COUPLING OF ELASTIC AND PLASTIC DEFORMATIONS OF BULK SOLIDS

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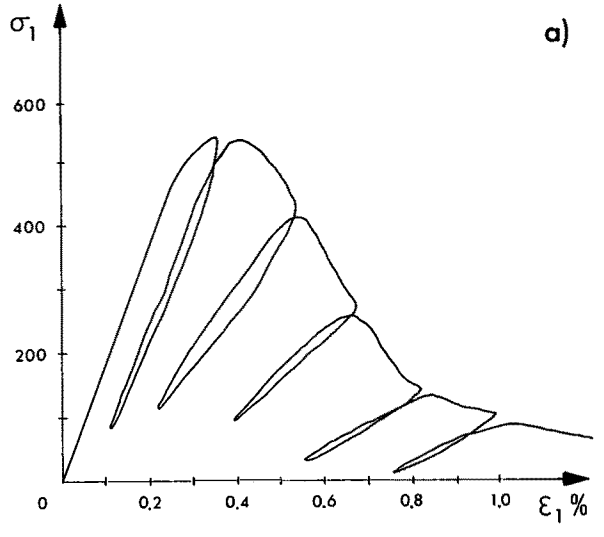
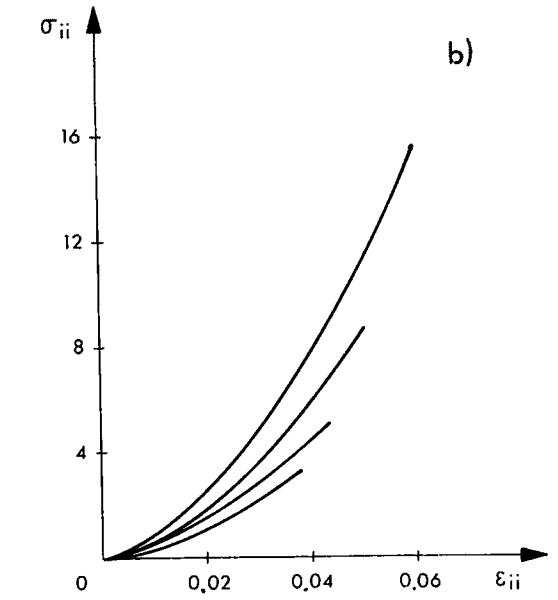
SOMMARIO: Il modulo di scarico elastico di solidi come le rocce, i suoli, i materiali ceramici e granulari dipende dalla entità delle deformazioni plastiche. Le conseguenze di questo fenomeno, chiamato nel seguito accoppiamento elastoplastico, vengono esaminate sia in relazione alla parte elastica che alla parte plastica della legge costitutiva. In particolare si dimostra che l'accoppiamento elastoplastico determina la non-normalità della legge di scorrimento plastico. La deviazione della normalità legata alla variazione del modulo elastico è studiata per mezzo di un potenziale di accoppiamento derivato da quello elastico.

SUMMARY: In such solids like rocks, soils, ceramics, grain en masse the plastic deformation strongly aects the current unloading modulus. The consequences of this effect referred to as the elastoplastic coupling both to the elastic and the plastic part of the constitutive law are examined. Particularly, it appears that such phenomenon induces a specific kind of the non-normality in the plastic flow law. The departure from the normality is studied in connection with the form of the elastic modulus variation basing on the notion of a coupling potential.

1. Introduction.

The phenomenon referred to as the elasto-plastic coupling is meant as the variation of the elastic unloading moduli caused by plastic deformations. This effect is characteristic for behaviour of such bulk materials as rocks, soils, ceramics, powders, grains etc. [1-8]. Fig. 1a shows a typical result of uniaxial stress compression of a rock [1],





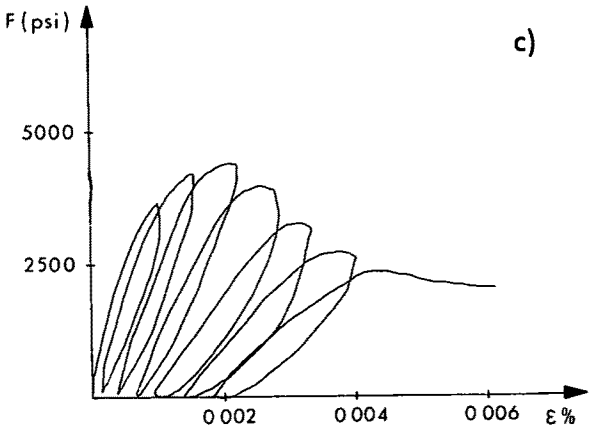


Fig. 1

and Fig. 1b the uniaxial strain compression of a wheat grain [5], while Fig. 1c presents a plot for the confined compression of concrete in the triaxial test. The fact of alternation of elastic moduli is sometimes connected with development of such microdefects like an irreversible microcracking in rocks and ceramics, or opening and closing of pores in granular and porous materials [6, 7, 8]. An interesting study of the influence of cracks on elasticity moduli of rock beams is given in [3]. For granular media the modulus variability has been studied theoretically and experimentally in [9] and [5]. The coupling through the plastic volumetric strain affecting elasticity moduli has been imple-

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mented into a constitutive incremental law for granular materials in [10].

This paper deals with a proposal of a consequent incorporation of the aforementioned phenomenon in the framework of the theory of plasticity in the broader context. The central role plays (sec. 2) the assumption that material is hyperelastic in its elastic behaviour, but the elastic potentials are additionally the functions of a plastic strain parameter. The consequences to the both parts of the constitutive relation [elasticity (sec. 3) and plasticity (sec. 4)] are examined in following sections. The special attention is focused on a scalar coupling (sec. 5); particular cases of plastic models are then reviewed in this context.

