ON THE THERMODYNAMIC POTENTIAL FOR CONTINUUMS WITH REVERSIBLE TRANSFORMATIONS - SOME POSSIBLE TYPES

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One of the most interesting problems in the mechanics of continuums with reversible transformations is that of the determination of the thermodynamic potential.

It presents two aspects, the physical one of the collecting of information from experiment and the mathematical one of the construction of an analytic expression compatible with the experimental indications.

These latter are in general largely insufficient. Hence the experimenter, in order to be able to work, must needs presuppose a certain type of analytic structure; then he does no more than proceed to the determination of the numerical values of the parameters on which the potential — in its presupposed form — depends.

The fundamental variables having been established, it often seems natural to assume as potential the expression constituted by the first terms of one of its expansions in series, but by doing this, certain fundamental properties which the potential must observe are verified only in an approximate fashion and without any idea being had of the error committed.

Of greater interest, on the other hand, seems to me the establishing beforehand of a certain type of analytic structure and the determining — if it exists — of the *exact* expression of the potential on the hypothesis that it has that structure. The criterion to conform to can be that of analytic simplicity, or also something different, as for example that of satisfying certain requisites in regard to the propagation of the waves (1).

By speaking of the *exact* expression of the potential I intend to express the fact that it verifies all the conditions that reasonably can and must be imposed a priori on the potential.

Here we shall adhere to the idea of analytic simplicity, after having described exactly — with reference to the case of finite deformations of homogeneous and isotropic elastic bodies without inner constraints — the principal requisites to which the thermodynamic potential must conform in a form that under a certain aspect is less restrictive — and more synthetic — than the habitual one, and showing that with the selection of suitable variables the Eulerian stress can assume simple expressions or outright linear ones in such variables (which, however, are not linear in the derivatives of the components of displacement).

For pure simplicity I shall consider only the isothermic case, it being well known, besides, how to adapt it to include the adiabatic case, even where finite deformations are concerned.

1. Fundamental remarks and premisses.

Let C' be the actual configuration of a continuum and C a configuration of reference which for the sake of simplicity we shall suppose to be in natural equilibrium, that is, exempt from stress.

With respect to a rectangular Cartesian frame I shall denote by x_r the coordinates of the generic point (element) P' of C', by y_r those of its correspondent P in C. Let

$$x_r = x_r(y_1, y_2, y_3; t) \tag{1}$$

be the analytic representation of the transformation between C and C′ which we suppose to be completely reversible and endowed with the well known conditions of regularity and possibly dependent on the time.

The transformation (1) requires the existence of certain fundamental matrices for the study of the behaviour of the continuum. Among them are those which I shall indicate by ε, $\varepsilon^{(t)}$, ε_p, whose elements are, respectively,

$$\begin{cases} \varepsilon_{rs} = \dfrac{1}{2}\left[\dfrac{\partial u_r}{\partial y_s} + \dfrac{\partial u_s}{\partial y_r} + \sum_i \dfrac{\partial u_i}{\partial y_r}\dfrac{\partial u_i}{\partial y_s}\right], \\ \varepsilon_{rs}{}^{(t)} = -\dfrac{1}{2}\left[\dfrac{\partial u_r}{\partial x_s} + \dfrac{\partial u_s}{\partial x_r} - \sum_i \dfrac{\partial u_i}{\partial x_r}\dfrac{\partial u_i}{\partial x_s}\right], \\ \varepsilon_{rs}{}^{(p)} = \dfrac{1}{2}\left[\dfrac{\partial u_r}{\partial y_s} + \dfrac{\partial u_s}{\partial y_r} + \sum_i \dfrac{\partial u_r}{\partial y_i}\dfrac{\partial u_s}{\partial y_i}\right] \end{cases} \tag{2}$$

where the u_r are the components of the displacement PP'.

