C^r-finite elements of Powell–Sabin type on the three direction mesh

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Received 1 April 1995; revised August 1996
Communicated by L.L. Schumaker

Let τ be the triangulation generated by a uniform three direction mesh of the plane. Let τ_6 be the Powell–Sabin subtriangulation obtained by subdividing each triangle $T \in \tau$ by connecting each vertex to the midpoint of the opposite side.

Given a smooth function u, we construct a piecewise polynomial function $v \in C^r(\mathbb{R}^2)$ of degree $n = 2r$ (resp. $2r + 1$) for r odd (resp. even) in each triangle of τ_6, interpolating derivatives of u up to order r at the vertices of τ.

Keywords: Triangular finite elements, bivariate Hermite interpolation.

AMS subject classification: 41A15, 41A05, 65D07, 65D05, 65N30.

1. Introduction

Let τ be the uniform Δ^1-type triangulation of the plane $\mathbb{R}^2$. For example that induced by integer translates of $x = 0$, $y = 0$ and $x - y = 0$. Let τ_6 be the PS subtriangulation of τ (after Powell–Sabin), obtained by connecting each vertex to the midpoint of the opposite side in each triangle $T \in \tau$. Let $S_n^r(\tau_6) = \{v \in C^r(\mathbb{R}^2)\colon\ v|_t \in \mathbb{P}_n,\ \forall t \in \tau_6\}$, where $\mathbb{P}_n$ is the space of bivariate polynomials of total degree at most n.

Given $u \in C^m(\mathbb{R}^2)$, $m \geqslant r$, we consider the following *Hermite interpolation problem* $H^r(u)$: construct $v \in S_n^r(\tau_6)$ satisfying $D^\alpha v(a) = D^\alpha u(a)$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, $|\alpha| = \alpha_1 + \alpha_2 \leqslant r$ and $a \in \tau$.

The procedure of construction consists in assembling composite finite elements $v_T = v|_T$ for $T \in \tau$, v_T depending only on interpolation data on T. For an arbitrary triangulation of $\mathbb{R}^2$, the classical Powell–Sabin element is a C^1-quadratic spline of dimension 9 (see, e.g., [21, 25]). Later Sablonnière [22] gave a C^r-generalized scheme in a subspace of splines of degree $3r-1$. This result was improved recently by the first author [16] in subspaces of degree $5s$ (resp. $5s+2$) for $r=2s$ (resp. $2s+1$). In this paper, due to the uniformity of the triangulation, we prove the existence of Hermite interpolation schemes in a subspace of $S_n^r(\tau_6)$ for lower degrees: respectively $n=2r+1$ for r even and $n=2r$ for r odd. These degrees are minimal, as is proved also in [16]. Another interesting feature is that the construction of our local finite elements needs only partial derivatives of order at most $r+[r/2]$ ($[x]$ is the integer part of x) at the vertices of the triangulation. So, there is no need of normal derivatives or derivatives at interior points of triangles. For arbitrary partitions and other types of finite elements, this cannot be avoided in general (see our related works [13–18] and [22–26], see also [19, 20, 29]).

The paper is organized as follows: in section 2, we recall and prove some results on the Bernstein–Bézier form of polynomials on triangles which is used for representing splines on the triangulation τ_6. In section 3 we give the construction of PS finite elements and of the solution of the Hermite interpolation problem of order r. Finally, in section 4, we give some error estimates.

2. Bernstein form of polynomials

2.1. *Representation of polynomials on triangles*

Let $T=A_1A_2A_3$ be an arbitrary triangle in the plane. Let $\mu=(\mu_1,\mu_2,\mu_3)$ be the barycentric coordinates of a point M with respect to T. We have

$$M=\sum_{i=1}^{3}\mu_iA_i,\quad |\mu|=\sum_{i=1}^{3}\mu_i=1,$$

and $M\in T$ if and only if $0\leqslant\mu_i\leqslant 1,\ i=1,2,3$.

For a multi-index $\beta=(\beta_1,\beta_2,\beta_3)\in\mathbb{N}^3$, we will use the notations

$$|\beta|=\sum_{i=1}^{3}\beta_i,\qquad \beta!=\prod_{i=1}^{3}\beta_i,\qquad \mu^\beta=\mu_1^{\beta_1}\mu_2^{\beta_2}\mu_3^{\beta_3},$$

$$B_\beta^n(\mu)=\frac{n!}{\beta!}\mu^\beta\quad\text{for } |\beta|=n \text{ and } n\in\mathbb{N}. \tag{1}$$

The $\binom{n+2}{2}$ polynomials (1) form the *Bernstein basis* of the space $\mathbb{P}_n$. Any polynomial $p\in\mathbb{P}_n$ can be written uniquely as

$$p(\mu)=\sum_{|\beta|=n}b(\beta)B_\beta^n(\mu),$$

which is called its Bézier representation with Bézier coefficients $b(\beta)$.

The set of B-vertices

$$\left\{\left(\sum_{i=1}^{3}\frac{\beta_i}{n}A_i, b(\beta)\right)\right\}$$

in $\mathbb{R}^3$ is the B-net (or control net) of p on T. In order to simplify the figures, instead of representing polynomials by their B-nets in $\mathbb{R}^3$, it is more convenient to display their B-coefficients on triangles which are their projections on the plane. We often make use of this convention.