2. Basic relations.

We shall consider an elastic-plastic material in the absence of thermal effects under the assumptions of homogeneity, isotropy and small deformations. The strain tensor ε_{ij} is supposed to decompose into an elastic, ε^e_{ij} and plastic, ε^p_{ij} parts.

The central constitutive novelty consists of the assumption, that since plastic strains modify the elastic properties, the latter are no more uniquely defined for the material itself but are assigned to its current plastically deformed configuration.

Thus the elastic stiffness tensor may be written as follows:

$$E_{ijkl} = E_{ijkl}(\varepsilon^e_{ij}, \mu_{ij}\,[\varepsilon^p_{kl}]) \tag{2.1}$$

while its inverse, the elastic compliance tensor correspondingly:

$$C_{ijkl} = C_{ijkl}(\sigma_{ij}, \mu_{ij}\,[\varepsilon^p_{kl}]) \tag{2.2}$$

Let $f = 0$ be a yield locus which defines the elastic domain $f \leqslant 0$. Assume, that the material response in this domain is hyperelastic and is uniquely determined through the elastic strain potential U or stress potential V. We postulate that the both potentials are homogeneous functions of strain and stress respectively and moreover depend on the plastic strain parameter $\mu_{kl} = \mu_{kl}(\varepsilon^p_{ij})$.

$$U = U(\varepsilon^e_{ij},\ \mu_{kl}\,[\varepsilon^p_{ij}]), \tag{2.3}$$

$$V = V(\sigma_{ij},\ \mu_{kl}\,[\varepsilon^p_{ij}]). \tag{2.4}$$

According to the definition of hyperelastic solids, stress and elastic strain are the gradients of corresponding potentials U and V:

$$\sigma_{ij} = \sigma_{ij}\,[\varepsilon^e_{kl},\ \mu_{kl}(\varepsilon^p_{ij})] = \frac{\partial U}{\partial \varepsilon^e_{ij}} \tag{2.5}$$

$$\varepsilon^e_{ij} = \varepsilon^e_{ij}\,[\sigma_{ij},\ \mu_{kl}(\varepsilon^p_{ij})] = \frac{\partial V}{\partial \sigma_{ij}} \tag{2.6}$$

The new variable $\mu_{kl}(\varepsilon^p_{ij})$ is the passive variable of U and V, with respect to Legendre transformation, see e.g. [11], (variables ε^e_{ij} and σ_{ij} correspondingly become active variables).

The function U as well as V represents then the one-parameter family of potentials, each of which is uniquely ascribed to a given state of plastic deformation (Fig. 2). A more general assumption could be considered also, by

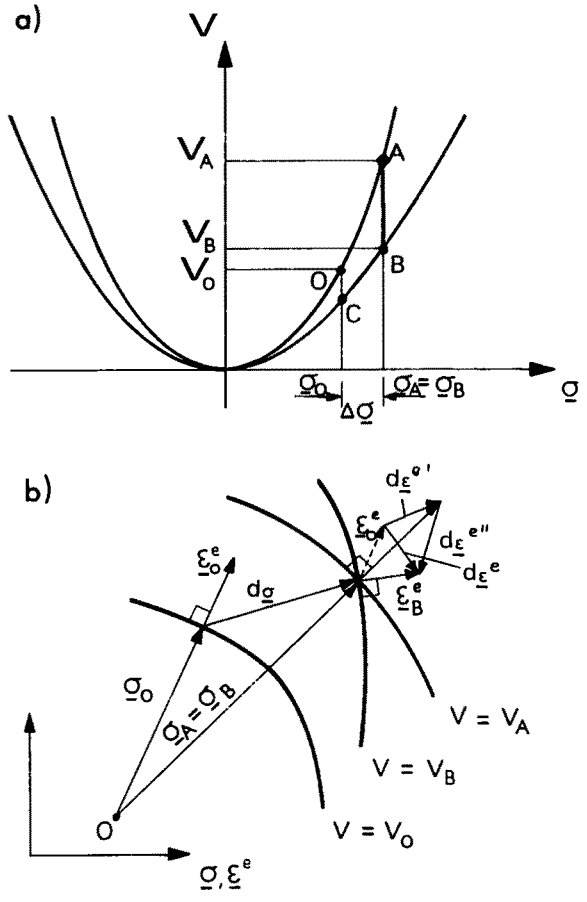

Fig. 2

taking U and V as functionals of ε^p_{ij}, therefore as some functions of the history of plastic strain.

This however, would be against the postulate made relating these potentials to the given plastically deformed configuration, but not to a path on which this was achieved.

The variation in stress about given elasto-plastic situation, at $f = 0$, can give rise both to the elastic unloading or the elasto-plastic loading. If we adopt the standard plasticity assumption of the progression of yield limit:

$$df = \frac{\partial f}{\partial \sigma_{ij}}\, d\sigma_{ij} + \frac{\partial f}{\partial \varepsilon^p_{ij}}\, d\varepsilon^p_{ij} \tag{2.7}$$

then for the former case $df < 0$, while for the latter $df = 0$. The plastic strain increment is assumed to be associated with yield locus and reads:

$$d\varepsilon^p_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}, \tag{2.8}$$

where $d\lambda$ is a plastic multiplier dependent on stress and plastic history

$$d\lambda = \frac{1}{H} \frac{\partial f}{\partial \sigma_{ij}}\, d\sigma_{ij}. \tag{2.9}$$

It takes on the following values:

$$d\lambda > 0, \quad \text{if} \quad f = 0, \quad df = 0,$$
$$d\lambda = 0, \quad \text{if} \quad f < 0; \quad \text{or} \quad f = 0, \quad df < 0. \tag{2.10}$$

H is the hardening modulus, positive if the product $\partial f / \partial \sigma_{ij}\, d\sigma_{ij}$ is so, negative (softening) if the product is negative, and equal to zero if $\partial f / \partial \sigma_{ij}\, d\sigma_{ij} = 0$; in this case $d\lambda$ is indetermined. The last case corresponds to the perfectly plastic behaviour, (critical state in soils and powders and brittle/ductile transition in rocks).

