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SYNTACTIC STRUCTURE AND SEMANTICAL REFERENCE I

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w 70 rocznicę Jego urodzin

The syntactical and semantical investigations in contemporary formal logic refer always to the languages with specified syntactic structure, as with respect to such languages one can formulate exactly and, subsequently examine with mathematical tools (1) the rules of transformation (axioms, rules of inference) and the systems based on these rules (formalized theories), (2) the relations of semantical reference which occur between linguistic expressions and elements of objective sphere.

Our considerations belong to that part of logical syntax and semantics which is independent of any assumptions concerning the rules of transformation.

The syntactic structure of some language $\mathcal{L}$ is determined 1° by the *vocabulary* of $\mathcal{L}$ i.e. by the list of simple (undecomposable) expressions in $\mathcal{L}$, and 2° by the *rules of construction* of $\mathcal{L}$ which state how the expressions of $\mathcal{L}$, especially the sentences in $\mathcal{L}$ are built of simple expressions.

In the first part of this paper we consider the general principles of the syntactic structure of languages. Namely, we shall formulate a scheme of the syntactic structure of language. This scheme will be called the *standard formalization*[1] and the languages which fall under this scheme will be called the *standard formalized languages*[1].

The scheme of standard formalization is based on a purely syntactical classification of expressions into so called *semantical categories.*

The standard formalization is an abstract from the concrete material of artificial symbolic languages which are considered in formal logic. It is general in the following sense: every symbolic language known in formal logic — after carrying some modifications in its calligraphy — falls directly under the scheme of standard formalization.

In the second part of this paper we consider the fundamental properties of semantic reference. First, we introduce a classification of objects into so called

[1] These terms are borrowed from A. TARSKI [7], p. 5, but they are used here in a different sense.

ontological categories. Further making use of some simple and quite natural connexion of conformity between semantical categories of simple expressions and ontological categories of corresponding objects, we can introduce the general notion of a *model* of any standard formalized language. Namely, for every standard formalized language $\mathcal{L}$ we define the family $M(\mathcal{L})$ of all models of $\mathcal{L}$. Every model of $\mathcal{L}$ is a totality to which the expressions of $\mathcal{L}$ *can* refer semantically and, conversely, every totality to which the expressions of $\mathcal{L}$ *can* refer semantically, belongs to the family $M(\mathcal{L})$. Thus, we obtain a general scheme of the relations of semantical reference which is quite closely connected with the scheme of standard formalization. This shows the ideographic character of standard formalized languages.

It may be a reasonable conjecture the content of this paper to be connected with the structural inquires in linguistcs and with some problems of the philosophy of language and of thinking. But, we do not discuss here these connexions.

In describing the scheme of standard formalization we make use of suitable indices. The method of applying indices in investigations of syntactic structure was introduced by K. Ajdukiewicz [1], [2]. We employ here such a modification of Ajdukiewicz's method which allows to take into account the syntactical role of variables and of operators binding the variables. The indices introduced below will be applied in the classification of expressions into semantical categories and in the classification of objects into ontological categories.

1. *The indices*

The indices which will be used in our considerations are divided into *three ranks*. Every index $\mathfrak{i}$ has a rank $r(\mathfrak{i}) = 0, 1, 2$. The indices are symbolic figures built in some manner of the following letters marked with natural numbers:

$$\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_k, \ldots$$

These letters are indices of rank 0. The letter $\mathfrak{t}_0$ is called the *principal index* and the remaining letters $\mathfrak{t}_k$ we call the *secondary indices*. When distinguishing the principal index we make only one distinction among the indices of rank 0.

If besides the principal index $\mathfrak{t}_0$ one secondary index $\mathfrak{t}_k$ will only occur in our considerations, then we write the letter

$$\mathfrak{t}$$

instead of this secondary index $\mathfrak{t}_k$ and the letter

$$\mathfrak{s}$$

instead of the principal index $\mathfrak{t}_0$.

The indices of rank 1 are fractions of the form

$$(1.1) \qquad \frac{\mathfrak{t}_k}{\mathfrak{t}_{k_1} \ldots\ldots \mathfrak{t}_{k_n}}$$

in which the numerators are arbitrary indices of rank 0 and the denominators are arbitrary finite sequences of indices of rank 0.

Finally, the indices of rank 2 are fractions of the form

$$\frac{\mathfrak{t}_k}{\mathfrak{i}_1 \,\ldots\ldots\, \mathfrak{i}_n} \tag{1.2}$$

in which the numerators are arbitrary indices of rank 0 and the denominators are such finite sequences of indices of rank 0 or 1 in which at least one index of rank 1 occurs.

We admit the terminol‑gy according to which the numerator of any index $\mathfrak{t}_k$ of rank 0 is this index itself.

Examples of indices of rank 1:

$$\frac{\mathfrak{s}}{\mathfrak{t}} \qquad \frac{\mathfrak{s}}{\mathfrak{t}\mathfrak{t}} \qquad \frac{\mathfrak{s}}{\mathfrak{t}\mathfrak{t}\mathfrak{t}}$$

$$\frac{\mathfrak{t}}{\mathfrak{t}} \qquad \frac{\mathfrak{t}}{\mathfrak{t}\mathfrak{t}} \qquad \frac{\mathfrak{t}}{\mathfrak{t}\mathfrak{t}\mathfrak{t}}$$

$$\frac{\mathfrak{s}}{\mathfrak{s}} \qquad \frac{\mathfrak{s}}{\mathfrak{s}\mathfrak{s}} \qquad \frac{\mathfrak{s}}{\mathfrak{s}\mathfrak{s}\mathfrak{s}}$$

$$\frac{\mathfrak{s}}{\mathfrak{s}\mathfrak{t}} \qquad \frac{\mathfrak{t}}{\mathfrak{t}\mathfrak{s}}$$

Examples of indices of rank 2:

$$\frac{\mathfrak{s}}{\frac{\mathfrak{s}}{\mathfrak{t}}} \qquad \frac{\mathfrak{s}}{\frac{\mathfrak{s}}{\mathfrak{t}}\frac{\mathfrak{s}}{\mathfrak{t}}}$$

$$\frac{\mathfrak{t}}{\frac{\mathfrak{s}}{\mathfrak{t}}} \qquad \frac{\mathfrak{t}}{\frac{\mathfrak{s}}{\mathfrak{t}}\frac{\mathfrak{s}}{\mathfrak{t}}} \qquad \frac{\mathfrak{t}}{\frac{\mathfrak{s}}{\mathfrak{t}\mathfrak{t}}} \qquad \frac{\mathfrak{t}}{\frac{\mathfrak{t}}{\mathfrak{t}}} \qquad \frac{\mathfrak{t}}{\frac{\mathfrak{t}}{\mathfrak{t}\mathfrak{t}}}$$

$$\frac{\mathfrak{s}}{\mathfrak{t}\frac{\mathfrak{s}}{\mathfrak{t}}} \qquad \frac{\mathfrak{s}}{\mathfrak{t}\frac{\mathfrak{s}}{\mathfrak{t}}\frac{\mathfrak{s}}{\mathfrak{t}}}$$

$$\frac{\mathfrak{t}}{\mathfrak{t}\frac{\mathfrak{s}}{\mathfrak{t}}} \qquad \frac{\mathfrak{s}}{\frac{\mathfrak{t}}{\mathfrak{s}\mathfrak{t}}} \qquad \frac{\mathfrak{s}}{\frac{\mathfrak{s}}{\mathfrak{t}\mathfrak{s}}}$$