We shall call ϱ the matrix characteristic of the local rotation and a the matrix $\left| \partial x_s / \partial y_r \right|$; it will be useful for the sequel to record the decomposition

$$a = \varrho\delta \tag{3}$$

where δ is a pure deformation connected with the matrix ε by the relation

$$\delta^2 = 1 + 2\varepsilon. \tag{4}$$

We shall denote by

$$X \equiv | X_{rs} |, \qquad Y \equiv | Y_{rs} | \tag{5}$$

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(1) To the first criterion conforms the elasticity of second grade of A. Signorini, *Trasformazioni termoelastiche finite*, Memoria II, Annali di Matematica pura e applicata, Serie IV, 30, 1-72, 1949. To the second, that of C. Tolotti, *Deformazioni elastiche: onde ordinarie di discontinuità e casi di solidi elastici isotropi*, Rend. Mat. Appl., Serie V, 4, 34-59, 1943. See also A. Bressan, *Sulla propagazione delle onde ordinarie di discontinuità nei sistemi a trasformazioni reversibili*, Rend. Sem. Mat. University of Padova. V. 33, 99-139, 1963.

the Eulerian and Lagrangian matrices of stress, connected by the relation

$$X = \frac{1}{D} a Y \bar{a} \tag{6}$$

where $\bar{a}$ indicates the transposed matrix of a, and we have set

$$D = \det \delta = \det \sqrt{1 + 2\varepsilon} = \det a. \tag{7}$$

We shall suppose the continuum to be of a simple type, that is, capable of reacting only to contact forces but not to couples, and the matrices (5) to be symmetrical.

For such a continuum the free energy I — the Helmholtz thermodynamic function — depends uniquely on the ε_{rs} and on the temperatures T, T' of C, C', as well as on the y_i in the case of absence of homogeneity.

I shall concern myself only with isothermic transformations, remarking only that a well known procedure reduces the adiabatic to these. Thus it will suffice to consider, rather than the free energy, the isothermic potential W that as a function of the ε_{rs} is defined by the equation

$$W(\varepsilon_{rs}) = I(\varepsilon_{rs}, T, T) - I(0, T, T), \tag{8}$$

where on the left hand side is no longer evidenced the dependence on T, inessential for the sequel.

It is well known that while the Y_{rs} can be expressed as functions of the ε_{rs}, the X_{rs} depend on the contrary, by reason of (6), also on the local rotation.

If the body is isotropic, as we shall suppose, W depends on the ε_{rs} (or on the $\varepsilon_{rs}^{(i)}$, or on the $\varepsilon_{rs}^{(\rho)}$) only by way of the three principal invariants of the matrix ε (or of $\varepsilon^{(i)}$, or of $\varepsilon^{(\rho)}$, or of three invariants with which they are in one to one correspondence. The matrices $\varepsilon, \varepsilon^{(\rho)}$ are connected by the relation $\varepsilon^{(\rho)} = \varrho\varepsilon\varrho^{-1}$ and have the same principal invariants, as well as the same principal coefficients which I shall indicate by E_1, E_2, E_3. In the sequel I shall refer to the three invariants I_1, I_2, D that as functions of E_1, E_2, E_3 have the expressions

$$\begin{cases} I_1 = \sum_{i=1}^{3} E_i, \qquad I_2 = \sum_{i=1}^{3} E_i E_{i+1} \\ D = \sqrt{(1 + 2E_1)(1 + 2E_2)(1 + 2E_3)} \end{cases} \tag{9}$$

In the case of a system with reversible transformations, isotropic, the Eulerian stress X_{rs} can be expressed in the form of a second degree polynomial in the characteristics of deformation, $\varepsilon_{rs}^{(i)}$, of the inverse displacement, with coefficients dependent only on the principal invariants or, also, as a second degree polynomial in the $\varepsilon_{rs}^{(\rho)}$, with coefficients dependent only on I_1, I_2, D.

We have, to be precise, the relations

$$X_{rs} = -\frac{1}{D^{(\rho)}} \left[l\delta_{rs} + 2m\varepsilon_{rs}^{(\rho)} + n\varepsilon_{ri}^{(\rho)}\varepsilon_{is}^{(\rho)} \right] \tag{10}$$

where δ_{rs} denotes Kronecker's symbol and we have set

$$\begin{cases} l = D^{(\rho)} \dfrac{\partial W}{\partial D^{(\rho)}} + \dfrac{\partial W}{\partial I_1^{(\rho)}} + I_1 \dfrac{\partial W}{\partial I_2^{(\rho)}} \\ m = \dfrac{\partial W}{\partial I_1^{(\rho)}} + \left(I_1^{(\rho)} - \dfrac{1}{2} \right) \dfrac{\partial W}{\partial I_2^{(\rho)}} \\ n = -2 \dfrac{\partial W}{\partial I_2^{(\rho)}}, \end{cases} \tag{11}$$

equivalent to those of *Finger*. The index ϱ indicates that the invariants are considered as expressed by means of the $\varepsilon_{rs}^{(\rho)}$, in particular by means of the $E_i^{(\rho)}$, according to (9).