For the construction of finite elements, we need the *expression of partial derivatives* and of *smoothness conditions* between two polynomials (the details of results can be found in Farin [10] or de Boor [2]). The partial derivatives of order $|\alpha| = \alpha_1 + \alpha_2 = r$, α_1 times in the direction A_jA_k and α_2 times in the direction A_jA_m, are given by

$$D^\alpha p(\mu)\big((A_jA_k)^{\alpha_1}, (A_jA_m)^{\alpha_2}\big) = \frac{n!}{(n-r)!}\sum_{|\beta|=n-r}\big(\Delta_{jk}^{\alpha_1}\Delta_{jm}^{\alpha_2}b(\beta)\big)B_\beta^{n-r}(\mu), \qquad (2)$$

where $\Delta_{jk}b(\beta) = b(\beta+e_k) - b(\beta+e_j)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ and $\Delta_{jk}^2 b(\beta) = \Delta_{jk}(\Delta_{jk}b(\beta))$, etc.

Now let $\widetilde{T} = \widetilde{A}_1A_2A_3$ be a neighboring triangle of T with $\widetilde{A}_1 \notin T$, $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \widetilde{\mu}_3)$ denotes the barycentric coordinates of a point with respect to $\widetilde{T}$ and $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*)$ those of $\widetilde{A}_1$ with respect to T.

Let

$$p(\mu) = \sum_{|\beta|=n} b(\beta)B_\beta^n(\mu) \in \mathbb{P}_n(T) \quad \text{and} \quad \widetilde{p}(\widetilde{\mu}) = \sum_{|\gamma|=n} c(\gamma)B_\gamma^n(\widetilde{\mu}) \in \mathbb{P}_n(\widetilde{T}),$$

then p and $\widetilde{p}$ are joined smoothly across the common edge A_2A_3 up to order r if and only if the following conditions hold:

$$c(s,i,j) = \sum_{|\beta|=s} b\big((0,i,j)+\beta\big)B_\beta^s(\mu^*), \qquad (3)$$

for $s = 0,\dots,r$ and $i + j = n - s$.

2.2. *Representation of splines on PS triangles*

For an arbitrary triangle T of τ, the vertices of the triangulation τ_6 of T are denoted as follows (see figure 1): the points A_{2i-1}, $i = 1,2,3$, are the vertices of T, the points A_{2i} are respectively the midpoints of the edges $\Gamma_i = A_{2i-1}A_{2i+1}$ of T and A_7 is the center of gravity of T. Let $t_{2i-1} = A_{2i-1}A_{2i}A_7$ and $t_{2i} = A_{2i+1}A_{2i}A_7$ $(1 \leqslant i \leqslant 3)$ be the micro-triangles of τ_6 in T. Here and in the following, each index relative to a vertex (resp. an edge) is counted modulo 6 (resp. 3).

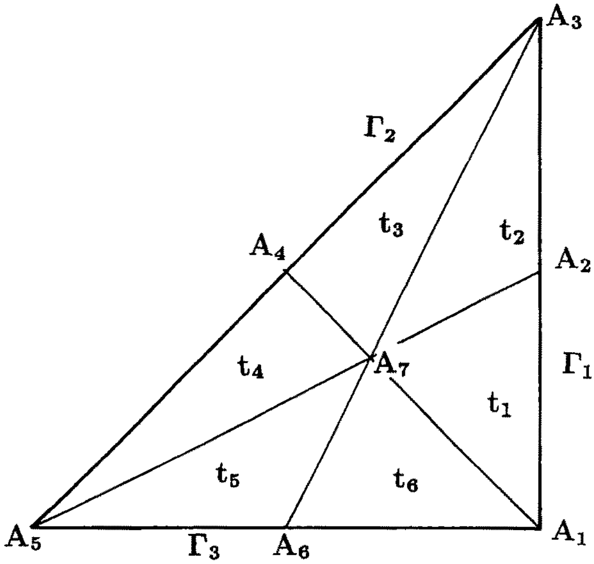

Figure 1.

For the definition of barycentric coordinates w.r.t. t_k, we label the vertices in the order $\{A_{2i-1}, A_{2i}, A_7\}$ for t_{2i-1} and $\{A_{2i+1}, A_{2i}, A_7\}$ for t_{2i}.

Let

$$\mathcal{S}_n^0(T, \tau_6) = \{f \in C^0(T)\colon\ f|_{t_k} \in \mathbb{P}_n,\ 1 \leqslant k \leqslant 6\}.$$

Let $p_k = f|_{t_k}$, $1 \leqslant k \leqslant 6$, then $\{b_k(\beta)\colon\ |\beta| = n\}$ denotes the B-coefficients of polynomial p_k on the triangle t_k and $\bigcup_{k=1}^{6}\{b_k(\beta)\colon\ |\beta| = n\}$ is the set of B-coefficients of the spline f on the triangle T. In the figures, since our splines are at least C^0, the B-coefficients situated on the interior edges of τ_6 are denoted by the same symbol.

From conditions (3) we get he following lemma, see, e.g., [5, 10].

Lemma 1. Let $\mathcal{S}_n^r(T, \tau_6) = \mathcal{S}_n^0(T, \tau_6) \cap C^r(T)$, $r \geqslant 1$. Then $f \in \mathcal{S}_n^r(T, \tau_6)$ if and only if the following relations hold for $i = 1, 2, 3$:

For $0 \leqslant s \leqslant r$ and $k + m = n - s$,

$$b_{2i}(s, k, m) = \sum_{j=0}^{s} \binom{s}{j} (-1)^j 2^{s-j} b_{2i-1}(j, k + s - j, m), \tag{4}$$

$$b_{2i}(k, s, m) = \sum_{|\beta|=s} \frac{s!}{\beta!} \frac{(-1)^{\beta_2} 3^{\beta_3}}{2^{s-\beta_2}} b_{2i+1}(k + \beta_1, \beta_2, m + \beta_3). \tag{5}$$

Let us now introduce some subsets of B-coefficients which are used later in the construction of local interpolants (see figure 2 for some examples).