Let $\mathfrak{j}$ be an index of rank 0, 1, 2. We define the *multiplicity* of an index $\mathfrak{t}_k$ of rank 0 in the index $\mathfrak{j}$ — symbolically $m(\mathfrak{t}_k, \mathfrak{j})$:

(0) if $r(\mathfrak{j}) = 0$ then $m(\mathfrak{t}_k, \mathfrak{j}) = 0$;

(1) if $r(\mathfrak{j}) = 1$ then $m(\mathfrak{t}_k, \mathfrak{j}) =$ the number of all occurences of the index $\mathfrak{t}_k$ in the denominator of the index $\mathfrak{j}$;

(2) if $r(\mathfrak{j}) = 2$ then $m(\mathfrak{t}_k, \mathfrak{j})$ = the greatest multiplicity of the index $\mathfrak{t}_k$ in the indices which are occurring in the denominators of the index $\mathfrak{j}$.

For example if $v \neq w$ then $m(\mathfrak{t}_v, \mathfrak{t}_v) = m\left(\mathfrak{t}_v, \frac{\mathfrak{t}_v}{\mathfrak{t}_w}\right) = 0$, $m\left(\mathfrak{t}_v, \frac{\mathfrak{t}_k}{\mathfrak{t}_v\mathfrak{t}_w}\right) = 1$, $m\left(\mathfrak{t}_v, \frac{\mathfrak{t}_w}{\mathfrak{t}_v\mathfrak{t}_v}\right) = 2$, $m(\mathfrak{t}_w, \mathfrak{t}_v) = m\left(\mathfrak{t}_w, \frac{\mathfrak{t}_w}{\mathfrak{t}_v\mathfrak{t}_v}\right) = 0$, $m\left(\mathfrak{t}_w, \frac{\mathfrak{t}_v}{\mathfrak{t}_w}\right) = m\left(\mathfrak{t}_w, \frac{\mathfrak{t}_k}{\mathfrak{t}_v\mathfrak{t}_w}\right) = 1$.

The notion of multiplicity $m(\mathfrak{t}_k, \mathfrak{j})$, however, will play much important role for our considerations in these cases in which $r(\mathfrak{j}) = 2$. For example, is the index $\mathfrak{j}$ of the following form:

(1.3)
$$\frac{\mathfrak{t}_k}{\mathfrak{t}_v \frac{\mathfrak{t}_v}{\mathfrak{t}_w} \frac{\mathfrak{t}_k}{\mathfrak{t}_v\mathfrak{t}_w} \frac{\mathfrak{t}_w}{\mathfrak{t}_v\mathfrak{t}_v}}$$

where $v \neq w$ then $m(\mathfrak{t}_v, \mathfrak{j}) = 2$ and $m(\mathfrak{t}_w, \mathfrak{j}) = 1$.

Let f be a function defined for each index and assigning to every index $\mathfrak{j}$ an index $f(\mathfrak{j})$ in such a way that:

$$\text{if } \mathfrak{j}_1 \neq \mathfrak{j}_2 \text{ then } f(\mathfrak{j}_1) \neq f(\mathfrak{j}_2).$$

If the function f fulfils the following conditions:

(a) $f(\mathfrak{t}_0) = \mathfrak{t}_0$,

(b) if $\mathfrak{t}_k$ is a secondary index of rank 0 then $f(\mathfrak{t}_k)$ is also a secondary index of rank 0,

(c) for $k = 0, 1, 2, \ldots$ and for any indices $\mathfrak{i}_1, \ldots, \mathfrak{i}_n$ of rank 0, 1:

$$f\left(\frac{\mathfrak{t}_k}{\mathfrak{i}_1 \ldots \mathfrak{i}_n}\right) = \frac{f(\mathfrak{t}_k)}{f(\mathfrak{i}_1) \ldots f(\mathfrak{i}_n)}$$

then the function f is called a *permutation of* (*secondary*) *indices.*

2. *The idea of the standard formalization*

The conception of the standard formalization is derived from the following general ideas concerning the syntactic structure of language.

(1) Among the expressions we distinguish the fundamental (categorematic) expressions and operators (syncategorematic expressions);

(2) The fundamental expressions are divided into fundamental semantical categories[2] among which there is the category of sentences and, eventually, some other fundamental semantical categories;

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[2] The terms *semantical category* (Bedeutungskategorie) was introduced by E. HUSSERL and, subsequently, used by ST. LEŚNIEWSKI and K. AJDUKIEWICZ in connexion with some relations of exchange (or substitution) of expressions in sentences; see for example [2]. In our paper we use this term in a different sense. The semantical categories in our sense will be defined later in a purely syntactical way, but they are closely connected with the kinds of semantical reference. This will be explained in the second part of our paper.

(3) The expressions may be simple i. e. single words or compound i. e. built from the simple expressions according to some rules of construction;

(4) There are two kinds of simple expressions: constants and variables; the variables are simple fundamental expressions;

(5) The operators are closely connected with the rules of construction and they are simple nonfundamental expressions;

(6) There are many semantical categories of operators, but all operators can be divided into two kinds: the operators of rank 1 do not bind the variables and the operators of rank 2 do.

We will describe the scheme of standard formalization by means of the syntactically marked and syntactically coherent vocabularies. The marking syntactically of a vocabulary replaces completely the rules of construction and, therefore, every standard formalized language is determined by a suitably marked and coherent vocabulary.

3. *Syntactically marked vocabularies*

Any nonempty set of arbitrary elements called *word-types*, when divided into *constant-types* and *variables-types*, is called a *vocabulary*. We assume that in every vocabulary the set of constant-types is nonempty and the set of variable-types is empty or infinite (i. e. denumerably infinite).

Let $\mathcal{V}$ be a vocabulary. Every function $\mathfrak{J}$ which assigns to each word-type ζ in $\mathcal{V}$ an index $\mathfrak{J}(\zeta)$ and fulfils two following conditions:

(a) if ζ is a variable-type in $\mathcal{V}$ then the corresponding index $\mathfrak{J}(\zeta)$ is of rank 0

(b) the set of all variable-types in $\mathcal{V}$ to which a common index of rank 0 is assigned, is empty or infinite,

is called a *syntactic-marking-function*.

We decide to represent in general considerations the variable-types marked with the index $\mathfrak{t}_k$, by the following signs:

$$\xi_1^{(k)}, \xi_2^{(k)}, \ldots, \xi_N^{(k)}, \ldots \tag{2.1}$$

where $k = 0, 1, 2, \ldots$.

