under the given loading programme* $\mathbf{F}(t)$, $\boldsymbol{\delta}(t)$ *would develop without plastic yielding in any element or instant.*

Proof. — Let τ_s be the instant at which the system shakes down. Because of the supposed periodicity, there exists a time $\tau \geqslant \tau_s$ such that, for any $t \geqslant \tau$,

$$\mathbf{F}(t) = \mathbf{F}(t - \tau) , \qquad \boldsymbol{\delta}(t) = \boldsymbol{\delta}(t - \tau) . \qquad (5.16)$$

Assume as initial conditions the displacements, velocities and plastic multipliers which are actually present at the instant τ:

$$\bar{\mathbf{f}}_0 = \mathbf{f}_\tau , \ \dot{\bar{\mathbf{f}}}_0 = \dot{\mathbf{f}}_\tau , \ \bar{\lambda} = \lambda_\tau . \qquad (5.17)$$

The resulting fictitious process for $t > 0$ coincides with the actual one for $t > \tau$; hence, it does not involve plastic deformations and satisfies for all $t > 0$ the inequality:

$$\bar{\boldsymbol{\varphi}}(t) = \tilde{\mathbf{N}}\mathbf{Q}^E(t) - (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\bar{\lambda} - \mathbf{K} \leqslant \mathbf{0} . \qquad (5.18)$$

(XIII) *When the workhardening matrix* $\mathbf{H}$ *is symmetric positive definite, the system shakes down under any loading programme, which produces a bounded elastic dynamic response.*

Proof. — If $\mathbf{H}$ is positive definite, also $\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N}$ is so. Then, according to the algebraic theorem used for proving (III) and (XI), for any fixed vector $\mathbf{a}$ there is a $\bar{\lambda} \geqslant \mathbf{0}$ such that

$$(\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\bar{\lambda} \geqslant \mathbf{a} \qquad \text{for any } t.$$

It is always possible, for bounded elastic response, to assume

$$\mathbf{a} > \tilde{\mathbf{N}}\bar{\mathbf{Q}}^E(t) - \mathbf{K}$$

and, hence, to comply with (5.6), which ensures shakedown by virtue of (XII).

5.3. *Specialization to quasi-static conditions.*

When the external actions vary so slowly that inertia and damping forces can be neglected, the preceding statements (XII) (XII*) still hold, provided that they do not mention initial conditions in terms of displacements and velocities, which become immaterial. Time plays the role of an "ordering variable" and may be replaced by any monotonic function of it. The loading programme can be defined through the intervals within which each component F_h, δ_h varies in whatever way and attains any value an infinite number of times. In this case it is possible to introduce the vector

$$\mathbf{M} \equiv [\dots M_j^1 \dots M_j^n \dots] \quad \text{where} \quad M_j^i \equiv \max \{\tilde{\mathbf{N}}_j^i \mathbf{Q}^{Ei}(t)\} \qquad (5.19)$$

and express the sufficient (XII) and the necessary (XII*) condition for shakedown, respectively, in the form:

$$\mathbf{M} - (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\bar{\lambda} - \mathbf{K} < \mathbf{0} \qquad (5.20)$$

$$\mathbf{M} - (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\bar{\lambda} - \mathbf{K} \leqslant \mathbf{0} . \qquad (5.21)$$

For perfectly plastic systems ($\mathbf{H} = \mathbf{0}$), reference can be made to selfstress states $\bar{\mathbf{Q}}^P$ instead of to $\bar{\lambda}$: thus statements (XII) and (XII*) reduce to part (a) and (b), respectively, of Melan's classical theorem (see e. g. [29] [14]).

5.4. *Safety analysis and programming problems.*

Let k be a common positive multiplier for all the straining effects $\mathbf{F}(t), \boldsymbol{\delta}(t)$ and the initial conditions $\mathbf{f}_0$, $\dot{\mathbf{f}}_0$ (which however are taken zero in most practical cases). The *safety factor* is defined as a value s such that for any $k \leqslant s$ the structure shakes down, and for $k > s$ it does not.

When the external actions are periodic, statements (XII) (XII*) leads to the evaluation of s as a constrained maximization problem in the variables k, $\bar{\lambda}$, $\bar{\mathbf{f}}_0$, $\dot{\bar{\mathbf{f}}}_0$; for generic loading programmes, this supplies only a lower bound to s. For perfectly plastic structures, periodic forces and absence of damping, the safety problem has been discussed in [30]. Only the quasi-static case will be briefly considered here below for hardening structures.