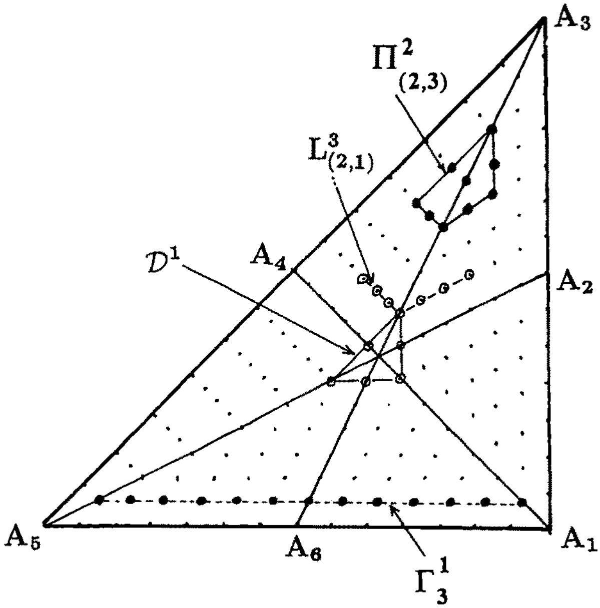

Figure 2.

Definition 1 (subsets of B-coefficients).

(a) For $0 \leqslant r \leqslant n$, $i = 1, 2, 3$ let

$$\Gamma_i^r = \{b_{2i-1}(j,k,r),\ b_{2i}(j,k,r):\ j+k = n-r\}$$

be the rth row of B-coefficients parallel to the external edge Γ_i.

(b) For $0 \leqslant r \leqslant n$, $0 \leqslant s \leqslant n-r$ and $i = 1, 2, 3$ the set

$$L_{(i,s)}^r = \{b_{2i-2}(\beta),\ b_{2i-1}(\beta):\ |\beta| = n,\ \beta_1 = s,\ \beta_2 \leqslant r\}$$

is called the sth level of order r opposite to the vertex A_{2i-1}. It is formed by the sth row parallel to the edges $A_{2i-2}A_7$ and $A_{2i}A_7$ respectively and bounded by the rows $\{b_{2i-2}(\beta):\ \beta_2 = r\}$ and $\{b_{2i}(\beta):\ \beta_2 = r\}$ which are themselves parallel to the edge $A_{2i-1}A_7$.

(c) For $0 \leqslant r \leqslant n$, $0 \leqslant s \leqslant n-r$ and $i = 1, 2, 3$ the set

$$\Pi_{(i,s)}^r = \bigcup_{k=0}^{r} L_{(i,s+k)}^{r-k}$$

is called the rth order plate at the level s associated with the vertex A_{2i-1}. It is formed by the $(r+1)^2$ B-coefficients situated in the region bounded by the rows Γ_{i-1}^{n-r-s}, Γ_i^{n-r-s} and $L_{(i,s)}^r$.

(d) Finally, for $0 \leqslant r \leqslant n$, let T_r be the subtriangle of T whose vertices are $b_1(r,0,n-r)$, $b_3(r,0,n-r)$, $b_5(r,0,n-r)$ and let $\mathcal{D}^r = \bigcup_{j=1}^{6}\{b_j(\beta)\colon n-r \leqslant \beta_3\}$ be the set of B-coefficients inside T_r (including edges).

Remark 1.

(1) We have $\Pi^{r_2}_{(i,s_2)} \subset \Pi^{r_1}_{(i,s_1)}$ for $r_2 < r_1$ and $s_2 \geqslant s_1$.

(2) For $f \in \mathcal{S}^r_n(T,\tau_6)$ and $0 \leqslant s \leqslant n-r$, Γ^s_i can be considered as the set of B-coefficients of a univariate spline of class C^r and of degree $n-s$ on a segment subdivided into two parts.

(3) For $r \leqslant s \leqslant n$, $\mathcal{D}^s$ can be considered as the set of B-coefficients of a bivariate spline in the space $\mathcal{S}^r_s(T_r,\tau_6)$.

In order to prove the unisolvency of the interpolation scheme, we need the well-known following lemma, see, e.g., [3, lemma 4.1] or [7]:

Lemma 2. Suppose that $f \in \mathcal{S}^{2s+1}_n(T,\tau_6)$.

(1) For fixed integers $j \in \{0,\dots,n-2s-1\}$ and $i \in \{1,2,3\}$, consider the following sets of B-coefficients:

$$\mathcal{A} = \left\{ \begin{matrix} b_{2i+1}(\beta)\colon\ |\beta| = n,\ j+1 \leqslant \beta_1 \leqslant j+2s+1,\ \beta_2 \leqslant 2s \\ \text{and } n-j-2s-1 \leqslant \beta_3 \leqslant n-j-1 \end{matrix} \right\},$$

$$\mathcal{B} = \left\{ b_{2i}(\beta),\ b_{2i+1}(\beta)\colon\ \beta_1 = j,\ s+1 \leqslant \beta_2 \leqslant 2s+1 \text{ and } |\beta| = n \right\}.$$

Then the B-coefficients in the set

$$\mathcal{C} = \left\{ b_{2i}(\beta),\ b_{2i+1}(\beta)\colon\ \beta_1 = j,\ \beta_2 \leqslant s \text{ and } |\beta| = n \right\}$$

can be uniquely computed from those in $\mathcal{A}$ and $\mathcal{B}$.