Thus, any *syntactically marked vocabulary* $\mathcal{V}$ may be represented in the following form:

$$(\mathcal{V}) \qquad \left\langle \begin{matrix} \zeta_1, \zeta_2, \ldots, \zeta_n, \ldots; & \Xi^{(0)}, \Xi^{(1)}, \ldots, \Xi^{(k)}, \ldots \\ \mathfrak{i}_1, \mathfrak{i}_2, \ldots, \mathfrak{i}_n, \ldots; & \mathfrak{t}_0, \mathfrak{t}_1, \ldots, \mathfrak{t}_k, \ldots \end{matrix} \right\rangle$$

where the word-types $\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots$ are all distinct constant-types in $\mathcal{V}$ which are marked with the indices $\mathfrak{i}_1, \mathfrak{i}_2, \ldots, \mathfrak{i}_n, \ldots$ correspondingly and every set $\Xi^{(k)}$ for $k = 0, 1, 2, \ldots$, is empty or identical with the set

$$\{\xi_N^{(k)}\}_{N=1,2,\ldots}$$

of all variable-types (2.1) marked with the index $\mathfrak{t}_k$.

Every word-type (constant-type or variable-type) marked with an index of rank 0 or 1 or 2 is called a *fundamental word-type* or an *operator-type of rank* 1 or an *operator-type of rank* 2 respectively.

We take the following vocabulary $\mathcal{V}^{\star}$ as an example of a syntactically marked vocabulary:

0)	t	$x, y, z, \ldots$ or $x_1, x_2, \ldots, x_3, \ldots$	(variables)
1)	t	**a**	Aristotle
2)	t	0	zero
3)	$\frac{t}{tt}$	$+$	plus
4)	$\frac{\mathfrak{s}}{t}$	**L**	a logician
5)	$\frac{\mathfrak{s}}{t}$	**N**	a number
6)	$\frac{\mathfrak{s}}{tt}$	**M**	more wise
7)	$\frac{\mathfrak{s}}{tt}$	$>$	greater
8)	$\frac{\mathfrak{s}}{tt}$	$=$	identical
9)	$\frac{\mathfrak{s}}{\mathfrak{s}}$	$\sim$	it is not the case that
10)	$\frac{\mathfrak{s}}{\mathfrak{s}\mathfrak{s}}$	$\wedge$	and
11)	$\frac{\mathfrak{s}}{\mathfrak{s}\mathfrak{s}}$	$\rightarrow$	if ... then ...
12)	$\frac{\mathfrak{s}}{\mathfrak{s}\mathfrak{s}}$	$\vee$	or
13)	$\frac{\mathfrak{s}}{\frac{\mathfrak{s}}{t}}$	$\bigwedge$	for every
14)	$\frac{\mathfrak{s}}{\frac{\mathfrak{s}}{t}}$	$\bigvee$	for some
15)	$\frac{t}{\mathfrak{s}}$	L	the object ... such that ...

16) $\dfrac{\mathfrak{s}}{\mathfrak{t}\ \frac{\mathfrak{s}}{\mathfrak{t}}}$ an object ... such that ...

The word-types of $\mathcal{V}^*$ are placed in the middle column and the corresponding indices are placed in the left column. The right column contains the transcriptions of word-types into the natural language.[3]

4. *The vocabularies and the rules of construction*

Every syntactically marked vocabulary $\mathcal{V}$ determines in an unambiguous manner *the set* $\mathcal{L}(\mathcal{V})$ *of all expressions which are constructible on the ground of* $\mathcal{V}$. Some sets $\mathcal{L}(\mathcal{V})$, for suitable vocabularies $\mathcal{V}$, will be identified with the *standard formalized languages.*

The operation which leads from any syntatically marked vocabulary $\mathcal{V}$ to the corresponding set $\mathcal{L}(\mathcal{V})$ will be described in strictly formal way in a later section. In the present one, however, we consider some details of this operation in a somewhat informal way.

It has been already remarked that the rules of construction of compound expressions can be replaced by marking syntactically a suitable vocabulary. It will be shown that every operator-type in a syntactically marked vocabulary determines a rule of construction. The elements of the set $\mathcal{L}(\mathcal{V})$, for a syntactically marked vocabulary $\mathcal{V}$, are (1) words or simple expressions which correspond directly to the word-types in $\mathcal{V}$ and (2) compound expressions which can be built from simple expressions according to the rules of construction determined

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[3] The four last lines of $\mathcal{V}^*$ contain four operator-types of rank 2 known in formal logic. They are successively: the general quantifier $\bigwedge$, the existential quantifier $\bigvee$, the singular descriptive operator $\llcorner$, and the general descriptive operator $\perp$.

The operator $\llcorner$ occurs in the phrases of the form

the object x such that so and so

or symbolically

$$\underset{x}{\llcorner}\,\alpha(x)$$

The rules of transformation concerning the operator $\llcorner$ were formulated by HILBERT-BERNAYS, FREGE and RUSSELL. These rules are discussed in the book [3], pp. 32—39.

The operator $\perp$ occurs in the phrases of the form

ϑ is an object x such that so and so

or symbolically

$$\vartheta\ \varepsilon\ \underset{x}{\perp}\,\alpha(x)$$

The customary rule of transformation concerning the operator $\perp$ allows to replace mutually the expressions of the form

$$\vartheta\ \varepsilon\ \underset{x}{\perp}\,\alpha(x) \qquad\qquad \alpha(\vartheta)$$

by the operator-types in $\mathcal{V}$. Every expression $\mathcal{L}(\mathcal{V})$ is supplied with exactly one index. Namely, we assign to the simple expressions in $\mathcal{L}(\mathcal{V})$ the indices with which the corresponding word-types in $\mathcal{V}$ are marked. The rules of construction, however, determined by the operator-types in $\mathcal{V}$ assign unambiguously an index of rank 0 to every compound expression belonging to $\mathcal{L}(\mathcal{V})$. Such expressions in $\mathcal{L}(\mathcal{V})$ to which the indices of rank 0 are assigned, are called fundamental expressions constructible on the ground of $\mathcal{V}$.

We want to show now in a somewhat informal way the connexion between operator-types and the rules of construction of compound expressicns. Namely, we will describe the syntactic role played by the operator-types in $\mathcal{V}$ in the process of building of compound expressions.

Let us assume that all word-types in a syntactically marked vocabulary $\mathcal{V}$ are graphic symbols of different forms which are put together on some list as in the case of the vocabulary $\mathcal{V}^\star$ given above.

Then, a graphic symbol is a simple expression or word belonging to the set $\mathcal{L}(\mathcal{V})$ if and only if it is of the same form as some word-type in $\mathcal{V}$. It follows that the simple expressions belonging to $\mathcal{L}(\mathcal{V})$ are divided into (1) fundamental simple expressions or fundamental words (spliting into constants and variables) with an index of rank 0, (2) operators of rank 1 with an index of rank 1 and (3) operators of rank 2 with an index of rank 2.