2. Some considerations concerning the thermodynamic potential. A fundamental condition.

I shall start to make some observations in relation to the analytic structure of W, first of all calling attention to some of the fundamental requirements that are customarily attributed to it.

Supposing the solid to be isotropic, homogeneous and elastic, and hence W a function of the $\varepsilon_{rs}^{(\rho)}$ by way of the invariants I_1, I_2, D, it is usual to require the thermodynamic potential to satisfy the following conditions:

a) For every non-rigid displacement starting from C it results that

$$W > 0$$

b) In uniform traction (compression) the linear coefficient of dilation is positive (negative);

c) In the simple extension of a cylindrical body the stress on a transverse section is an increasing function of the elongation per unit length;

d) In the problem mentioned in *c*) the longitudinal lengthening increases more rapidly than the corresponding tension;

e) W tends towards infinity if and only if at least one of the E_i tends towards $-\frac{1}{2}$ or towards ∞.

In the isotropic case W can indifferently be considered as a function of I_1, I_2, D or a symmetrical function of E_1, E_2, E_3, or also of the principal elongations Δ_1, Δ_2, Δ_3, connected to the E_i by the equations

$$1 + \Delta_i = \sqrt{1 + 2E_i}, \quad (i = 1, 2, 3). \tag{12}$$

In a three-dimensional Euclidean space referring to a rectangular Cartesian frame let Q be the point whose coordinates are Δ_1, Δ_2, Δ_3. The properties enunciated are usually assumed in consideration of the variation of Q in the three-dimensional region V defined by the limitations

$$-1 < \Delta_i < \infty, \qquad (i = 1, 2, 3). \tag{13}$$

Otherwise it is evident that no natural body is elastic in all V. One thinks, apart from anything else, of the limit of plasticity which defines a threshold of elasticity. For example, in the theory of Von Mises it imposes on a quadratic form of the stress the condition of not reaching a certain experimental value. It therefore seems reasonable to impose the conditions *a*), *b*), *c*), *d*) not in all of V but in a three-dimensional region V' contained in V and containing the point O, reducing the condition *e*) to the requirement that W remains finite in all of V'.

Thus is enlarged the possibility of selection of the functions W and we can presume that the conditions enunciated — if verified in a region V' of the type considered — are sufficient to ensure that any natural body admits a given W as elastic potential.

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It is worth observing that the requirements *a*), *b*), *c*) can be derived from a single global condition. In order to achieve this I begin with the observation that the surface tensions ascribed to the state of reference are characterized by the matrix

$$t = -aY = -\varrho\delta Y. \tag{14}$$

In every problem of homogeneous deformations the matrices a, δ, Y do not depend on the y_r and hence the same applies to the local rotation that can be assumed equal to the identity. Consequently (14) gives

$$t = -\delta y. \tag{15}$$

Granted the hypothesis of isotropy, the matrices ε, δ, Y have the same united directions and the same applies for t which like δ and Y does not depend on the y_r. On the grounds of (15), the principal coefficients, T_r, of t are expressed by the equalities

$$T_r = -\sqrt{1+2E_r}\,Y_r = \sqrt{1+2E_r}\,\frac{\partial W}{\partial E_r} = \frac{\partial W}{\partial \Delta_r}. \tag{16}$$

In the state of reference, represented by the point $O \equiv (0,0,0)$, W can be assumed to be equal to zero and, on the contrary, C being a state of natural equilibrium, we have furthermore

$$\frac{\partial W}{\partial E_r} = \frac{\partial W}{\partial \Delta_r} = 0, \quad (\text{when } \Delta_1 = \Delta_2 = \Delta_3 = 0) \tag{17}$$

There exists the theorem: *A sufficient condition for the requirements a*), *b*), *c*) *to be verified in a threedimensional region* V' *contained in* V *and containing* O *is that in every point of* V' *the quadratic form*

$$A = \sum_{i,h=1}^{3} \frac{\partial^2 W}{\partial \Delta_i \partial \Delta_h}\,\xi_i \xi_h \tag{18}$$

be definite positive.