(XIV) *If the hardening matrix* $\mathbf{H}$ *is symmetric positive semidefinite, the safety factor with respect to repeated external actions variable within given intervals, is the optimal value of the following dual linear programming problems in the variables* k, λ:

(*P*) *maximize* k

subject to (7):

$$k\mathbf{M} - (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\lambda \leqslant \mathbf{K} \qquad (5.22)$$

(*the constraint* $\lambda \geqslant \mathbf{0}$ *is immaterial*)

(*D*) *minimize* $\tilde{\mathbf{K}}\lambda'$

subject to:

$$\tilde{\mathbf{M}}\lambda' = 1, \quad (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\lambda' = \mathbf{0} , \quad \lambda' \geqslant \mathbf{0} . \qquad (5.23)$$

Proof. — The primal program (*P*) clearly derives from (XII) and (XII*) via specialization and (5.20) (5.21). Only the statement in brackets requires a formal proof. Suppose that the inequality (5.22) be satisfied, for a given k, by a vector λ with some negative component. By identifying $k\mathbf{M}$ with $\tilde{\mathbf{N}}\mathbf{Q}^E$, this inequality can be thought of as defining the feasible domain of problem (VII). This is solvable for $k\mathbf{M}$, since any quadratic program is so, if the objective function is bounded below in the feasible region [26] (8). Because of the duality relation, problem (VI) is solvable as a consequence, and its solution certainly complies with both $\lambda \geqslant \mathbf{0}$ (3.19) and (3.21). Therefore, if (5.22)

(7) The constraint $k \geqslant 0$ is clearly superfluous, as long as $\mathbf{K} \geqslant \mathbf{0}$; it may be convenient in the dualization (cf. [14]).

(8) A new interesting proof of this known concept of programming theory is given by Eaves in Techn. Rep. 69-4, Op. Res. House, Stanford Un. July 1969.

is fulfilled by a generic $\boldsymbol{\lambda}$, it is so also by a nonnegative $\boldsymbol{\lambda}$: this justifies the statement in brackets.

The linear program (D) is obtained by means of a dualization process and of some remark, which are completely analogous to those expounded in a previous paper in this Journal [14], and which, therefore, do not need to be given here. A prime on $\boldsymbol{\lambda}$ in (D) recalls that $\boldsymbol{\lambda}$ has in (D) values and meaning different from those in (P).

Through the same mechanical interpretation as in [14] Sec. 6, it is easy to recognize in (D) a formulation of the kinematic shakedown theorem extended to hardening structures. The only new point to be clarified is the meaning of the second constraint (5.23). By premultiplying this equation by $\boldsymbol{\lambda}'$ we have:

$$\tilde{\boldsymbol{\lambda}}'\mathbf{H}\boldsymbol{\lambda}' + \tilde{\boldsymbol{\lambda}}'\tilde{\mathbf{N}}(-\mathbf{Z})\mathbf{N}\boldsymbol{\lambda}' = \mathbf{0}\,. \tag{5.24}$$

Since both quadratic forms are nonnegative definite, both must vanish and, hence (see e. g. [26] p. 32):

$$\mathbf{H}\boldsymbol{\lambda}' = \mathbf{0} \tag{5.25}$$

$$\mathbf{Z}\mathbf{N}\boldsymbol{\lambda}' = \mathbf{0}\,. \tag{5.26}$$

Eq. (5.26) combined with $\boldsymbol{\lambda}' \geqslant \mathbf{0}$, means that the minimization must be performed *within the class of the* $\boldsymbol{\lambda}'$ *which define stressless (compatible) plastic strain systems* ("mechanisms"), as in the nonhardening case. Eq. (5.25) shows that feasible plastic multiplier distributions must *leave unaltered all the plastic potentials and, hence, the yield loci* (9). A plastic strain set $\mathbf{p}' = \mathbf{N}\boldsymbol{\lambda}'$ is the result of an "admissible plastic strain rate cycle", in the sense of Koiter [29] but with the additional property (5.25).

6. Generalized limit analysis.

6.1. *Structures with unbounded plastic strain capacity.*

When the external actions do not vary in time, the linear programs of Sec. 5.4 reduce via (5.19) to the form

(XV-P) maximize k subject to:

$$k\tilde{\mathbf{N}}\mathbf{Q}^E - (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\boldsymbol{\lambda} \leqslant \mathbf{K} \tag{6.1}$$

(XV-D) minimize $\tilde{\mathbf{K}}\boldsymbol{\lambda}'$ subject to:

$$\tilde{\mathbf{Q}}^E\mathbf{N}\boldsymbol{\lambda}' = 1\,, \quad (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\boldsymbol{\lambda}' = \mathbf{0}\,, \quad \boldsymbol{\lambda}' \geqslant \mathbf{0}\,. \tag{6.2}$$