The compound expressions, however, belonging to the set $\mathcal{L}(\mathcal{V})$ are the inscriptions which are built in the following inductive manner. We consider two cases of construction of compound expressions: by means of an operator of rank 1 and by means of an operator of rank 2.

We consider the first case of construction. Let the graphic symbol $\overset{*}{\eta}$ be an operator of rank 1 marked by the index (1.1) of rank 1 (p. 214). Then, the operator η is called a n-ary operator of rank 1. If the inscriptions

$$\omega_1, \ldots\ldots, \omega_n$$

are any fundamental expressions, simple or compound, to which the indices $\iota_{k_1}, \ldots, \iota_{k_n}$ of rank 0 being placed in the denominator of (1.1) had been already assigned, then the inscription

$$\overset{*}{\eta}\,(\omega_1, \ldots\ldots, \omega_n) \tag{4.1}$$

is the fundamental compound expression to which we assign the index ι_k of rank 1 placed in the numerator of (1.1).

We consider now the second case of construction; it is more complicated than the first one.[4] Let the graphic symbol $\overset{**}{\eta}$ be an operator of rank 2 marked with the index (1.2) of rank 2 (p. 215). Then, the operator $\overset{**}{\eta}$ is called a n-ary operator

[4] Namely, in the second case we must make use of the notions of *free* occurence and of *bound* occurence of a variable sin an expression. These notions are familiar to formal logicians. They will be explained in a trictly formal way and in full generality in a later section.

of rank 2. Suppose that $\mathfrak{t}_{k_1}, \ldots, \mathfrak{t}_{k_r}$ are all such indices of rank 0 that their multiplicities in the index (1.2) are different from 0 and let these multiplicities be equal to the numbers $m_1, \ldots, m_r$ respectively. Then the construction of compound expressions by means of the operator $\overset{**}{\eta}$ proceeds as follows.

If the graphic symbols

$$\xi^{(k_1)}_{N^1_1}, \ldots\ldots, \xi^{(k_1)}_{N^1_{m_1}}$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\xi^{(k_r)}_{N^r_1}, \ldots\ldots, \xi^{(k_r)}_{N^r_{m_r}}$$

are variables in number $m_1, \ldots, m_r$ and with indices $\mathfrak{t}_{k_1}, \ldots, \mathfrak{t}_{k_r}$ respectively, then we construe firstly the inscription

$$\frac{\overset{**}{\eta}}{\xi^{(k_1)}_{N^1_1} \ldots\ldots \xi^{(k_r)}_{N^r_{m_r}}}$$

which is called a *prefix* and contains the operator $\overset{**}{\eta}$ and all the variables considered just now. Further, if the inscriptions

$$\bar{\omega}_1, \ldots, \bar{\omega}_n$$

are "suitable" fundamental expressions, simple or compound, to which such indices of rank 0 had been already assigned that are numerators of indices $\mathfrak{i}_1, \ldots, \mathfrak{i}_n$ respectively, being placed in the denominator of (1.2), then the inscription

$$(4.2) \qquad \frac{\overset{**}{\eta}}{\xi^{(k_1)}_{N^1_1} \ldots\ldots \xi^{(k_r)}_{N^r_{m_r}}} [\bar{\omega}_1, \ldots\ldots, \bar{\omega}_n]$$

is a fundamental compound expression to which we assign the index $\mathfrak{t}_k$ of rank 0 being placed in the numerator of (1.2).

Yet it remains to explain the "suitableness" of the expressions $\bar{\omega}_1, \ldots, \bar{\omega}_n$. For this purpose it will be enough to consider one of them, namely the expression $\bar{\omega}_l$ where $1 \leqslant l \leqslant n$ and the corresponding index $\mathfrak{i}_l$ being placed in the denominator of (1.2). Let the numbers $m_{l,1}, \ldots, m_{l,r}$ be the multiplicities of the indices $\mathfrak{t}_{k_1}, \ldots, \mathfrak{t}_{k_r}$ respectively in the index $\mathfrak{i}_l$. The expression $\bar{\omega}_l$ is "suitable" if and only if for $s = 1, \ldots, r$ it occurs free in $\bar{\omega}_l$ exactly $m_{l,s}$ variables among

$$\xi^{(k_s)}_{N^s_1}, \ldots\ldots, \xi^{(k_s)}_{N^s_{m_s}}.$$

To illustrate the case of construction of compound expressions by means of operators of rank 2, we consider an example. Suppose that the operator $\overset{**}{\eta}$ of rank 2 has been marked with the index (1.3) of rank 2 (p. 216). Then, the construction runs as follows. Consider the variables

$$\xi^{(w)}_{N0},\ \xi^{(v)}_{N1},\ \xi^{(v)}_{N2}$$

where $N_1 \neq N_2$ and construe the prefix

$$\frac{\overset{**}{\eta}}{\xi^{(w)}_{N0}\ \xi^{(v)}_{N1}\ \xi^{(v)}_{N2}}.$$

Consider now any four such expressions

$$\omega_1,\quad \omega_2(\xi^{(w)}_{N0}),\quad \omega_3(\xi^{(v)}_{N}, \xi^{(w)}_{N0}),\quad \omega_4(\xi^{(v)}_{N1}, \xi^{(v)}_{N2})$$

to which the indices t_v, t_v, t_k, t_w respectively are assigned and which fulfil the following conditions:

(1) None of the variables $\xi^{(w)}_{N0}$, $\xi^{(v)}_{N1}$, $\xi^{(v)}_{N2}$ occurs free in the expression ω_1.

(2) The variable $\xi^{(w)}_{N0}$ occurs free in the expression $\omega_2(\xi^{(w)}_{N0})$, but the variables $\xi^{(v)}_{N_1}$, $\xi^{(v)}_{N_2}$ do not occur in the expression $\omega_2(\xi^{(w)}_{N0})$.

(3) The variable $\xi^{(w)}_{N0}$ and exactly one of the variables $\xi^{(v)}_{N1}$, $\xi^{(v)}_{N2}$ occur free in the expression $\omega_3(\xi^{(v)}_{N}, \xi^{(w)}_{N0})$.

(4) The variables $\xi^{(v)}_{N1}$, $\xi^{(v)}_{N2}$ occur free in the expression $\omega_4(\xi^{(v)}_{N1}, \xi^{(v)}_{N2})$, but the variable $\xi^{(w)}_{N0}$ does not occur free in the expression $\omega_4(\xi^{(v)}_{N1}, \xi^{(v)}_{N2})$.