Let l be any curve of V' emerging from O and s a system of abscissas on l with the origin in O. Let l be such that to every value of s there corresponds a single configuration of the body and such that every one of its possible configurations, on the contrary, corresponds at most to one point of l. Given l, there comes to be defined a (continuous) succession of configurations in correspondence to which we have

$$\Delta_i = \Delta_i(s). \tag{19}$$

It is supposed that in every point of l exist the first and second derivatives of the $\Delta_i(s)$ with respect to s and, on the other hand, that the first derivatives of Δ_i are never simultaneously equal to zero. On l furthermore

$$\sum_{i=1}^{3} \frac{\partial W}{\partial \Delta_i}\,\Delta_i'' = 0, \tag{20}$$

where the apex denotes derivation with respect to s. For example, every straight line emerging from O satisfies the conditions mentioned.

On l it results that

$$\frac{dW}{ds} = \sum_{i=1}^{3} \frac{\partial W}{\partial \Delta_i}\,\Delta_i', \tag{21}$$

$$\frac{d^2W}{ds^2} = \sum_{i,h=1}^{3} \frac{\partial^2 W}{\partial \Delta_i \partial \Delta_h}\,\Delta_i' \Delta_h' \tag{22}$$

and, A being by hypothesis positive definite, we have

$$\frac{d^2W}{ds^2} > 0. \tag{23}$$

From this it follows that $\frac{dW}{ds}$ is an increasing function of s and becomes equal to zero only in O, assuming elsewhere the sign of s. Since every point of V' is attainable by a curve l — for example, with a segment of a straight line — and W is equal to zero at O, we can conclude that requirement *a*) is certainly satisfied in V'.

Furthermore, it is clear that the three derivatives of W with respect to Δ_i cannot simultaneously be equal to zero. The same circumstance arises in consequence for the Eulerian principal tensions which are proportional to them, and — in a problem of homogeneous deformations — for the three T_r.

In the problem of uniform traction or pressure we have

$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta, \qquad T_1 = T_2 = T_3 = T \tag{24}$$

If we assume as the curve l the straight line of the equations

$$\Delta_1 = \Delta_2 = \Delta_3 = s \tag{25}$$

from (16), (21) it follows that

$$T = \frac{\partial W}{\partial \Delta_1} = \frac{\partial W}{\partial \Delta_2} = \frac{\partial W}{\partial \Delta_3} = \frac{dW}{3ds}. \tag{26}$$

This signifies that T is an increasing function of s, that is, of the value Δ common to the three principal elongations, and has the same sign as they have: requirement *b*) is then satisfied.

In the simple traction of a cylindrical body with generatrices parallel to the axis of index three, T_1, T_2 are equal to zero and we obtain, therefore,

$$\frac{\partial W}{\partial \Delta_1} = \frac{\partial W}{\partial \Delta_2} = 0 \tag{27}$$

The curve l of equations

$$\Delta_1 = \Delta_2 = \varphi(s), \qquad \Delta_3 = s \tag{28}$$

verifies condition (20). We have, furthermore,

$$T_3 = \frac{\partial W}{\partial \Delta_3} = \frac{dW}{ds} \tag{29}$$

which by reason of (16), (27) gives

$$\frac{dT_3}{d\Delta_3} > 0, \tag{30}$$

From this it follows that T_3 increases with Δ_3 and has the same sign as it. Requirement *c*) is therefore verified and the theorem enunciated is proved.

Requirement *d*) is not, on the contrary, a consequence of the conditions imposed on the quadratic form (18). It requires that

$$\frac{d^2T_3}{d\Delta_3^2} < 0 \qquad \text{when } \Delta_3 > 0 \tag{31}$$

which on the grounds of (28), (29) is equivalent to

$$\frac{d^3W}{ds^3} < 0 \qquad \text{when } s > 0 \tag{32}$$

on the curve of equations (28).