The elastic strain increment becomes the ordinary differential of two tensorial components according to the definition (2.6), (Fig. 2b)

$$d\varepsilon^e_{ij} = \frac{\partial^2 V}{\partial \sigma_{ij}\, \partial \sigma_{kl}}\, d\sigma_{kl} + \frac{\partial^2 V}{\partial \sigma_{ij}\, \partial \varepsilon^p_{kl}}\, d\varepsilon^p_{kl}, \tag{2.11}$$

if

$$f = 0, \quad df = 0;$$

and

$$d\varepsilon^e_{ij} = \frac{\partial^2 V}{\partial \sigma_{ij}\, \partial \sigma_{kl}}\, d\sigma_{kl}, \tag{2.12}$$

if

$$f < 0, \quad \text{or} \quad f = 0, \quad df < 0.$$

The Hessian tensors $\partial^2 V / \partial \sigma_{ij}\, \partial \sigma_{kl}$ and $\partial^2 V / \partial \sigma_{ij}\, \partial \varepsilon^p_{kl}$ are functions of stress and plastic strain amount. In consequence of (2.5) the elastic strain increment can be dualy expressed in terms of elastic strain potential (see Appendix)

$$d\varepsilon^e_{ij} = D_{ijkl}\, d\sigma_{kl} - D_{ijmn}\, \frac{\partial^2 U}{\partial \varepsilon^e_{mn}\, \partial \varepsilon^p_{kl}}\, d\varepsilon^p_{kl}; \quad f = 0, \quad df = 0,$$

$$d\varepsilon^e_{ij} = D_{ijkl}\, d\sigma_{kl}; \quad f < 0 \quad \text{or} \quad f = 0, \quad df < 0. \tag{2.13}$$

The tensor D_{ijkl} is the inverse of Hessian $\partial^2 U / \partial \varepsilon^e_{ij}\, \partial \varepsilon^e_{kl}$.

We are now at the position to write the complete incremental stress strain relation for elasto-plastic coupling in the loading condition:

$$d\varepsilon_{ij} = \frac{\partial^2 V}{\partial \sigma_{ij}\, \partial \sigma_{kl}}\, d\sigma_{kl} + \frac{\partial^2 V}{\partial \sigma_{ij}\, \partial \varepsilon^p_{kl}}\, \frac{\partial f}{\partial \sigma_{kl}}\, d\lambda + \frac{\partial f}{\partial \sigma_{ij}}\, d\lambda \tag{2.14}$$

or for brevity:

$$d\varepsilon_{ij} = d\varepsilon^{e'}_{ij} + d\varepsilon^{e''}_{ij} + d\varepsilon^p_{ij} \tag{2.15}$$

Note that introduction of the variation of elasticity moduli relevant to the amount of plastic deformation leads to the additional, "coupling" term $d\varepsilon^{e''}_{ij}$ in the incremental constitutive law. As can clearly be seen from the flow rules (2.10), this term is irreversible on the incremental level though it enters the elastic part of increment. Its direction in ε_{ij} space coincides with neither the reversible strain increment $d\varepsilon^{e'}_{ij}$ nor the plastic one. It is rather related to the mode of the rearrangement of the elasticity tensor due to the plastic flow (Fig. 2b).

3. Elastic properties of material with elasto-plastic coupling.

In the coupled elastic-plastic process the elastic behaviour is no more insensitive to the yield locus. The material response within the instantaneous yield limit, where $\varepsilon^p_{ij} = \text{cons.}$, is purely elastic and characterized by the potential function

$$V[\sigma_{ij}, \mu_{ij0}],$$

$\mu_{ij0}(\varepsilon^p_{ij0})$ being a dead parameter.

For the stress paths at the yield limit μ_{kl} becomes the live parameter and the elastic tensor is affected by the both variables σ_{ij}, ε^p_{ij}. The variation of the elastic potential during a loading from the stress point O to B in the presence of a progression of plastic deformation from ε^p_{ij} (1) to ε^p_{ij} (2) is shown schematically in Fig. 2a. The process can be conceived as composed of the hypothetic purely elastic part from the point O on the equipotential surface $V = V_0$ to the point A on V_A with the dead parameter μ_{kl} and the perfectly inelastic part from A to B with constant stress. Assume now that an unloading takes place from the stress point A to the point $C \equiv O$ always inside the yield surface.

In the above closed stress cycle the value of the potential is not recovered. The difference of the potential V between two points O and C, does not depend on the path on which the two states were reached within an individual potential surface $V = V\ (\sigma_{ij},\ \mu = \text{const.})$ and results therefore from the inelastic portion of the loading process only; thus:

$$\Delta V_{0c} = V(\sigma_{ij},\ \mu_0) - V(\sigma_{ij},\ \mu_c) \tag{3.1}$$

This difference corresponds to a dissipated (or adsorbed) part of elastic energy due to the irreversible deformation of the material in the elastic-plastc process.