Finally, construe the fundamental compound expression:

$$\frac{\overset{**}{\eta}}{\xi^{(w)}_{N0}\ \xi^{(v)}_{N1}\ \xi^{(v)}_{N2}}\left[\omega_1,\ \omega_2(\xi^{(w)}_{N0}),\ \omega_3(\xi^{(v)}_{N}, \xi^{(v)}_{N0}),\ \omega_4(\xi^{(v)}_{N1}, \xi^{(v)}_{w2})\right]$$

to which assign the index t_k of rank 0.[5]

5. *The diagrams of expressions*

The preceding discussion on the connexion between the operators and the rules of construction contains no precise formulation of these rules. Usually, the formulations of these rules for artificial symbolic language contain not only the description of syntactic relations which hold between simple expressions occurring in compound expressions, but also take into account some conventions of calligraphy for compound expressions. It is clear to see that we have also adopted in the preceding section some conventions of this kind.

The calligraphy, however, for compound expressions has a little importance from our point of view. Namely, we are interested only in the syntactic struc-

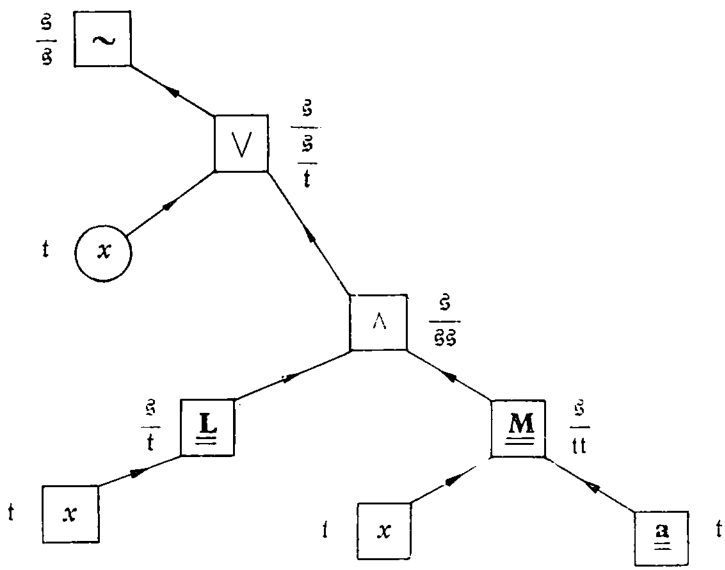
[5] We leave to the reader to verify that the customary constructions of compound expressions by means of quantifiers, descriptive operators and other operators binding the variables, fall under the second case of construction just described.

ture of expressions and, therefore, we inted to describe the standard formalized languages quite independently of any conventions of calligraphy assumed in these languages.[6]

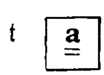
To our purpose we will apply the method of geometric diagrams; comp. [8], pp. 225—226. Namely, we will represent the expressions which are constructible on the ground of some syntactically marked vocabulary, as suitable geometric diagrams which take into account but the simple expressions and the purely syntactic relations between expressions occurring in compound expressions.

To illustrate the method of geometric diagrams we give below, as examples, the diagrams of five expressions which are constructible on the basis of vocabulary $\mathcal{V}^{\star}$ given in the section 3.

I. $\sim \bigvee_{x} (x \, \varepsilon \, \underline{\underline{\mathbf{L}}} \wedge x \, \varepsilon \, \underline{\underline{\mathbf{M}}}(\underline{\underline{\mathbf{a}}}))$ It is not the case that some logician is more wise than Aristotle.



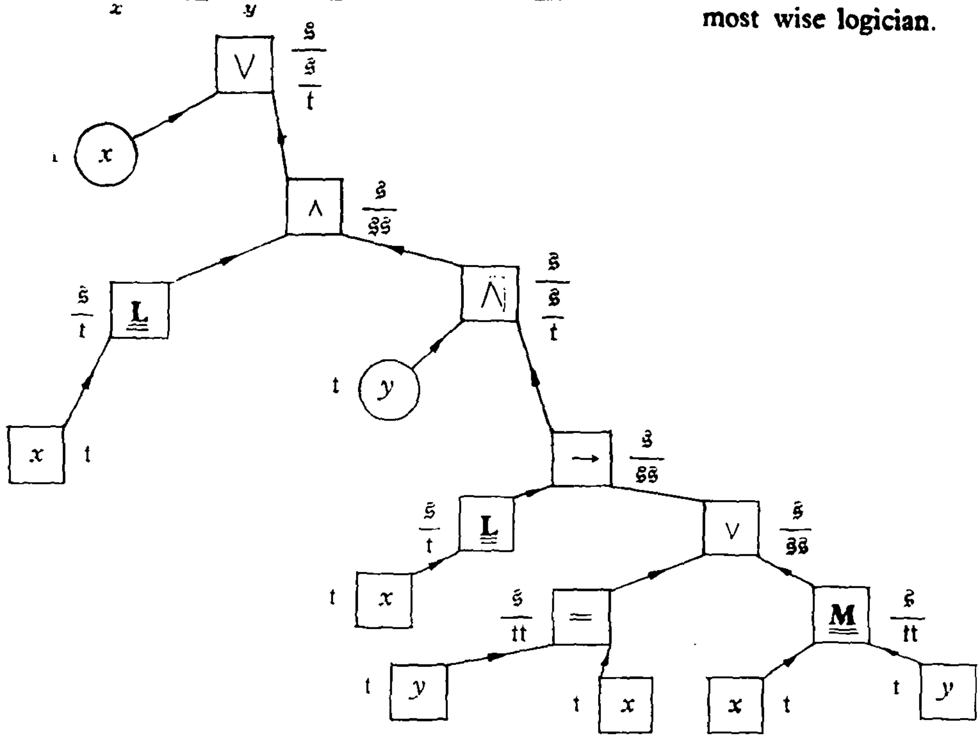
II. $\underline{\underline{\mathbf{a}}}$ Aristotle



[6] Compare the following v. NEUMANN'S remark: „...würde ...die Bezeichnungsfrage in irgendeiner Beziehung wesentliche Schwierigkeiten machen, so wäre es ein Leichtes, sie in trivialer Weise ein für allemal aus der Welt zu schaffen. Es würde genügen, statt die Formeln fertig hinzuschreiben, bei jeder Formel ihre Entstehungsgeschichte... ausführlich anzugeben." [5] p. 333.