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The condition that the quadratic form (18) is positive definite, associated with (32), imposes on the potential W a behaviour in a certain sense consonant with the same requirement *e*), within V'. In reality the condition that A is positive definite is superabundant with respect to the requirements *a*), *b*), *c*), for the verification of which it is only sufficient, but it is worth while to illustrate its significance in relation to any deformations whatsoever, even if not homogeneous. Setting

$$\bar{t} = \varrho^{-1} t \qquad (33)$$

from (14) it follows that

$$\bar{t} = -\varrho^{-1} a y = -\delta y. \qquad (34)$$

$\bar{t}$ is, therefore, symmetrical and has the same united directions as δ and y. The principal coefficients, $\bar{T}_r$, are expressed by (16) and therefore on every line l emerging from O and described by the point

$$Q \equiv [\Delta_1(s), \Delta_2(s), \Delta_3(s)]$$

it results that

$$\frac{dW}{ds} = \sum_{i=1}^{3} \bar{T}_i \Delta_i'. \qquad (35)$$

If l verifies condition (20), then (22) is valid. From this it is deduced that the condition that the quadratic form (18) be positive definite is necessary and sufficient in order that, subordinately to condition (20), the scalar

$$M = \sum_{i=1}^{3} \bar{T}_i \Delta_i' \qquad (36)$$

be an increasing function of s and have the same sign as it.

We have in reality been dealing with a requirement of the same type, but in a more general form, as that indicated in the case of problems of uniform traction and simple extension, which satisfy it as particular cases.

This encourages the idea of substituting the ensemble of conditions *a*)... *e*) with the following *fundamental property: An expression of the thermodynamic potential W is acceptable from the mathematical point of view if there exists a region V' of V, containing* 0, *such that in every one of its points the quadratic form A proves to be positive definite and, furthermore,* (32) *is verified on every curve of equations* (28).

It is worth observing that if the *fundamental property is* verified at O where, in fact, it is necessary as well as sufficient for the verification of conditions *a*)... *d*), reasons of continuity imply the existence of a region V' contained in V and containing O where the specified requirement is verified. Furthermore, it is evident that the admitted reasons of continuity show the necessity of the existence of a region V' of the type considered where the *fundamental property* is satisfied.

After what has been said, it can also be affirmed that *the necessary and sufficient condition for the fundamental property to be verified in a certain region V' of the type considered is that it maintains at* O.

3. On possible types of thermodynamic potential.

The complexity of the analytic problems of the theory of finite deformations induces us to look for expressions of W which lead to expressions of the stresses that are not too complicated, even if, in such a case, the small number of the coefficients on which the potential comes to depend limits its validity as regarding the variation of natural bodies and the increase in their deformations.

The Eulerian matrix of the stress (5,1) is the one which presents the greatest practical interest and it is therefore natural to wonder, in relation to the possibility of expressing it by means of the characteristics $\varepsilon_{rs}^{(i)}$ of the inverse displacement, if the X_{rs} can depend linearly on the $\varepsilon_{rs}^{(i)}$. The answer, as is known, is in the negative. Signorini (2) has demonstrated that in the hypothesis of linearity we are not able to satisfy condition *a*) for the variation of $Q^{(i)} \equiv (E_1^{(i)}, E_2^{(i)}, E_3^{(i)})$ in the region $V^{(i)}$ defined by the limitations $-\frac{1}{2} < E^{(i)}_s < \infty$. It is worth observing that even with less restrictive conditions — analogously to what was said in the preceding section — such as accepting an expression of the thermodynamic potential provided that it satisfies conditions *a*)... *e*) in a more restricted region of V in the neighbourhood of $Q^{(i)} \equiv (0, 0, 0)$, we find the impossibility described by Signorini. In fact, the hypothesis of linearity concerned implies for W the expression (3)

$$W = \frac{\mu}{D^{(i)}} (I_1^{(i)} + 1) - \mu \quad , \quad (\mu > 0), \qquad (37)$$

where $D^{(i)}$ and $I_1^{(i)}$ are deduced from (9) by the substitution of $E_s^{(i)}$ for E_s and μ is a constant.