Their common optimal value is the safety factor s_p against plastic collapse, characterized by the possibility of *simultaneous* compatible unlimited plastic yielding at constant loads and fixed yield planes, as appears from (6.2), (5.25) (5.26). By the virtual work principle:

$$\tilde{\mathbf{Q}}^E\mathbf{N}\boldsymbol{\lambda}' = \tilde{\mathbf{Q}}^{EF}\mathbf{N}\boldsymbol{\lambda}' = \tilde{\mathbf{Q}}^o\mathbf{N}\boldsymbol{\lambda}' = \tilde{\mathbf{F}}\mathbf{f}' \tag{6.3}$$

where $\mathbf{Q}^o$ represents any stress state equilibrated with the loads $\mathbf{F}$, and $\mathbf{f}'$ represents the displacement set which governs, through (3.1), the strain set $\mathbf{q}' = \mathbf{p}' = \mathbf{N}\boldsymbol{\lambda}'$. Eq. (6.3) shows that dislocations $\boldsymbol{\delta}$ do not influence s_p and allow to rewrite (XV-D) making explicit the classical notion of kinematically admissible multipliers:

(XV-D') minimize k subject to:

$$\tilde{\mathbf{K}}\boldsymbol{\lambda}' (= k\tilde{\mathbf{F}}\mathbf{f}') = k\mathbf{Q}^E\mathbf{N}\boldsymbol{\lambda}', \qquad (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\boldsymbol{\lambda}' = \mathbf{0}, \quad \boldsymbol{\lambda}' \geqslant \mathbf{0}\,. \tag{6.2'}$$

The first equation in (6.2') expresses the balance between external and dissipated work, the latter being represented by $\tilde{\mathbf{K}}\boldsymbol{\lambda}'$, since $\mathbf{H}\boldsymbol{\lambda}' = \mathbf{0}$.

The preceding formulations can be readily reduced to the traditional ones for perfectly plastic behaviour.

It may be interesting to observe that the linear programs (XV) for calculating s_p, can be re-established on the basis of the finite minimum theorems (VI) and (VII). In fact consider the totality of the solutions to (VI) for a given k, i. e., $\boldsymbol{\lambda}^o$ being a single holonomic solution, the vector set characterized as:

$$\boldsymbol{\lambda}^o + \boldsymbol{\lambda}' \geqslant \mathbf{0}\,, \quad (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\boldsymbol{\lambda}' = \mathbf{0}\,, \qquad \tilde{\boldsymbol{\lambda}}'(k\tilde{\mathbf{N}}\mathbf{Q}^E - \mathbf{K}) = \mathbf{0}\,. \tag{6.4}$$

A comparison between (6.2') and (6.4) shows that (6.2') expresses the condition that, *if* a holonomic solution exists, there is also an unbounded solution set, i. e. $k = s_p$. Since, if no holonomic solution exists, there cannot be any nonholonomic solution, theor. (XI), it is justified to use (6.2) or (6.2') in the search for the limit multiplier with also nonholonomic laws.

Similarly the constraints (6.1) can be regarded as the necessary and sufficient condition [see proof (III)] for the solvability of the analysis problem with holonomic and, hence, through (XI), also with nonholonomic laws.

6.2. *Structures with limited plastic strain capacity.*

In many structures the basic assumption of unlimited deformability is invalid; local failures may preceed plastic collapse so that predictions founded on the classical limit theorems become unsafe, sometimes quite seriously. Step-by-step calculations are an obvious but lengthy way of dealing with these cases. Procedures of practical value have been devised for the analysis and design of reinforced concrete beams and frames in view of the above circumstances (cf. e. g. surveys [15] [16]). However direct methods for evaluating or bounding the safety factor s_L with allowance for local failure, seem desirable but have not been proposed so far, to the author's knowledge. The present approach and the above conclusions lend themselves to a preliminary discussion of this question.

For one-component stress-strain relations (e. g. flexural characteristics of beams) it is reasonable to refer to extreme values $\lambda^i{}_{jR}$ of plastic multipliers (e. g. rotation capacity). In multicomponent cases the analogous assumption $\boldsymbol{\lambda}^i \leqslant \boldsymbol{\lambda}_R{}^i$ is convenient though questionable; however any

(9) If either $\mathbf{H}$ or $-\tilde{\mathbf{N}}\mathbf{Z}\mathbf{N}$ are definite, the feasible domain of (D) is empty; since that of (P) is not so ($k = 0$, $\boldsymbol{\lambda} = \mathbf{0}$ are always feasible) the objective of (P) is unbounded on it according to a known programming theorem. Hence $s = \infty$. In the former hypothesis, (XIII) is thus confirmed in a narrower context. The latter may hold when not all elements of the structure are considered plastic.