(2) For fixed integers $m \leqslant n-2s-1$ and $i \in \{1,2,3\}$, the set of B-coefficients

$$\mathcal{B}' = \left\{ b_{2i-1}(\beta),\ b_{2i}(\beta)\colon\ \beta_1 \leqslant s,\ \beta_3 = m \text{ and } |\beta| = n \right\}$$

can be computed uniquely from the ones in the set

$$\mathcal{A}' = \left\{ b_{2i-1}(\beta),\ b_{2i}(\beta)\colon\ s+1 \leqslant \beta_1 \leqslant 2s+1,\ \beta_3 = m \text{ and } |\beta| = n \right\}.$$

From theorem 2.1 of Schumaker [27], we deduce

Lemma 3.

$$\dim \mathcal{S}^r_n(T,\tau_6) = 6\binom{n-r+1}{2} + \binom{r+2}{2} + \sum_{j=1}^{n-r}(r+1-2j)_+,$$

where $x_+ = \max(x,0)$.

We need this lemma in the following section.

3. Construction of PS finite elements

We describe in this section the construction of the generalized C^r-scheme of type PS on any macro-triangle T of τ. We construct the finite elements in spaces of generalized super-splines $\mathcal{S}_{n(r)}^{r,r+[r/2]}(T,\tau_6)$, where $n(r)=2r$ (resp. $2r+1$) for r odd (resp. even). These spaces of functions v_T are defined as follows (see Chui and Lai [6], Schumaker [28] and Ibrahim and Schumaker [12]):

Definition 2 (subspaces of finite elements).

(i) If r is odd: $r=2s+1,\ s\geqslant 0$, we impose that v_T be of class C^{3s+1} (instead of C^{2s+1}) at the vertices of T. We thus obtain the subspace $\mathcal{S}_{4s+2}^{2s+1,3s+1}(T,\tau_6)$ of $\mathcal{S}_{4s+2}^{2s+1}(T,\tau_6)$.

(ii) If r is even: $r=2s,\ s\geqslant 1$, we first reinforce the smoothness of v_T by imposing $v_T\in\mathcal{S}_{4s+1}^{2s+1}(T,\tau_6)$. Then we impose that v_T be of class C^{3s} (instead of C^{2s+1}) at the vertices of T. We thus obtain the subspace $\mathcal{S}_{4s+1}^{2s+1,3s}(T,\tau_6)$ of $\mathcal{S}_{4s+1}^{2s+1}(T,\tau_6)$.

For the construction we need the following lemma:

Lemma 4.

(1) $\dim\mathcal{S}_{3s+1}^{2s+1}(T,\tau_6)=3(s+1)(2s+1)$.

(2) Any function $w\in\mathcal{S}_{3s+1}^{2s+1}(T,\tau_6)$ is uniquely determined by the data

$$D^\alpha w(A_{2i-1})\quad\text{for } |\alpha|\leqslant 2s \text{ and } i=1,2,3.$$

Proof. By lemma 3,

$$\dim\mathcal{S}_{3s+1}^{2s+1}(T,\tau_6)=6\binom{s+1}{2}+\binom{2s+3}{2}+\sum_{j=1}^{s}(2s+2-2j)_+=3(s+1)(2s+1).$$

The number of data being equal to the dimension of $\mathcal{S}_{3s+1}^{2s+1}(T,\tau_6)$, it suffices to show that $w\equiv 0$ when the data are zero. We prove the result by induction on s.

(a) For $s=1$, consider the space $\mathcal{S}_4^3(T,\tau_6)$ (see figure 3), whose dimension is 18. We assume that $D^\alpha w(A_{2i-1})=0$ for $|\alpha|\leqslant 2$ and $i=1,2,3$. Thus from (2) and (5) the B-coefficients marked by "•" are zero. Using part 2 of lemma 2, those marked by "o" are zero because C^3-smoothness across $A_7A_{2i},\ i=1,2,3$ determines these B-coefficients from the black ones. Now the B-coefficients "$*$" are zero by using part 1 of lemma 2. Those marked "×" are zero by C^1-continuity across the edges A_7A_{2i}. Finally the C^1-continuity at A_7 determines the central B-coefficient. Thus $w\equiv 0$, q.e.d.

(b) For $s=2$, in a similar way, we can show that $w\in\mathcal{S}_7^5(T,\tau_6)$ is zero when $D^\alpha w(A_{2i-1})=0$ for $|\alpha|\leqslant 4$ and $1\leqslant i\leqslant 3$.

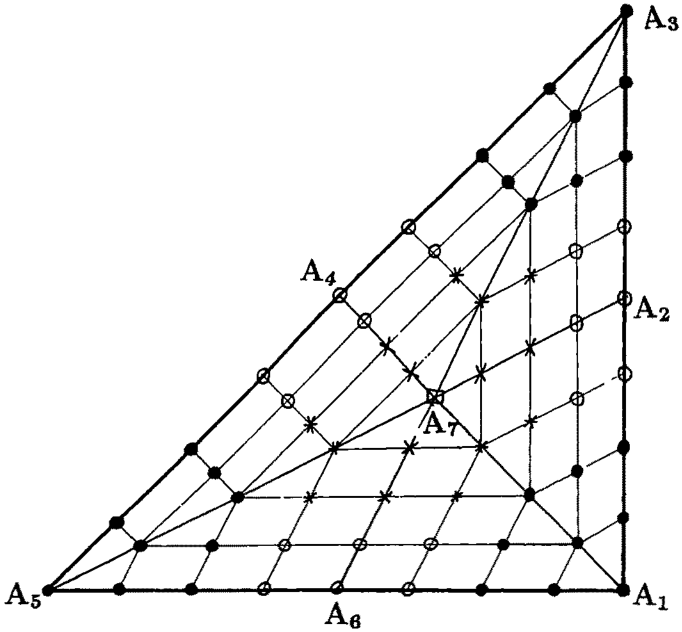

Figure 3.