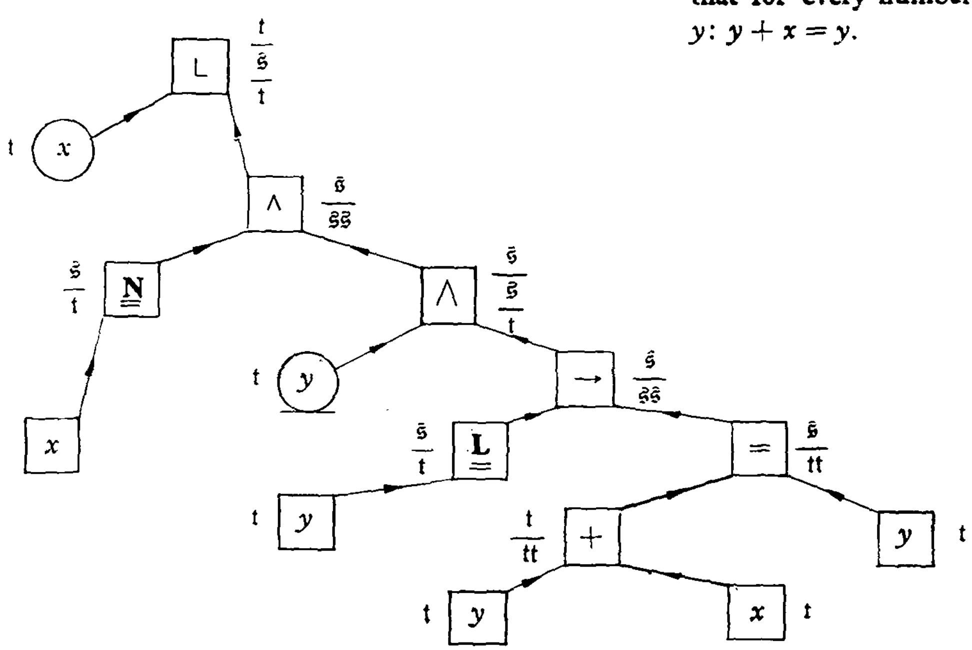
III. $\bigvee_{x} (x \, \varepsilon \, \underline{\underline{\mathbf{L}}} \wedge \bigwedge_{y} (y \, \varepsilon \, \underline{\underline{\mathbf{L}}} \to y = x \vee x \, \varepsilon \, \underline{\underline{\mathbf{M}}}(y)))$

Some logician is the most wise logician.



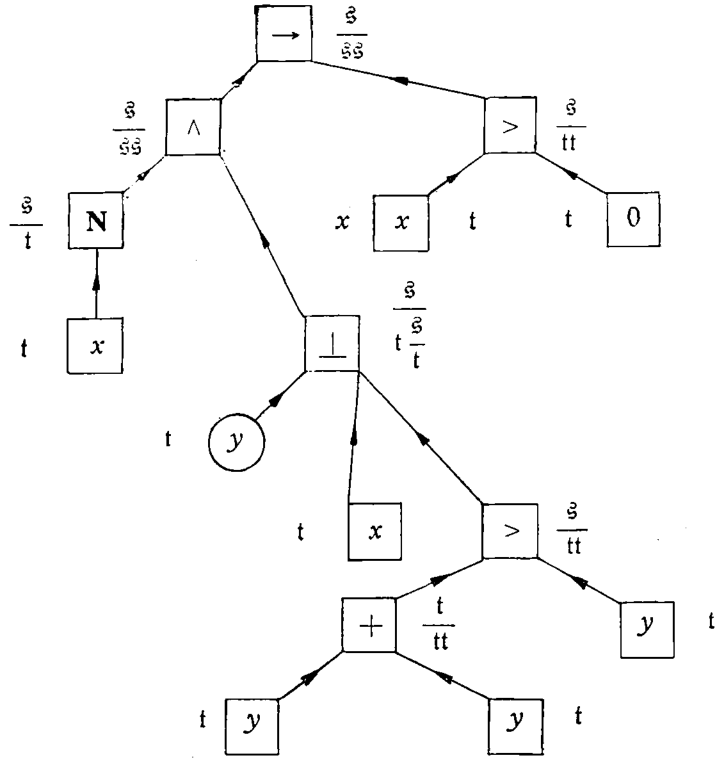
IV. $\mathop{\llcorner}_{x} (x \, \varepsilon \, \underline{\underline{\mathbf{N}}} \wedge \bigwedge_{y} (y \, \varepsilon \, \underline{\underline{\mathbf{N}}} \to y + x = y))$

This number x such that for every number y: $y + x = y$.



V. $(x \,\varepsilon\, \mathbf{N} \wedge x \,\varepsilon \underset{y}{\perp} (y + y > y)) \rightarrow x > 0$

If x is a number y such that $y + y > y$, then $x > 0$.



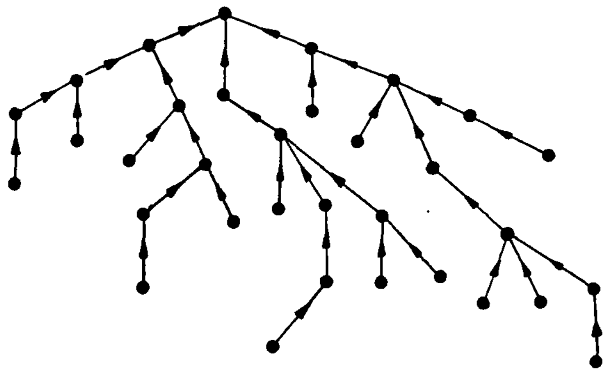
These examples show that we can get rid of any problem of calligraphy of compound expressions if we will represent these expressions by suitable diagrams. At the same time we see that these diagrams are finite sets of plane points conveniently connected by vectors (arrows) and marked with suitable word-types and corresponding indices. In other words, every such diagram can be decomposed into a suitable plane figure and a function which assigns in some way the word-types of some vocabulary and their indices to the points of this figure. Therefore, will describe at first these figures and, further, we will characterize the corresponding functions.

6. *The graphs*

We consider a fixed plane and we define a kind of geometrical plane diagrams which we will call *graphs*.[7]

[7] The term *graph* is borrowed from the general mathematical theory of graphs. But we use here this term in a much narrowed sense.

At first let us call attention to such plane figures called here *compound graphs*



which have the following properties:

(1) They are finite sets of points containing more than one point; these points are called *nodes*.

(2) Their points, i. e. the nodes are interconnected by means of arrows (vectors) called *arms*, in such a way that

(3) every node is an initial point or an endpoint of some arm, but

(4) there is exactly one node called the *vertex* which is not an initial point of any arm and

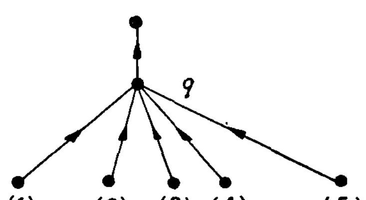
(5) every node distinct from the vertex is an initial point of exactly one arm.

It follows that in any compound graph there are nodes which are not endpoints of any arm; these nodes are called *fundamental nodes*.

A node p is called a *directly subordinate* node to the node q if the node p is an initial point and the node q is an endpoint of the same arm.

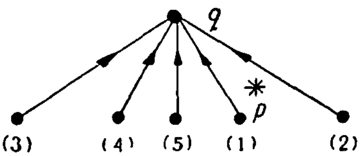
One can enumerate, i. e. mark with numbers 1,2,... all directly subordinate nodes to some nonfundamental node q. There are of course many such enumerations.

But if the nonfundamental node q is not identical to the vertex then we will take into account only one natural and univocally determined enumeration:



This is the counterclockwise enumeration of all directly subordinate nodes to the given nonfundamental node q distinct from the vertex.