We shall introduce a notion of the materials with fixed natural state ($V = 0$), which despite of their sensitivity of the elastic properties to the plastic deformation, exhibit total reversibility of the value of elastic potential in any closed stress cycle around the stress-free state. The fundamental feature of the materials with fixed natural state is that the strain remaining after stress unloading to $\sigma_{ij} = 0$ from any elastoplastic state, is to be identified with plastic strain only, no matter what would be the history of plastic and coupling strain. This fact seems to be consistent with macroscopic experimental approach, since there is little hope that a criterion could be found to distinguish between plastic and "coupled" parts in measured irreversible strain state. On the other hand, after a closed stress incremental cycle the coupled strain increment accompanies the elastic and plastic strain increments. The above distinction, make it possible to apply the framework of plasticity theory. From now on we shall deal mostly with materials with the fixed natural state.

The phenomenalogical observations quoted in the Introduction furnishe the examples of an increase as well as of a decrease of the elastic stiffness due to the plastic deformation. The experience shows moreover that both cases may refer to the behaviour of the same material in different circumstances [5]. Consider a function $\sigma = \sigma(\varepsilon^e)$ for an uniaxial elastic-plastic coupled process and a stress increment from a given stress point A. The slope of

the actual curve i.e. the elastic stiffness may increase or decrease with referrence to the hypothetically pure elasticity. By the analogy to the terminology of plasticity the former case will be referred to as "elastic hardening" while the latter as "elastic softening". In the hardening response the increase of the potential V is less then if the material were purely elastic. The oposite conclusion may be applied to the softening behaviour. In a multi-dimensional process a criterion for the elastic hardening can be formulated as follows:

$$dV_e = \frac{\partial V}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial V}{\partial \varepsilon^p_{ij}} d\varepsilon^p_{ij} \qquad (3.2)$$

$$\leqslant dV_e = \frac{\partial V}{\partial \sigma_{ij}} d\sigma_{ij}$$

The converse inequality can be referred to the softening response.

Summing up and combining the above with (2.8), (2.9), (2.10) we can write the condition for elastic hardening and softening respectively at a given point σ_{ij}

for elastic hardening:

$$\frac{\partial V}{\partial \varepsilon^p_{kl}} \frac{\partial f}{\partial \sigma_{kl}} < 0;$$

and

$$\frac{\partial U}{\partial \varepsilon^p_{hk}} \frac{\partial f}{\partial \sigma_{hk}} > 0.$$

for elastic softening

$$\frac{\partial V}{\partial \varepsilon^p_{kl}} \frac{\partial f}{\partial \sigma_{kl}} > 0;$$

and

$$\frac{\partial U}{\partial \varepsilon^p_{hk}} \frac{\partial f}{\partial \sigma_{hk}} < 0 \qquad (3.3)$$

The inequalities for the potential function U can be immediately derived from the relations given in the Appendix [see (A.2)].

4. Plastic properties of material with elasto-plastic coupling.

The two irreversible increments of strain appearing in the equation (2.14) have different meanings. The latter represents the plastic strain rate and is directed along the normal to the yield locus. The former is due to irreversible changes of the elastic moduli. The joint irreversible strain increment can be written in the form:

$$d\varepsilon^{(i)}_{ij} = d\lambda \left(\frac{\partial^2 V}{\partial \sigma_{ij} \partial \varepsilon^p_{kl}} + \delta_{ik}\delta_{lj} \right) \frac{\partial f}{\partial \sigma_{kl}} \qquad (4.1)$$

or dually by virtue of (A.3)

$$d\varepsilon^{(i)}_{ij} = d\lambda \left(- \frac{\partial^2 U}{\partial \varepsilon^e_{rs} \partial \varepsilon^p_{kl}} E_{ijrs} + \delta_{ik}\delta_{lj} \right) \frac{\partial f}{\partial \sigma_{kl}} \qquad (4.2)$$

The resultant vector of the irreversible strain increment is thus non-orthogonal to the yield locus, and the deviation from the normality is governed through the Hessian matrix

$$\frac{\partial^2 V}{\partial \sigma_{ij} \partial \varepsilon^p_{kl}}$$

or

$$\frac{\partial^2 U}{\partial \varepsilon^e_{ij} \partial \varepsilon^p_{kl}}.$$

The following conclusion may be drawn, that the elasto-plastic coupling effect leads to the specific kind of the non associated flow rule [12]. The natural question may arise now, whether it would not be a reasonable simplification to skip the intermediate term in the constitutive law, and either to associate the resultant $d\varepsilon^{(i)}_{ij}$ with the yield locus gradient or to reduce the case to the classical non-associated rule defining appropriately the flow potential.

The first of such approaches was adopted in [5], [9], [13], [14]. There are however several arguments which show certain inconsistencies of the proposed simplifications. Consider first the incremental stress cycle around some stress point O, presented for the case of the elastic softening and plastic hardening for uniaxial stress and strain and linear elasticity in Fig. 3.