As a result of (37) we recognise easily that however small the neighbourhood of the origin choseu it is not possible to satisfy the fundamental requirement enunciated, in particular the condition that A be positive definite (4). In fact, more simply, it is sufficient to observe that in correspondence with the non-rigid displacement characterized by the equalities $E_1^{(i)} = E_2^{(i)}$, $E_3^{(i)} = 0$, (37) gives $W = 0$.

It seems to me still more interesting to observe that we can have a linear Eulerian stress, but in the variables $\varepsilon_{rs}^{(i)}$. In such a case we get (5)

$$W = \mu D (I_1^{(\rho)} - 1) + \mu, \quad (\mu > 0), \qquad (38)$$

I mean to say that the stress deriving from the expression (38) of W on the grounds of (10), (11) is linear in the $\varepsilon_{rs}^{(\rho)}$ and that there exists a region V' contained in V and containing O where W verifies the fundamental property. For the moment I renounce the examination of the expression (38) of W but observe that it can be considered as a particular case of a W corresponding to an Eulerian stress

(2) Signorini A., Loc. cit.: note (1), p. 34.

(3) Expression (37) is an immediate consequence of what is said in loc. cit. in note (2). See also G. Grioli, *Mathematical Theory of Elastic Equilibrium* (*Recent Results*), Ergebnisse der Angewandten Mathematik, 7. Springer-Verlag 1962, p. 24.

(4) With reference to the impossibility that the Eulerian stress is linear in $\varepsilon_{rs}^{(i)}$, P. G. Bordoni has demonstrated the possibility that the X_{rs} are expressed by the product of linear functions of the $\varepsilon_{rs}^{(i)}$ by a same function of the invariants $I_1^{(i)}$, $D^{(i)}$: *Sopra le trasformazioni termostatiche finite di certi solidi omogenei ed isotropi*, Rend Mat. pura e appl. V. XIII, S.V., 237-266, 1953.

(5) In the case of incompressible bodies (38) reduces to the expression proposed by D. C. Treloar, *The elasticity of a network of long chain molecules*, Trans Faraday SOC. 39, 1943, pp. 36-41 and 241-246.

expressed by the product of a power of D for a polynomial in the $\varepsilon_{rs}^{(\rho)}$. In particular, I shall consider the problem of a stress of the type

$$X_{rs} = D_{(\rho)}{}^{p-1} P_2(\varepsilon_{rs}{}^{(\rho)}), \tag{39}$$

where $P_2(\varepsilon_{rs}{}^{(\rho)})$ is a second degree polynomial in the $\varepsilon_{rs}{}^{(\rho)}$, while p is any real number or zero.

On the grounds of (10), (11), for (39) to be valid, it must be true that

$$\begin{cases} l = D_{(\rho)}^{\ p}(a + bI_1{}^{(\rho)} + cI_1{}^{(\rho)2} + dI_2{}^{(\rho)}) \\ m = D_{(\rho)}^{\ p}(e + fI_1{}^{(\rho)}) \\ n = gD_{(\rho)}^{\ p} \end{cases} \tag{40}$$

where a, b, ..., g are constants.

From the combination of (11) and (40) it follows that

$$\begin{cases} \dfrac{\partial W}{\partial I_1{}^{(\rho)}} = D_{(\rho)}^{\ p}\left[e - \dfrac{g}{4} + \left(f + \dfrac{1}{2}g\right) I_1{}^{(\rho)}\right] \\ \dfrac{\partial W}{\partial I_2{}^{(\rho)}} = -\dfrac{g}{2} D_{(\rho)}^{\ p} \\ \dfrac{\partial W}{\partial D^{(\rho)}} = D_{(\rho)}^{\ p-1}\left[a + \dfrac{g}{4} - e + (b - f)\, I_1{}^{(\rho)} + \right. \\ \qquad\qquad\qquad \left. + cI_1{}^{(\rho)2} + dI_2{}^{(\rho)}\right] \end{cases} \tag{41}$$

which imply the conditions of integrability

$$\begin{cases} g = -\dfrac{2d}{p}, \quad f = b - ep - \dfrac{d}{2}, \\ c = \dfrac{1}{4}\,[2p(b - ep) - d(2 + p)] \end{cases} \quad \text{if} \ \ p \neq 0, \tag{42}$$

or

$$q - f = 0,\ c = 0, \qquad d = 0,\ \text{if}\ p = 0. \tag{43}$$

I shall consider first of all the case $p \neq 0$.