If the nonfundamental node q is identical to the vertex, then we consider the whole set of all counterclockwise enumerations of all directly subordinate nodes to the vertex q. It is clear that the choice of one directly subordinate node p^* to the vertex q distinguishes exactly one such counterclockwise enumeration of all directly subordinate nodes to the vertex q according to which the chosen node p^* is marked with number 1:



Besides compound graphs we also consider the *simple graphs*, i. e. the single points. They have, of course, no arms. Every simple graph, however, contains only one node which at the same time is the vertex and the only one fundamental node of this simple graph.

Any graph Γ such that

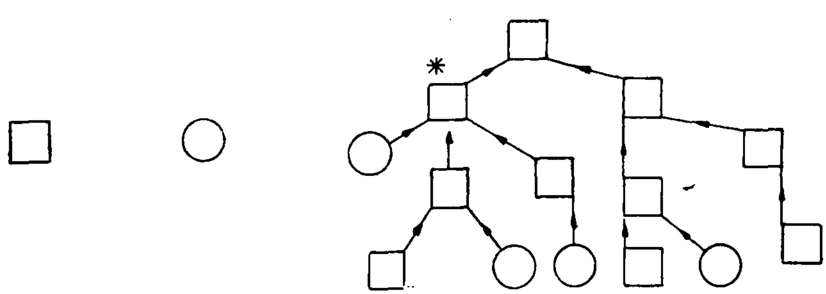
(1) the fundamental nodes of Γ are divided into *auxiliary* fundamental nodes and *proper* fundamental nodes, and

(2) if Γ is compound then exactly one of the directly subordinate nodes do the vertex of Γ is distinguished just as the first in some distinguished counterclockwise enumeration of these nodes,

— any such graph (simple or compound) is called a *special* graph (simple or compound).

It is clear that in each special compound graph for every given nonfundamental node (inclusive the vertex) there is exactly one distinguished counterclockwise enumeration of all directly subordinate nodes to the given node. It follows that in every compound special graph each node distinct from the vertex may be supplied with an univocally determined number 1,2,... .

We assume the convention according to which the auxiliary fundamental nodes will be marked with circles, the other nodes with squares and the distinguished directly subordinate node to the vertex — with an asterisk. Thus the special graphs may be represented as diagrams of the following examplary form:



simple special graphs compound special graph

In the following we will apply only the notion of a special graph and its theory and, therefore, we assume the terminology according to which we will write henceforth *graph* instead of *special graph.*

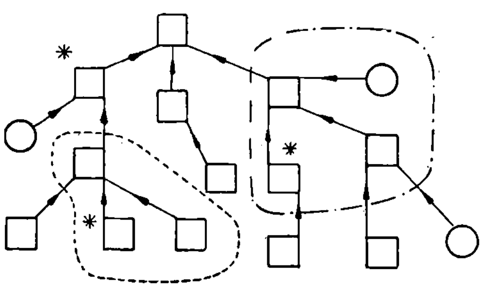
Let Γ_1 and Γ_2 be graphs. We say that Γ_1 is *contained* (or *included*) in Γ_2 or that Γ_1 is a *subgraph* of Γ_2 if the graphs Γ_1 and Γ_2 fulfil the following conditions:

(1) every node of Γ_1 is a node of Γ_2,

(2) if p is in Γ_1 directly subordinate to q, then p is in Γ_2 directly subordinate to q, and conversly, if p and q are such nodes of Γ_1 that p is in Γ_2 directly subordinate to q then p is in Γ_1 directly subordinate to q,

(3) if p is an auxiliary fundamental node of Γ_1 then p is an auxiliary fundamental node of Γ_2, and conversely, if p is such a node of Γ_1 which is an auxiliary fundamental node of Γ_2 then p is an auxiliary fundamental node of Γ_1,

(4) if Γ_1 is compound then the distinguished subordinate node to the vertex in Γ_1 is identical to this directly subordinate node to the vertex in Γ_2 which according to the enumeration of nodes in Γ_2 is supplied with least number.



Every graph Γ_1 which is such a subgraph of a graph Γ_2 that every fundamental node in Γ_1 is at the same time a fundamental node in Γ_2 is called a *fundamental subgraph* of Γ_2.

In the set of all such fundamental subgraph of graph Γ which have a node p of Γ as a common vertex, there exists one greatest, i. e. containing each remained; it is called *the fundamental subgraph determined in* Γ *by the node* p.

We say that Γ_1 is a *direct subgraph* of Γ_2 if Γ_1 is the fundamental subgraph determined in the graph Γ_2 by some directly subordinate node to the vertex of Γ_2. Of course, the simple graphs have no direct subgraph. We observe that in every compound graph Γ there is exactly one distinguished enumeration of all direct subgraphs of Γ.

Let Γ_1 be a fundamental subgraph of Γ_2. If the graphs Γ_1 and Γ_2 have a common vertex and if every nonfundamental node of Γ_1 is an endpoint of exactly one arm in Γ_1, then the subgraph Γ_1 is called a *branch* of Γ_2. It is obvious that every node of any graph Γ is situated on a branch of Γ and, further, that every fundamental node of Γ is situated on exactly one branch of Γ and, finally, that on every branch of Γ there is exactly one fundamental node.

7. *The printed graphs and the sets* $\mathfrak{E}(\mathfrak{V})$

Making use of the notion of a graph (i. e. of a special graph) as introduced in the preceding section, we can represent the purely syntactic structure of expressions by diagrams and determine the corresponding sets $\mathfrak{E}(\mathfrak{V})$ in a strictly formal way and independently of any principle of calligraphy for compound expressions.

Let $\mathfrak{V}$ be a syntactically marked vocabulary. Every ordered pair

$$\langle \Gamma, \Phi \rangle$$

which consists of a graph Γ and a function Φ assigning to every node p in Γ a word-type $\Phi(p)$ belonging to $\mathfrak{V}$, is called a *printed graph*. The graph Γ is called the *position* of $\langle \Gamma, \Phi \rangle$ and the function Φ is called the *printing-function* of $\langle \Gamma, \Phi \rangle$. It may be said that the printed graph $\langle \Gamma, \Phi \rangle$ is the result of printing the word-types of $\mathfrak{V}$ according to Φ on the nodes of Γ.

If $\mathfrak{J}$ is the syntactic-marking-function of $\mathfrak{V}$ then to every word-type ζ in $\mathfrak{V}$ an index $\mathfrak{J}(\zeta)$ is assigned. Therefore, in every printed graph $\langle \Gamma, \Phi \rangle$ to every node p of its position Γ an index $\mathfrak{J}(\Phi(p))$ is assigned.

Every element of the set $\mathfrak{E}(\mathfrak{V})$ of all expressions which are constructible on the basis of $\mathfrak{V}$, is a printed graph. It will appear later that it is characteristic for the printed graphs which are expressions belonging to the set $\mathfrak{E}(\mathfrak{V})$, some special connexion between indices assigned to nodes.

It is very easy to see that any printed graph $\langle \Gamma, \Phi \rangle$ can be represented by a suitable diagram, namely by the graph Γ, the nodes of which are marked according to the function Φ with word-types and corresponding indices. Some examples of such diagrams are just given in the section 5.