(c) General case: we suppose that any $v \in \mathcal{S}^{2s-1}_{3s-2}(T, \tau_6)$, $s \geqslant 3$, is determined by the data $D^\alpha v(A_{2i-1})$, $|\alpha| \leqslant 2s-2$, $1 \leqslant i \leqslant 3$. Consider $w \in \mathcal{S}^{2s+1}_{3s+1}(T, \tau_6)$ (see figure 4 for $s = 4$) and suppose that the data are zero. By (2) and (5) we obtain:

$$b = 0 \quad \text{for all } b \in \bigcup_{i=1}^{3} \Pi^{2s}_{(i,s+1)}. \tag{6}$$

Using C^{2s+1}-continuity across the edges $A_{2i}A_7$ (as in lemma 2) we have:

$$b = 0 \quad \text{for all } b \in \bigcup_{i=1}^{3} \bigcup_{k=0}^{s} \Gamma^k_i. \tag{7}$$

Consider the disk $\mathcal{D}^{3s-2}$. It can be considered as the set of B-coefficients of a spline $w^* \in \mathcal{S}^{2s+1}_{3s-2}(T_{3s-2}, \tau_6)$ (see remark 1). The $2s+2$ B-coefficients

$$\{b_{2i-1}(\beta),\ b_{2i-2}(\beta)\colon\ |\beta| = 3s+1,\ \beta_1 = s,\ \beta_3 \leqslant s\}$$

are in the level $L^{2s+1}_{(i,s)}$ because the second component of β satisfies $s+1 \leqslant \beta_2 \leqslant 2s+1$ (see definition 1 in section 2). From (7) they are zero because they are also in Γ^k_{i-1} (or Γ^k_i), $k \leqslant s$. Thus using lemma 2, we deduce that $b = 0$ for all $b \in L^{2s+1}_{(i,s)}$. By (6) we have also $b = 0$ for all $b \in \Pi^{2s+1}_{(i,s)}$.

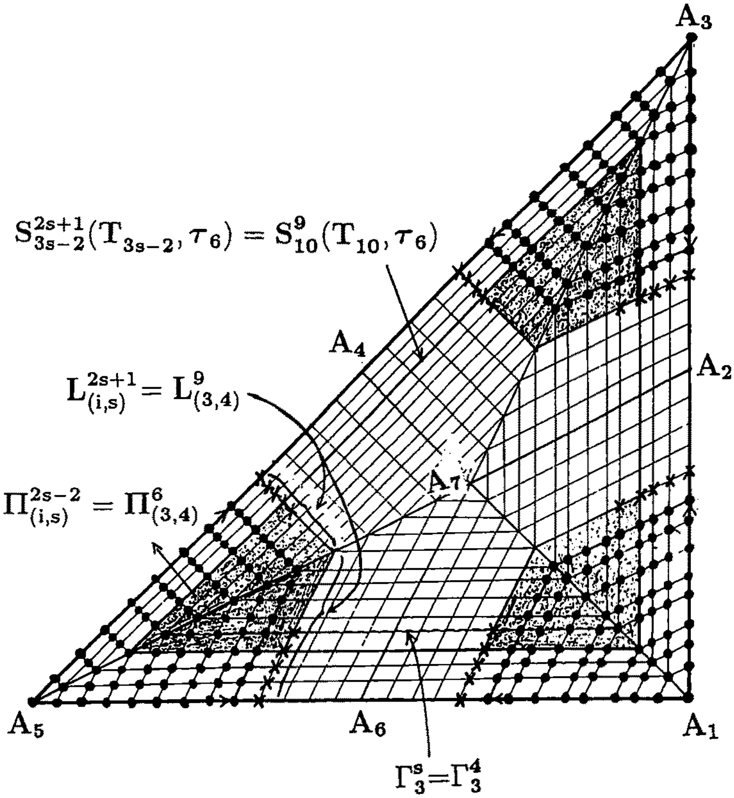

Figure 4.

Let us remark that the plate $\Pi^{2s+1}_{(i,s)}$ contains $\Pi^{2s-2}_{(i,s)}$. The later inclusion implies that the partial derivatives of w^* of order up to $2s-2$ at the vertices of the subtriangle T_{3s-2} are zero. We have $\mathcal{S}^{2s+1}_{3s-2}(T_{3s-2}, \tau_6) \subset \mathcal{S}^{2s-1}_{3s-2}(T_{3s-2}, \tau_6)$. From the induction hypothesis we get $w^* \equiv 0$. Therefore all B-coefficients of w are zero. □

Definition 3. Let

$$\mathcal{S}^{r,r+[r/2]}_{n(r)}(\tau) = \left\{ \begin{array}{l} v \in C^r(\mathbb{R}^2)\colon\ v|_T \in \mathcal{S}^{r,r+[r/2]}_{2r}(T, \tau_6) \text{ (resp. } \mathcal{S}^{r+1,r+[r/2]}_{2r+1}(T, \tau_6)) \\ \qquad \text{for } r \text{ odd (resp. even) for all } T \in \tau \end{array} \right\}.$$

Theorem 1.

(1) Given $u \in C^m(\mathbb{R}^2)$, $m \geqslant r + [r/2]$, there exists a unique function $v_T \in \mathcal{S}^{r,r+[r/2]}_{2r}(T, \tau_6)$ (resp. $\mathcal{S}^{r+1,r+[r/2]}_{2r+1}(T, \tau_6)$) for r odd (resp. even) satisfying the following interpolation conditions:

$$D^\alpha v_T(A_{2i-1}) = D^\alpha u(A_{2i-1}) \quad \text{for all } |\alpha| \leqslant r + [r/2] \text{ and } i = 1, 2, 3. \qquad (8)$$

(2) The global interpolant v defined on $\mathbb{R}^2$ by $v|_T = v_T$ for all $T \in \tau$ is an element of the space $\mathcal{S}^{r,r+[r/2]}_{n(r)}(\tau)$.