Setting

$$\nu = e + \frac{d}{2p}, \qquad a = \frac{1}{4}\left[2(b - ep) - \frac{2d}{p} - d\right], \tag{44}$$

(41), integrated on the hypothesis that C is a state of natural equilibrium, give for W the expression

$$W = D_{(\rho)}^{\ p}\left[\nu\left(I_1 - \frac{1}{p}\right) + aI_1{}^{(\rho)2} + \frac{d}{p} I_2{}^{(\rho)}\right] + \frac{\nu}{p} \tag{45}$$

With some calculations, taking (12) into account, we find that in $O \equiv (0, 0, 0)$ we obtain

$$\frac{\partial^2 W}{\partial \Delta_i \partial \Delta_h} = \lambda + 2\mu\delta_{rs}, \tag{46}$$

where δ_{rs} is Kronecker's symbol and we have set

$$\lambda = 2a + p\nu + \frac{d}{p}, \qquad \mu = \nu - \frac{d}{2p} \tag{47}$$

It is easily verified that the necessary and sufficient condition for the quadratic form (18) to be positive definite in O is that the coefficients λ and μ (of Lamé) verify the well known conditions

$$\mu > 0, \qquad 3\lambda + 2\mu > 0. \tag{48}$$

From (47) it follows that

$$d = 2p(\nu - \mu), \qquad a = \frac{\lambda + 2\mu - (2 + p)\nu}{2} \tag{49}$$

and the expression (45) of W becomes [6]

$$W = D_{(\rho)}^{\ p}\left\{\frac{\lambda + 2\mu - (2 + p)\nu}{2} I_1{}^{(\rho)2} + 2(\nu - \mu)I_2{}^{(\rho)} + \right. \\ \left. + \nu\left(I_1{}^{(\rho)} - \frac{1}{p}\right)\right\} + \frac{\nu}{p}. \tag{50}$$

After some expansion it is seen that all the conditions imposed by the fundamental property are verified in O, *whatever are the values of* ν *and* $p \neq 0$. We conclude that there exists a region V' of the type considered where for W expressed by (50) the fundamental property is verified, whatever are the values of ν and if $p \neq 0$. This assures the *mathematical* acceptability of W. It is clear, on the other hand, that for $\nu = 0$, W satisfies condition a) in the whole region V.

With $\nu = \mu, p = -1$, (50) becomes

$$W = \frac{1}{D^{(\rho)}}\left[\frac{\lambda + \mu}{2} I_1{}^{(\rho)2} + \mu(I_1{}^{(\rho)} + 1)\right] - \mu, \tag{51}$$

formally analogous (but substantially different) to the expression that we have in the case of an Eulerian stress depending on two parameters in the characteristics of deformation of the inverse displacement [7]. In such a case it is seen that the necessary and sufficient condition in order for $W > 0$ in the whole of V is that we have

$$9\lambda + 5\mu > 0, \qquad \mu > 0, \tag{52}$$

slightly more restrictive than (48). Conditions (52) coincide with those necessary and sufficient for the acceptability of the theory depending on two parameters of an Eulerian stress of the second degree in the characteristics of deformation of the inverse displacement, already established by *Signorini*, but the greater restriction of the conditions on λ and μ are due to the search for a potential which is acceptable in the entire region V.

In so far as concerns the dependence of the expression (50) of W on p, we observe that if $p > 0$, W tends toward zero in accordance with the tendency of even a single one of the E_i towards $-\dfrac{1}{2}$. This indicates that if $p > 0$ the acceptability of the expression (50) of W is limited to a region V' in which the Δ_i are greater than quantities greater than -1, otherwise it is not certain that the fundamental property is verified. We can observe besides that an analogous fact holds where we look for a polynomial Eulerian stress in the $\varepsilon_{rs}{}^{(\rho)}$, of a whole degree however high. In this case we find for W an expression of the type

$$W = D^{(\rho)} P(I_1{}^{(\rho)}, I_2{}^{(\rho)}) + \mu, \tag{53}$$

where $P(I_1, I_2)$, is a whole degree polynomial in I_1, I_2 and μ is a constant. This strengthens the idea that the condition that the thermodynamic potential should satisfy the requirements a)... e) in the whole V, in particular e), is too restrictive and also superfluous in practice.