function of $\boldsymbol{\lambda}^i$ might be chosen, with computational but without conceptual consequences. By associating to (6.1) the further constraints

$$\boldsymbol{\lambda} \leqslant \boldsymbol{\lambda}_R \text{ where } \tilde{\boldsymbol{\lambda}}_R \equiv [\tilde{\boldsymbol{\lambda}}^1{}_R \dots \tilde{\boldsymbol{\lambda}}^n{}_R] \tag{6.5}$$

the maximum k^o provided by the linear program (XV-*P*) is clearly an upper bound to s_L:

$$s_L \leqslant k^o \leqslant s_p \tag{6.6}$$

By dualizing the so modified primal program, the same value k^o can also be obtained by means of the following optimization (details on the dualization process can be found in Ref. [26] and [14]; it is convenient to put $\boldsymbol{\lambda} = \boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-$, $\boldsymbol{\lambda}^+ \geqslant \mathbf{0}$, $\boldsymbol{\lambda}^- \geqslant \mathbf{0}$, write the primal problem in tableau form and "transpose" this tableau according to the rules given in Ref. [14], Sec. 5):

minimize

$$\tilde{\mathbf{K}}\boldsymbol{\lambda}' + \tilde{\boldsymbol{\lambda}}_R(\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\boldsymbol{\lambda}'$$

subject to:

$$\tilde{\mathbf{Q}}^E\mathbf{N}\boldsymbol{\lambda}' \geqslant 1\,, \qquad (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\boldsymbol{\lambda}' \geqslant \mathbf{0}\,, \qquad \boldsymbol{\lambda}' \geqslant \mathbf{0} \tag{6.7}$$

In contrast to s_p, the safety factor s_L clearly depends on loading history, dislocations and preexisting selfstresses.

However a "radial" loading path is implied by the amplification of the given external action through the common factor k. This path often gives rise to regularly progressive yielding in Hodge's sense: every yield plane, if activated, does not loose contact with the stress point (i. e. no local unloading). If this happens, the actual structural response for increasing k is defined by a sequence of *holonomic* solutions $\boldsymbol{\lambda}^o(k)$ such that $\boldsymbol{\lambda}_2{}^o(k_2) \geqslant \boldsymbol{\lambda}_1{}^o(k_1)$ if $k_2 > k_1$: then if $\boldsymbol{\lambda}_1{}^o(k_1)$ violates inequality(6.5), any $\boldsymbol{\lambda}_2{}^o(k_2)$ for $k_2 > k_1$ will violate it as well. Therefore if (6.5) forms a constraint system together with all the relations (3.7) (3.12) which govern the holonomic solutions, and if k is maximized subject to such a system, the maximum represents s_L, i. e. denotes either a plastic collapse or a local failure situation.

This conclusion can be expressed as follows, by introducing the slack variable vector $\mathbf{v}$:

(XVII) *If plastic yielding progresses regularly throughout as the factor k increases, the safety factor s_L is provided by the programming problem in the variables k, $\boldsymbol{\lambda}$, $\boldsymbol{\varphi}$, $\mathbf{v}$:*

maximize $k \geqslant 0$ *subject to:*

$$k\tilde{\mathbf{N}}\mathbf{Q}^E - (\mathbf{H} - \tilde{\mathbf{N}}\mathbf{Z}\mathbf{N})\boldsymbol{\lambda} - \boldsymbol{\varphi} = \mathbf{K}\,, \qquad \boldsymbol{\lambda} + \mathbf{v} = \boldsymbol{\lambda}_R \tag{6.6}$$

$$\boldsymbol{\varphi} \leqslant \mathbf{0}\,, \qquad \boldsymbol{\lambda} \geqslant \mathbf{0}\,, \qquad \mathbf{v} \geqslant \mathbf{0}\,, \tag{6.7}$$

$$\tilde{\boldsymbol{\varphi}}\boldsymbol{\lambda} = 0 \tag{6.8}$$

Like all results of the deformation theory, the above statement is to be applied with caution. If the hypothesis is not verified the maximization (XVII) furnishes merely an upper bound for s_L, anyway a better one in general than that supplied by the above linear programs. However, computational experience shows, as is well known, that in most practical cases under proportionally increasing loads, "local unloading" either does not intervene or has little influence. It is reasonable to expect that the higher the workhardening coefficients H_{rr} the more likely this is to happen. However no restrictive hypothesis on matrix $\mathbf{H}$ was needed for (XVII).

The complementarity condition (6.8) make the programming problem nonlinear and its feasible domain generally not convex. Nevertheless the problem is only marginally nonlinear: it has been pointed out in [31] for a special class of cases and in [32] for a general case including the present one, that an optimization of this kind can be achieved by means of a numerical procedure essentially founded on Dantzig's simplex technique, implemented in each pivotal step by an additional rule which reflects Eq. (6.8).

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