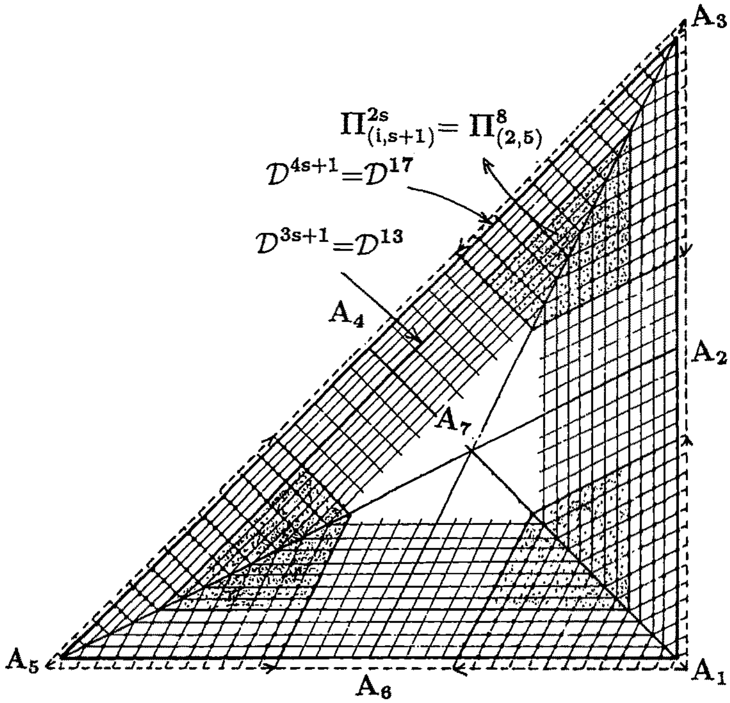

Figure 5.

Proof. (1) We prove the local interpolation scheme for the space $\mathcal{S}^{2s+1,3s}_{4s+1}(T,\tau_6)$ (see figure 5). The derivative data (8) determine the $3s$th order plates $\Pi^{3s}_{(i,s+1)}$ $(1 \leqslant i \leqslant 3)$ (i.e., the black coefficients). The C^{2s+1}-continuity of v_T across each edge A_7A_{2i} determines the non-marked B-coefficients on all rows Γ^k_i, $k \leqslant 2s$: they can be computed from the corresponding black ones (see lemma 2). In particular, all rows Γ^k_i, $k \leqslant s$, $1 \leqslant i \leqslant 3$ are known. The remaining parameters form the disk $\mathcal{D}^{3s+1}$ which is associated with a spline $w \in \mathcal{S}^{2s+1}_{3s+1}(T_{3s+1},\tau_6)$. Since $\Pi^{3s}_{(i,s+1)}$ is in $\mathcal{D}^{3s+1}$, hence $\Pi^{2s}_{(i,s+1)}$ is included in $\mathcal{D}^{3s+1}$, therefore the derivatives $D^\alpha w$, $|\alpha| \leqslant 2s$ at the vertices of the subtriangle T_{3s+1} are known. By lemma 4, w is uniquely determined.

The scheme of the space $\mathcal{S}^{2s+1,3s+1}_{4s+2}(T,\tau_6)$ is a straightfoward consequence of the preceding result. Indeed, we remark that the disk $\mathcal{D}^{4s+1}$ situated in the set of B-coefficients of $v_T \in \mathcal{S}^{2s+1,3s+1}_{4s+2}(T,\tau_6)$ is associated with a spline $w_1 \in \mathcal{S}^{2s+1,3s}_{4s+1}(T,\tau_6)$. So the data $D^\alpha v_T(A_{2i-1})$, $|\alpha| \leqslant 3s+1$, $1 \leqslant i \leqslant 3$, determine the derivatives $D^\alpha w_1$, $|\alpha| \leqslant 3s$, $1 \leqslant i \leqslant 3$, at the vertices of T_{4s+1}.

(2) In order to prove the second part of the theorem it suffices to show that the global function v belongs to $C^r(\mathbb{R}^2)$.

Let v_T and v_{T^*} be the two finite elements uniquely defined on adjacent triangles T and T^* with $T \cap T^* = \Gamma_3$ for example, by interpolating the same function u. Let A_7^* be the corresponding center of gravity of T^*. Since A_7A_2 and $A_7^*A_2$ are colinear and not parallel to Γ_3, it suffices to show that $g_k = D^k(v_T - v_{T^*}).(A_7A_2)^k|_{\Gamma_3}$ is identically zero for $k \leqslant r$.

Indeed, g_k is a univariate spline in the space $\mathcal{S}^{2s+1}_{n(s)-k}(\Gamma_3)$ for $r = 2s$ (resp. $2s+1$) and $n(s) = 4s+1$ (resp. $4s+2$) (it is a polynomial of degree $2s+1$ for $k = r$). We have dim $\mathcal{S}^{2s+1}_{n(s)-k}(\Gamma_3) = 2(3s+1-k)$ for $r = 2s$ (resp. $2(3s+2-k)$ for $r = 2s+1$) and g_k is uniquely determined by the classical degrees of freedom $g_k^{(m)}(A_1)$, $g_k^{(m)}(A_3)$ for $m \leqslant r + [r/2] - k$ (here $g_k^{(m)}$ denotes the mth derivative of g_k). The values $g_k^{(m)}(A_i)$, $i = 1,3$ are computed from total derivatives of order $k + m \leqslant r + [r/2]$ of $v_T - v_{T^*}$ which are zero because v_T and v_{T^*} interpolate the same function u at the ends of Γ_3. Thus $g_k^{(m)}(A_1) = g_k^{(m)}(A_3) = 0$ and therefore $g_k \equiv 0$. □

Remark 2. For r even, the local finite element is of class C^{r+1}, but the global interpolant is only of class C^r.