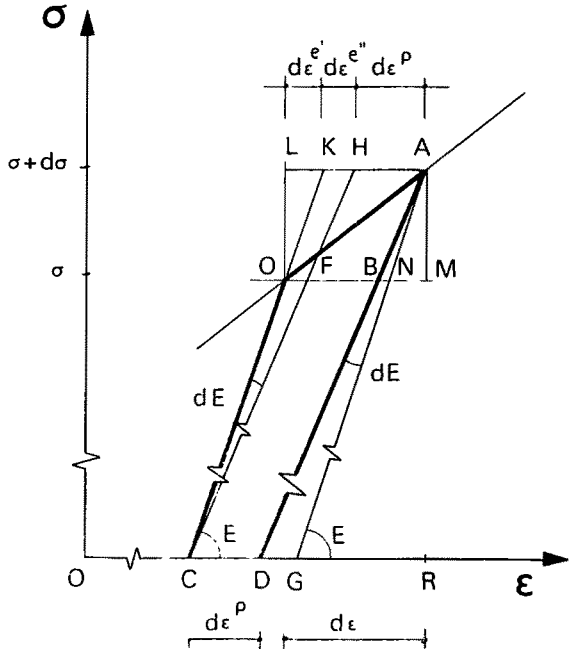

Fig. 3

Let DO be an elastic unlcading curve from the given stress state and OA the respective loading curve due to the stress increment $d\sigma$. Let the unloading curve AD be affected by the elastic modulus reduction dE which corresponds to the angle between the lines 1 and 2, 2 being parallel to the unloading line 3. The modulus deviation produces the strain increment $dE\,\sigma$, $OF = HK = DG$, which actually equals to $(\partial^2 V/\partial\sigma\,\partial\varepsilon^p)\,d\varepsilon^p$, if the fixed natu-

ral state postulate (sec. 3) is supposed to be flulfilled. In fact, the strain increment CD remaining after the unloading to the zero stress, is equal to that denoted by $FB = HA$, which is identified with the plastic strain rate. The reversible part of the elastic strain increment is equal (with accuracy of the order $dE\,d\sigma$) to $E\,d\sigma$, what means that the distance between the points B and C is negligeable, ($LK \cong BM$). From the above consideration it is readily visible that as far as the difference, DG, between the strain remaining after unloading, CD, and the hypothetical purely elastic one, NM, is of the same order as the total strain increment, the coupling term, $KH = DG$, should not be neglected in the incremental law.

The incorporation of the coupling term into the plastic one, jointly colinear with some plastic potential gradient, leads to the ambigueous definition of the plastic strain increment confusing when compared with residual strain increment on finite stress cycle, $KA \neq CD$. Moreover, there is no guaranty of the elastic strain recovery at zero stress state. This is particularly relevant when the elastic hardening occurs compensating for the plastic flow as it takes place for instance in the plastic consolidation of granular media, see. [5], [10].

The direction of the resultant vector of the irreversible strain increment in eq. (2.14) is affected by change in Hessian matrix. The simplification by taking a non-associated flow law would then require a specific variation of plastic potential.

Despite of the afore-mentioned differences there exists the clear analogy in the formal structure between coupled law and the " frictional " non-associated flow rule on the incremental level and will be found helpfull in the foregoing analysis, [16], [17].

5. Scalar coupling.

Consider the general tensorially linear form of the elasticity relation (2.6) under the hypothesis of isotropy:

$$\varepsilon^e_{ij} = M_0\delta_{ij} + M_1\sigma_{ij} + M_2\varepsilon^p_{ij} \tag{5.1}$$

Where M_0, M_1, M_2 ar scalar functions of the invariants of stress and plastic strain, including mixed invariants:

$$\begin{aligned} Z_1 &= \sigma_{ij}\varepsilon^p_{ij}\ ; \\ Z_2 &= \sigma_{kl}\varepsilon^p_{lm}\sigma_{mk}\ ; \\ Z_3 &= \varepsilon^p_{ij}\sigma_{jk}\varepsilon^p_{ki} \end{aligned} \tag{5.2}$$

As postulated at the beginning, the material in the virgin elastic state is isotropic. This means that tensor σ_{ij} and ε^e_{ij} are coaxial i.e. their principal directions coincide. We assume now that the plastic deformations do not disturbe this coaxiality. Since there is no reasons to suppose that the plastic deformation tensor ε^p_{ij} is coaxial both with the σ_{ij} and ε^e_{ij}, the condition $M_2 = 0$ must be set in (5.1).

Under the above assumptions the general form of the potential V reduces to the expression:

$$V = V\,[I_1, I_{2d},\ \mu(\mathscr{I}^p_1, \mathscr{I}^p_{2d})], \tag{5.3}$$

where the following invariants are employed:

$$\begin{aligned} I_1 &= \sigma_{ll}\,, & \mathscr{I}^p_1 &= \varepsilon^p_{ll} \\ I_{2d} &= s_{ij}s_{ij}\,, & \mathscr{I}^p_{2d} &= e^p_{ij}e^p_{ij} \end{aligned} \tag{5.4}$$

and μ is the scalar coupling parameter, s_{ij} and e^p_{ij} are the deviators of σ_{ij} and ε^p_{ij} respectively.

This form of the coupling will be referred to as the scalar coupling.

It is of use to introduce a coupling potential function $g = g(\mu, \sigma_{ij})$ derived from the elastic potential:

$$g(\mu, \sigma_{ij}) = \frac{\partial V}{\partial \mu} \tag{5.5}$$

The variable g has the sense of the thermodynamic force acting on the internal variable μ.

The irreversible strain increment (4.1) takes now the form:

$$d\varepsilon^{(i)}_{ij} = d\lambda\left(\zeta\,\frac{\partial g}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \sigma_{ij}}\right), \tag{5.6}$$

where $d\lambda$ is given by (2.9); ζ, a scalar function dependent on stress and plastic strains is defined as:

$$\zeta(\sigma_{ij}, \varepsilon^p_{ij}) = \frac{\partial \mu}{\partial \varepsilon^p_{hk}}\,\frac{\partial f}{\partial \sigma_{hk}}\,. \tag{5.7}$$

The resultant vector $d\varepsilon^{(i)}_{ij}$ can be conceived as the sum of two gradient vectors multiplied by scalar factors. The scalar coupling leads in effect to a corner at the intersection point of the yield locus $f = 0$, and the equipotential surface $g = g_0$; $g_0 = \text{cons}$, being the current value of the potential, Fig. 4.