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[6] If we compel in (50) the constant ν to reduce to zero the coefficient of $I_1{}^{(\rho)2}$ we get the expression for the incompressible bodies as proposed by M. Mooney, *A Theory of Large Elastic Deformations*, J. Appl. Phys. XI, 1940, pp. 582-592.

[7] Signorini, Loc. Cit. in note (1), p. 37.

It is worth observing that for $\nu = 0$, (50) becomes

$$W = D^{p}_{(\rho)} \left[\frac{\lambda + 2\mu}{2} I_1^{(\rho)2} - 2\mu I_2^{(\rho)} \right] \qquad (54)$$

This is the case of an elastic potential that differs by the factor D^p from that of the linearized theory. The same cannot be said, however, for the stress.

On the other hand, for $\lambda = p\mu$, $\nu = \mu$, from (50) we obtain

$$W = \mu D^{p}_{(\rho)} \left[I_1^{(\rho)} - \frac{1}{p} + \frac{1}{p D^{p}_{(\rho)}} \right] \qquad (55)$$

This is now the case of a linear Eulerian stress in the $\varepsilon_{rs}^{(\rho)}$ less the factor $D^{p-1}_{(\rho)}$. In the special case $p = 1$ we find again the expression (38) and we have a linear Eulerian stress in the $\varepsilon_{rs}^{(\rho)}$:

$$X_{rs} = -\mu \, [I_1^{(\rho)} \delta_{rs} + 2\varepsilon_{rs}^{(\rho)}]. \qquad (56)$$

Taking into consideration (12), (16), it becomes clear that in the problem of uniform traction and pressure we have, on the grounds of (38),

$$T = 5\mu \, (1 + \Delta)^2 \frac{(1 + \Delta)^2 - 1}{2}, \qquad (57)$$

while in the case of simple extension the result is

$$\Delta_1 = \Delta_2 = \frac{\sqrt{5 - (1 + \Delta_3)^2}}{2} - 1 \qquad (58)$$

$$T_1 = T_2 = 0,$$

$$T_3 = \frac{5}{16} \mu \, [(1 + \Delta_3)^2 - 1] \, [5 - (1 + \Delta_3)^2] \, . \qquad (59)$$

By analysing the expression (38) of W it becomes clear that requirement *a*) is verified in the entire region V. The same cannot be said for the fundamental property that is decisive for the acceptability of the expression of W that has been found. For the reasons given there certainly exists a V' of the type considered where the specified fundamental property is verified. It is not difficult to recognise that V' has no points external to the region $V_{\prime\prime}$ defined by the limitations

$$\frac{1}{\sqrt{2}} - 1 < \Delta_i < \sqrt{3} - 1 \qquad (60)$$

It is worth observing that if the theory expressed by (38), (56) is linearized, we find the formulas of the classical linear theory of elasticity exactly for the case in which the value of the Poisson coefficient is $\frac{1}{4}$. In other words it can be presumed that for deformations which are not too large and materials having a Poisson coefficient near $\frac{1}{4}$ — quite a concrete case — the theory depending on the thermodynamic potential (38) may prove to be useful.

* * *

I shall dedicate only a few words to the *case* $p = 0$. On such a hypothesis, easy calculations show that we have

$$W = \left(\frac{g}{4} - \mu \right) (\ln D^{(\rho)} - I_1^{(\rho)}) + \\ + \frac{1}{2} \left(\lambda + \frac{g}{2} \right) I_1^{(\rho)2} - \frac{g}{2} I_2^{(\rho)}, \qquad (61)$$

where g, λ and μ are constants. It is possible to demonstrate that the conditions expressed in (48) are also now necessary and sufficient for the existence of a region V' contained in V and containing O where the fundamental property is satisfied. It may be observed that for $g = 4\mu$, (61) reduces — only formally, however — to the classical expression of the linearized theory.

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