4. Hermite basis and interpolation error

Let $n(r) = 2r$ for r odd (resp. $n(r) = 2r+1$ for r even). For any triangle $T = a_1a_2a_3$ in τ, let $y_i = a_ia_{i+1}$, $z_i = a_ia_{i-1}$ (where the index i is counted modulo 3) be the directions of the edges of T. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ with $|\alpha| = \alpha_1 + \alpha_2 \leqslant r + [r/2]$ and $i \in \{1,2,3\}$, we define the local Hermite basic splines $\Psi_{i,\alpha} \in \mathcal{S}^{r,r+[r/2]}_{2r}(T, \tau_6)$ (resp. $\mathcal{S}^{r+1,r+[r/2]}_{2r+1}(T, \tau_6)$) for r odd (resp. even) by the following conditions:

(1) $D^{p+q}\Psi_{i,\alpha}(a_i).(y_i^p, z_i^q) = \delta_{p\alpha_1}\delta_{q\alpha_2}$ for $p + q \leqslant r + [r/2]$, where

$$\delta_{kl} = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases}$$

(2) $D^\eta\Psi_{i,\alpha}(a_j) = 0$ for $j \neq i$ and $|\eta| \leqslant r + [r/2]$.

The local expression of the interpolant $\pi_T u$ of u on T (defined by data (8) in theorem 1) can be written in the local Hermite basis as follows:

$$\pi_T u(x) = \sum_{i=1}^{3} \sum_{|\alpha| \leqslant r+[r/2]} D^{|\alpha|}u(a_i).\left(y_i^{\alpha_1}; z_i^{\alpha_2}\right)\Psi_{i,\alpha}(x). \tag{9}$$

Now suppose that the vertices of τ are the points of $\mathbb{Z}^2$. Another global expression is given as follows: for $\alpha \in \mathbb{N}^2$, $|\alpha| \leqslant r + [r/2]$, let Φ_α be the spline defined in $\mathcal{S}^{r,r+[r/2]}_{n(r)}(\tau)$ by the data

$$D^\eta\Phi_\alpha(a) = \delta_{0a}\delta_{\alpha\eta} \quad \text{for } \eta \in \mathbb{N}^2,\ |\eta| \leqslant r + [r/2] \text{ and } a \in \mathbb{Z}^2.$$

The support of Φ_α is the hexagon centered at the origin (see figure 6).

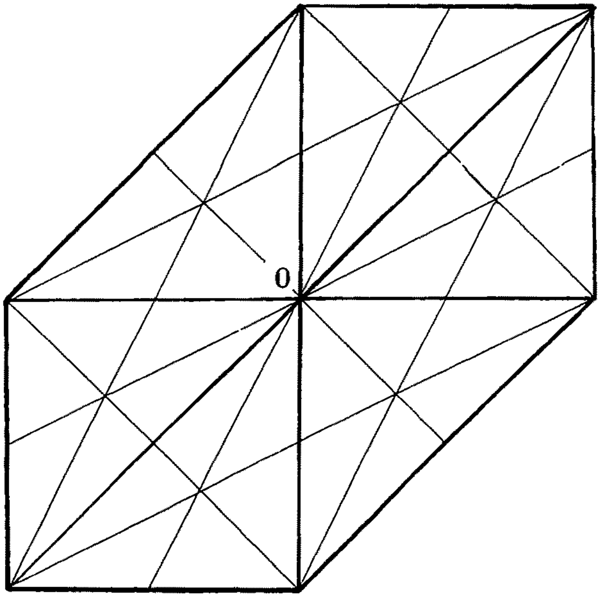

Figure 6.

The global interpolant πu of u obtained by assembling PS finite elements can be written as a linear combination of translates of Φ_α, i.e.,

$$\pi u(x) = \sum_{a\in\mathbb{Z}^2} \sum_{|\alpha|\leqslant r+[r/2]} D^\alpha u(a).\Phi_\alpha(x-a).$$

Since our problem is local, we give in the following theorem the interpolation error corresponding to the form (9) using the following notations: for a function f defined and bounded on a subset $X \subset \mathbb{R}^2$ we set

$$\|f\|_{\infty,X} = \sup_{x\in X} \left|f(x)\right|.$$

Let $D^m f$ be the total derivative of order m of $f \in C^m(X)$ and let

$$\left\|D^m f(x)\right\|_\infty = \sup\left\{\left|D^m f(x).(v_1,\dots,v_m)\right|,\ v_i \in \mathbb{R}^2,\ \|v_i\| = 1,\ 1 \leqslant i \leqslant m\right\},$$

where $\|v\|$, $v \in \mathbb{R}^2$ is the usual Euclidean norm in $\mathbb{R}^2$. Finally we set

$$M_m(f) = \max_{x\in X} \left\|D^m f(x)\right\|_\infty.$$

Theorem 2. Given $u \in C^{n(r)+1}(\mathbb{R}^2)$, there holds

$$\|\pi_T u - u\|_{\infty,T} \leqslant C_r h^{n(r)+1} M_{n(r)+1}(u),$$

where C_r is a positive constant depending only on r and h is the diameter of T.