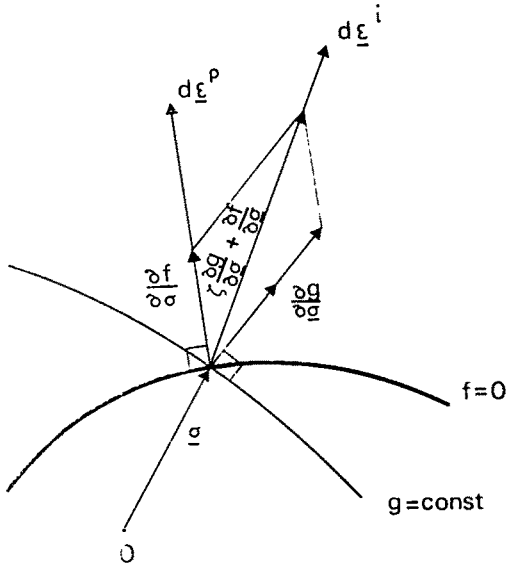

Fig. 4

In the dual formulation the scalar coupling results in the following constitutive law:

$$d\sigma_{kl} = E_{ijkl}\, d\varepsilon_{ij} - \frac{\left(\frac{\partial f}{\partial \sigma_{rs}} E_{rskl} - \zeta \frac{\partial h}{\partial \varepsilon^e_{kl}}\right) \frac{\partial f}{\partial \sigma_{ij}} E_{ijmn}}{H + \left(\frac{\partial f}{\partial \sigma_{ut}} E_{utrs} - \zeta \frac{\partial h}{\partial \varepsilon^e_{rs}}\right) \frac{\partial f}{\partial \sigma_{rs}}} d\varepsilon_{mn} \tag{5.8}$$

Where the dual coupling potential $h(\varepsilon^e_{rs}, \mu)$ defined as:

$$h(\varepsilon^e_{rs}, \mu) = \frac{\partial U}{\partial \mu} \tag{5.9}$$

is related to the primary potential by the expression [see (A.3) and (A.4)].

$$-\frac{\partial g}{\partial \sigma_{rs}} \frac{\partial^2 U}{\partial \varepsilon^e_{rs}\, \partial \varepsilon^p_{mn}} = \frac{\partial h}{\partial \varepsilon^e_{mn}}. \tag{5.10}$$

The multiplier ζ (5.7) is a function of stress and plastic strains. At the beginning of the plastic process it may be zero or have a finite value, directly depending whether the elasticity moduli function derivative with respect to the parameter vanishs at this very moment or not. In the former case the deviation of the vector $d\varepsilon^{(t)}_{ij}$ from the yield locus normal is gradual, being zero at the yield initiation. In the second case the non-normality is initially present in the flow rule. The multiplier ζ plays moreover the role of a discriminant between the elastic hardening and softening. Rewriting the inequality (3.3) we arrive at the conclusion that

$$\begin{aligned} &g\zeta < 0, \quad \text{(for elastic hardening)}, \\ &g\zeta > 0, \quad \text{(for elastic softening)} \end{aligned} \tag{5.11}$$

The central novelty in the coupled elasto-plasticity viewed from the stand point of non-associated flow rules is the fact that the departure from the normality is governed by the another phenomenon, i.e. the elasticity variation. It is of interest to examine in terms of the equation (5.6) what the deviation should be expected, given the moduli variation. Consider with this scope a polynomial representation of the potential (5.3). Following the hypothesis of purely elastic behaviour of the material before reaching the virgin yield surface, we can split the potential into two parts as below

$$V = V_0(\sigma_{ij}) + V_1(\sigma_{ij}, \mu) \tag{5.12}$$

For the sake of simplicity let us restrict ourself to the stress-linear elasticity.

Then

$$V = \frac{1}{2}(\varkappa I_1^2 + \omega I_{2d}), \qquad \varkappa = \varkappa_0 + \varkappa_1(\mu). \qquad \omega = \omega_0 + \omega_1(\mu). \tag{5.13}$$

Where $\varkappa$ and ω are the bulk and shear compliance moduli, and $\varkappa_0$, ω_0, $\varkappa_1$, ω_1, their constant and variable parts respectively. Elasticity relations read now:

$$\varepsilon^e_{ij} = \varkappa(\mu)\, \sigma_{mm}\, \delta_{ij} + \omega(\mu)\, s_{ij} \tag{5.14}$$

$$d\varepsilon^{e'}_{ij} = \varkappa(\mu)\, d\sigma_{rr}\, \delta_{ij} + \omega(\mu)\, ds_{ij} \tag{5.15}$$

$$d\varepsilon^{e''}_{ij} = \left(\frac{\partial \varkappa}{\partial \mu} \sigma_{rr}\, \delta_{ij} + \frac{\partial \omega}{\partial \mu} s_{ij}\right) d\mu \tag{5.16}$$

The coupling potential $g = \partial V / \partial \mu$ is again the quadratic form in stresses. Its any equipotential section forms in the plane of stress invariants the family of ellipses parametrized with respect to μ:

$$g(\sigma_{ij}, \mu) = \frac{1}{2}\left[\frac{\partial \varkappa_1}{\partial \mu} I_1^2 + \frac{\partial \omega_1}{\partial \mu} (\sqrt{I_{2d}})^2\right]. \tag{5.17}$$

The components of the coupling potential gradient are in this case:

$$\frac{\partial g}{\partial I_1} = \frac{\partial \varkappa_1}{\partial \mu} I_1\,; \qquad \frac{\partial g}{\partial \sqrt{I_{2d}}} = \frac{\partial \omega_1}{\partial \mu} \sqrt{I_{2d}} \tag{5.18}$$

An example of the coupling equipotential and the yield locus allowing for plastic hardening as well as for softening is depicted in Fig. 5, in the case of the positive multiplier ζ.

Let us assume now that only the volumetric elastic compliance modulus varies due to plastic deformations; the shear modulus remains constant. Then the terms in the equations (5.14, 5.16, 5.18) containing the factor $\partial\omega_1/\partial\mu$ vanish. The equipotential $g = g_0$ reduces to the form of the straight line parallel to the I_{2d}-axis and the coupling strain rate term gives the contribution only to the volumetric component of irreversible strain rate Fig. 5a. When the shear elastic compliance alone is affected by the plastic strain, the additional term in the incremental law has only the deviatoric component. In fact, $\partial\varkappa/\partial\mu = 0$; the equipotential $g = g_0$ degenerates to the straight line parallel to the first invariant axis. Note that in the case of the elastic hardening, presented in Fig. 2, when the value of the potential V decreases with the increase of the plastic volumetric strain, the coupling strain rate vector is directed toward the interior of the coupling potential in the plastic consolidation range. It is directed to the exterior in the dilatation range where the elastic softening occurs, Fig. 5b.

Basing on the described properties of the coupling potential a particular approximative model for the coupling in granular and rocklike materials was proposed in [10]. The experimental results [5] indicate that in the plastic consolidation range the elastic volumetric stiffness increases while the shear stiffness remains constant. The opposite refers to the plastic dilatation range. The coupling equipotential surface in principal stress space composes of a cylinder and a plane lid in the above ranges respectively intersecting along the critical surface [10].

6. Simplified elastoplastic coupling models.

Let us focus our attention on two limiting cases of the material property types.

First case refers to the characteristic properties of hard rocks which initially have relatively high stiffness, so that in the virgin elastic domain $V_0 = 0$ (5.12) for any stress and the material responses as a rigid one.

Once the yield locus is achieved and plastic strains appear, the elastic stiffness gradually decreases. $V = V_1$ is no more zero and reversible and coupling strain rates come into the picture in the constitutive incremental law.

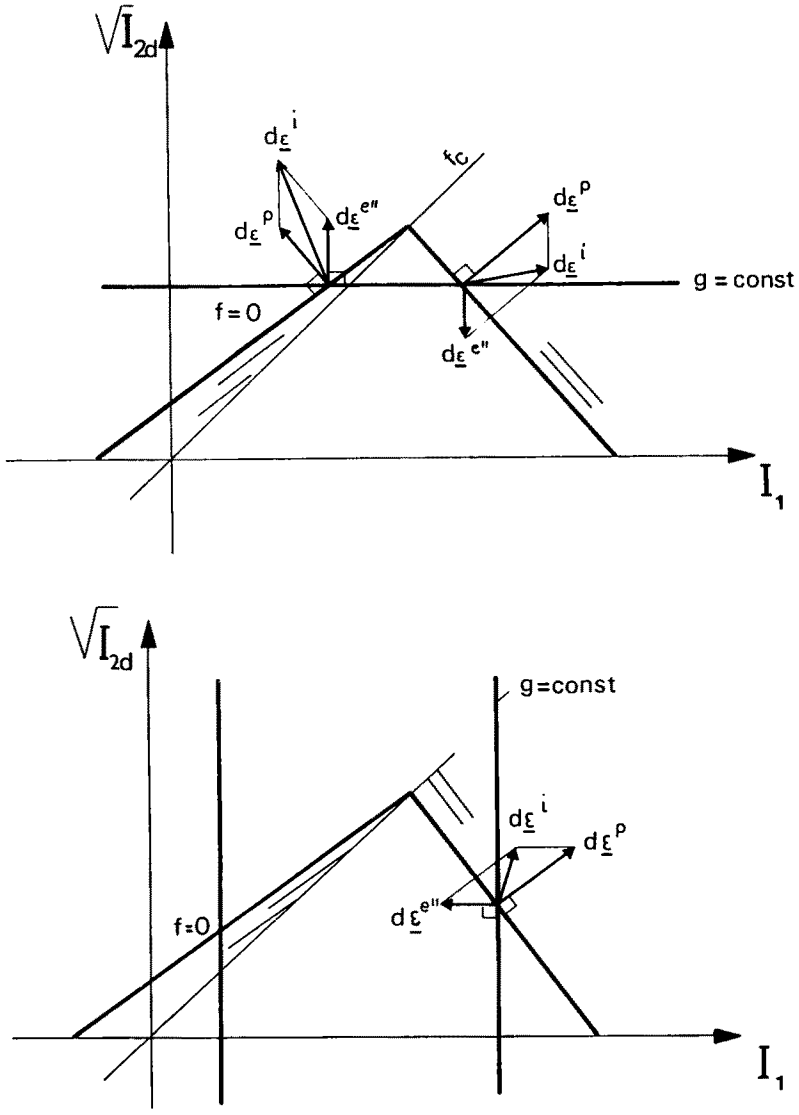

Fig. 5

It may appear that the elatstic, stiffness is still very high and $d\varepsilon_{ij}^{e'}$ may be neglected while the coupling terms is of the comparative order with the plastic strain rate, $\partial(\partial V_1/\partial\sigma_{ij})\, d\sigma_{ij} \ll \partial(\partial V_1/\partial\sigma_{kl})/\partial\mu \cdot (\partial\mu/\partial\varepsilon_{ij}^{p})$; the constitutive relation then reduces to

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{(i)} = d\lambda \left(\zeta \frac{\partial g}{\partial\sigma_{ij}} + \frac{\partial f}{\partial\sigma_{ij}}\right). \tag{6.1}$$

In other words, despite of the initial lack of elasticity the elasto-plastic coupling takes place, and in particular case leads to non-associated flow rule of rigid-plastic hardening material.