Proof. We use the Taylor formula for u at the vertex a_i $(i = 1, 2, 3)$:

$$u(a_i) = \sum_{k=0}^{n(r)} \frac{1}{k!} D^k u(x).(x\, a_i)^k + \frac{1}{n(r)!} J^0(u, a_i)(x),$$

where

$$J^0(u, a_i)(x) = \int_0^1 (1-t)^{n(r)} D^{n(r)+1} u\big(\gamma_i(x)\big).(x\, a_i)^{n(r)+1}\, \mathrm{d}t,$$
$$\gamma_i(x) = \theta_i x + (1-\theta_i) a_i, \quad 0 < \theta_i < 1.$$

Similarly, we have for $1 \leqslant |\alpha| = \alpha_1 + \alpha_2 \leqslant r + [r/2]$:

$$D^{|\alpha|} u(a_i).\big(y_i^{\alpha_1}; z_i^{\alpha_2}\big) = \sum_{k=|\alpha|}^{n(r)} \frac{1}{(k-|\alpha|)!} D^k u(x).\big((x\, a_i)^{k-|\alpha|};\ y_i^{\alpha_1};\ z_i^{\alpha_2}\big)$$
$$+ \frac{1}{(n(r)-|\alpha|)!} J^\alpha(u, a_i)(x).\big(y_i^{\alpha_1}; z_i^{\alpha_2}\big),$$

where

$$J^\alpha(u, a_i)(x).\big(y_i^{\alpha_1}; z_i^{\alpha_2}\big)$$
$$= \int_0^1 (1-t)^{n(r)-|\alpha|} D^{n(r)+1} u\big(\gamma_i(x)\big).\big((x\, a_i)^{n(r)+1-|\alpha|}; y_i^{\alpha_1}; z_i^{\alpha_2}\big).$$

Using the technique of multipoint Taylor formulas (see Ciarlet [9, chapter 3], Arcangeli and Gout [1] and Gout [11]), we get

$$\pi_T u(x) - u(x) = \frac{1}{n(r)!} \left[\sum_{i=1}^{3} J^0(u, a_i)(x) \Psi_{i,\alpha}(x) \right]$$
$$+ \sum_{i=1}^{3} \sum_{1 \leqslant |\alpha| \leqslant r+[r/2]} \frac{1}{(n(r)-|\alpha|)!} J^\alpha(u, a_i)(x).\big(y_i^{\alpha_1}; z_i^{\alpha_2}\big) \Psi_{i,\alpha}(x).$$

Clearly, we have for all $x \in T$,

$$\big|J^0(u, a_i)(x)\big| \leqslant \frac{1}{n(r)+1} M_{n(r)+1}(u) h^{n(r)+1}.$$

Similarly, for $1 \leqslant |\alpha| \leqslant r + [r/2]$

$$\big|J^\alpha(u, a_i)(x).\big(y_i^{\alpha_1}; z_i^{\alpha_2}\big)\big| \leqslant \frac{1}{n(r)+1-|\alpha|} M_{n(r)+1}(u) h^{n(r)+1}.$$

Table 1

$\|\Psi_{1,0}\|_{\infty,T}$	1
$\|\Psi_{1,(1,0)}\|_{\infty,T}$	2.411×10^{-1}
$\|\Psi_{1,(2,0)}\|_{\infty,T}$	3.063×10^{-2}
$\|\Psi_{1,(1,1)}\|_{\infty,T}$	2.441×10^{-2}
$\|\Psi_{1,(3,0)}\|_{\infty,T}$	1.800×10^{-3}
$\|\Psi_{1,(2,1)}\|_{\infty,T}$	1.485×10^{-3}

Thus we obtain

$$\begin{aligned}
&\|\pi_T u - u\|_{\infty,T} \\
&\quad \leqslant \frac{1}{(n(r)+1)!}\left(\sum_{i=1}^{3}\|\Psi_{i,0}\|_{\infty,T}\right) M_{n(r)+1}(u) h^{n(r)+1} \\
&\quad + \left(\sum_{i=1}^{3}\sum_{1\leqslant|\alpha|\leqslant r+[r/2]} \frac{1}{(n(r)-|\alpha|+1)!}\|\Psi_{i,\alpha}\|_{\infty,T}\right) M_{n(r)+1}(u) h^{n(r)+1}.
\end{aligned}$$

Therefore the constant of the theorem is given by

$$C_r = \frac{1}{(n(r)+1)!}\sum_{i=1}^{3}\|\Psi_{i,0}\|_{\infty,T} + \sum_{i=1}^{3}\sum_{1\leqslant|\alpha|\leqslant r+[r/2]} \frac{1}{(n(r)-|\alpha|+1)!}\|\Psi_{i,\alpha}\|_{\infty,T}.$$

□

Examples.

(1) For $r = 1$ it is shown in Sablonnière [25] that $C_1 \leqslant 1/3$.

(2) For $r = 2$ and $n(2) = 5$, we estimate C_2 by direct computation of norms of functions $\Psi_{i,\alpha}$ of the Hermite basis. The result is obtained by a computer code. In table 1 norms of $\Psi_{1,0}$; $\Psi_{1,(1,0)}$; $\Psi_{1,(2,0)}$; $\Psi_{1,(1,1)}$; $\Psi_{1,(3,0)}$; $\Psi_{1,(2,1)}$ are given. We have the same norms for the other functions by permutation of indices $i = 1, 2, 3$.

From table 1 we deduce the following estimate for C_2:

$$C_2 \simeq 0.03021.$$

5. Conclusions

(1) The only required data are *partial derivatives at the vertices of the triangulation* τ. Hence there is no degree of freedom of type 2 (according to the terminology of Ženíšek [29]), i.e., derivatives at points inside T or inside edges of T. Thus, the above finite elements are perhaps more interesting for use on the uniform

mesh τ than the HCT type elements, which need partial derivatives at the vertices together with normal derivatives at some interior points of the edges of τ (see [16, 17]).

(2) The functions forming the basis of the interpolation space have the same support as displayed in figure 6. These functions have smaller supports than some box-splines constructed on the mesh τ by several authors (see, e.g., [4]). But of course the structure of the local space is more complicated.

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