The formulation presented in the preceding sections is not restricted by the assumption of $H \neq 0$. In fact, we show below the limiting case of rigid-perfectly plastic model with variable dilatancy due to the coupling. Such a model [15] is oriented at the description of the positive and negative dilatancy, defined as the ratio $d\mathscr{I}_1/d\mathscr{I}_{2d}^{1/2}$ ($\mathscr{I}_1$, $\mathscr{I}_{2d}^{1/2}$ being the first and second deviatoric invariants of strain), asymptotically ceasing for the advanced flow. The flow rule is the same as (6.1), but $d\lambda$ is now an undetermined magnitude. Assume that the plastic flow produces shear deformations only, whereas volumetric strain rate is due to the elastic volumetric modulus variation. Consequently:

$$f = \frac{1}{2} s_{ij}s_{ij} - k_0; \qquad \mu = (e_{ij}^{p}e_{ij}^{p})^{\frac{1}{2}} \tag{6.2}$$

k_0 being constant.

The complementary potential takes on the following form:

$$V_0 = 0, \qquad V_1 = \frac{1}{2}\varkappa(\mu)\, I_1^2 \tag{6.3}$$

while the coupling potential and the multiplier ζ are equal:

$$g = \frac{1}{2}\frac{\partial\varkappa}{\partial\mu} I_1^2; \qquad \zeta = s_{ij}e_{ij}^{p} \tag{6.4}$$

The complete strain rate-stress relation reads:

$$d\varepsilon_{ij}^{(l)} = d\lambda \left[s_{kl} \frac{e_{kl}^{p}}{\sqrt{\mathscr{J}_{2d}^{p}}} \frac{\partial \varkappa}{\partial \mu} I_1 \delta_{ij} + s_{ij} \right] \tag{6.5}$$

Note that the dilatation function is variable and depends on the stress and strain:

$$\frac{d\mathscr{J}_1}{d\mathscr{J}_{2d}^{\frac{1}{2}}} = s_{kl} \frac{e_{kl}^{p}}{\sqrt{\mathscr{J}_{2d}^{p}}} \frac{\partial \varkappa}{\partial \mu} \frac{I_1}{\varkappa_0} \tag{6.6}$$

Let us assume the function $\varkappa$ in the form:

$$\begin{aligned} &\varkappa = \varkappa_0 (1 - \mu)^m (\varrho_0/\varrho_{cr} - 1)^n , \\ &\varkappa_0 = \varkappa_0(\varrho_0/\varrho_{cr}) > 0 \end{aligned} \tag{6.7}$$

where m, n are integers (n-necessarily odd) while ϱ_0 and ϱ_{cr} are the initial and critical density of the material. Then for $\varrho_0 > \varrho_{cr}$ the material dilates, while for $\varrho_0 < \varrho_{cr}$ it consolidates, and both processes asymptotically cease if the shear deformations are large enough.

7. Conclusions.

The alternation of the elastic moduli due to the plastic deformation casted into the framework of incremental elasto-plasticity theory leads to a specific kind of the non-associated flow rule. The substantial difference appears then between the residual strain on incremental and on global stress cycle. On the latter cycle the normality of remaining (= plastic) strain vector to the yield locus is conserved. One of the crucial phenomenologically observed effects of presence of coupling is its influence on the dilatation function. The direct connection between type of elastic moduli variation and deviation from normality gives clear indication to the possible experimental rules in the examinations of coupling phenomenon. The limiting cases of coupling suggest the eventual less "a priori" interpretation of the non-associated flow, laws. The existence and uniqueness of response and stability of behaviour of materials with coupling as well as incremental boundary value problems are expounded elsewhere [16], [17].

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Appendix.

Let U and V be a pair of dual potentials interralated through the Legendre transformation with respect to their active variables ε_{ij}^{e}, σ_{ij} correspondingly and ε_{ij}^{p} the common passive variable [11].

$$U(\varepsilon_{ij}^{e}, \varepsilon_{ij}^{p}) + V(\sigma_{ij}, \varepsilon_{ij}^{p}) = \sigma_{ij}\varepsilon_{ij}^{e} \tag{A.1}$$

Three straightforward results are useful to be reminded:

$$\frac{\partial U}{\partial \varepsilon_{ij}^{p}} = - \frac{\partial V}{\partial \varepsilon_{ij}^{p}} \text{ at given } \varepsilon_{ij}^{e}, \sigma_{ij} \tag{A.2}$$

$$\frac{\partial^2 V}{\partial \sigma_{ij} \partial \varepsilon_{hk}^{p}} = - \frac{\partial^2 U}{\partial \varepsilon_{ij}^{e} \partial \varepsilon_{rs}^{p}} \frac{\partial^2 V}{\partial \sigma_{rs} \partial \sigma_{hk}} \tag{A.3}$$

$$\frac{\partial^2 U}{\partial \varepsilon_{ij}^{e} \partial \varepsilon_{hk}^{p}} = - \frac{\partial^2 V}{\partial \sigma_{ij} \partial \varepsilon_{mn}^{p}} \frac{\partial^2 U}{\partial \varepsilon_{mn}^{e} \partial \varepsilon_{hk}^{p}} \tag{A.4}$$

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