Let $\langle \Gamma, \Phi \rangle$ be a printed graph. If Γ_0 is a subgraph of Γ then the symbolic notation

$$\Phi \mid \Gamma_0$$

denotes the function Φ restricted to the nodes of Γ_0 ; the pair $\langle \Gamma_0, \Phi \mid \Gamma_0 \rangle$ is also a printed graph, of course.

We say that the variable-type $\xi_N^{(k)}$ of $\mathfrak{V}$ *occurs free* in the printed graph $\langle \Gamma, \Phi \rangle$ if there is in Γ a proper fundamental node p such that $\Phi(p) = \xi_N^{(k)}$ and there does not exist such a node q on the branch on which the node p is situated that $\Phi(q)$

is an operator-type of rank 2 and $\Phi(q^{\star}) = \xi_N^{(k)}$ for some auxiliary fundamental node $q^{\star}$ directly subordinate to the node q.

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The elements of the set $\mathcal{L}(\mathcal{V})$, i. e. the expressions which are constructible on the basis of a syntactically marked vocabulary $\mathcal{V}$, can be determined in the following inductive way. We define, firstly, the simple expressions belonging to $\mathcal{L}(\mathcal{V})$ and, subsequently, we will determine the conditions under which some printed graphs built from expressions being just in $\mathcal{L}(\mathcal{V})$, are compound expressions belonging to $\mathcal{L}(\mathcal{V})$; these conditions correspond to the rules of construction determined by the operator-types in $\mathcal{V}$, as discussed in section 4.

At the same time we define what is meant by the index of an expression belonging to $\mathcal{L}(\mathcal{V})$.

Let $\mathcal{V}$ be a syntactically marked vocabulary and let $\mathfrak{I}$ be the syntactic-marking-function of $\mathcal{V}$.

A printed graph $\langle \Gamma, \Phi \rangle$ is a *simple expression* (or *word*) belonging to $\mathcal{L}(\mathcal{V})$ if and only if Γ is such a simple graph that the only one node p of it is proper and Φ is such a function that $\Phi(p)$ is any word-type of $\mathcal{V}$. The index $\mathfrak{I}(\Phi(p))$ assigned to the node p is taken as the index of the simple expression $\langle \Gamma, \Phi \rangle$. If this index is of rank 0 or 1 or 2 then the simple expression $\langle \Gamma, \Phi \rangle$ is called a *fundamental word* (*constant* or *variable*) or an *operator of rank* 1 or an *operator of rank* 2.

If $\Phi(p)$ is a variable-type or a constant-type, then the simple expression $\langle \Gamma, \Phi \rangle$ is called a *variable* or a *constant*. Any operator is a constant and any variable is a fundamental word.

We pass now to the compound expressions in $\mathcal{L}(\mathcal{V})$. We distinguish two cases according to the division of operator-types in $\mathcal{V}$ and operators in $\mathcal{L}(\mathcal{V})$.

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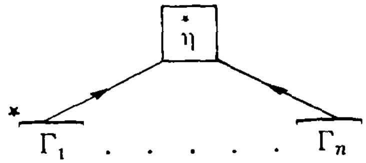
For the first case let us assume that $\overset{*}{\eta}$ is an operator-type of rank 1 in $\mathcal{V}$ marked with the index (1.1) of rank 1 (p. 214) i. e.

$$\mathfrak{I}(\overset{*}{\eta}) = \frac{t_k}{t_{k_1} \ldots\ldots t_{k_n}}$$

We consider a printed graph $\langle \Gamma, \Phi \rangle$ which has two following properties:
(1) Γ is a compound graph having exactly n direct subgraphs $\Gamma_1, \ldots, \Gamma_n$ — enumerated just according to the enumeration distinguished in Γ and

(2) $\Phi(p) = \overset{*}{\eta}$, where p is the vertex of Γ.



Now we state: the printed graph $\langle \Gamma, \Phi \rangle$ is a *compound expression* belonging to $\mathfrak{L}(\mathfrak{V})$ and its index is t_k if the printed graphs $\langle \Gamma_1, \Phi | \Gamma_1 \rangle, \ldots, \langle \Gamma_n, \Phi | \Gamma_n \rangle$ are expressions (simple or compound) belonging to $\mathfrak{L}(\mathfrak{V})$ and if their indices are $t_{k_1}, \ldots, t_{k_n}$ respectively.

Notice. It is to see immediately that for $l = 1, \ldots, n$ any variable-type occurring free in $\langle \Gamma_l, \Phi | \Gamma_l \rangle$ occurs free also in $\langle \Gamma, \Phi \rangle$.

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For the second case let us assume that $\overset{**}{\eta}$ is an operator-type of rank 2 in $\mathfrak{V}$ marked with the index (1.2) of rank 2 (p. 215) i. e.

$$\mathfrak{I}(\overset{**}{\eta}) = \frac{t_k}{i_1 \ldots\ldots i_n}$$

Let us make the following assumptions:

(a) the indices $t_{k_1}, \ldots\ldots, t_{k_r}$ are all such indices of rank 0 that their multiplicities in the index (1.2) are different from 0;

(b) $k_1 < \ldots\ldots < k_r$;

(c) the multiplicities of the indices $t_{k_1}, \ldots\ldots, t_{k_r}$ in the index (1.2) are equal to the numbers $m_1, \ldots\ldots, m_r$ respectively;

(d) $m = m_1 + \ldots\ldots + m_r \neq 0$;

(e) the multiplicities of the indices $t_{k_1}, \ldots\ldots, t_{k_r}$ in the index i_l are equal (for $l = 1, \ldots\ldots, n$) to the numbers $m_{l,1}, \ldots\ldots, m_{l,r}$ respectively;

(f) the indices $t_{j_1}, \ldots\ldots, t_{j_n}$ are the numerators of the indices $i_1, \ldots\ldots, i_n$ respectively.

We consider a printed graph $\langle \Gamma, \Phi \rangle$ which has the following properties:

(1) Γ is a compound graph which has exactly $m + n$ direct subgraphs $\Gamma_1, \ldots,$ $\Gamma_m, \Gamma_{m+1}, \ldots, \Gamma_{m+n}$ — enumerated just according to the enumeration distinguished in Γ;

(2) the first m direct subgraphs $\Gamma_1, \ldots, \Gamma_m$ are auxiliary fundamental nodes in Γ, the last n direct subgraphs $\Gamma_{m+1}, \ldots, \Gamma_{m+n}$ are not auxiliary fundamental nodes;

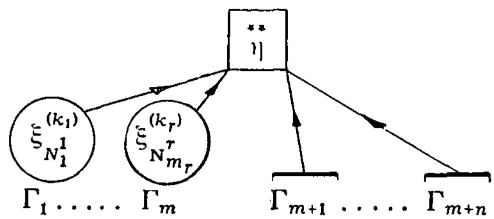
(3) the function Φ assigns the distinct variable-types in $\mathfrak{V}$ to the auxiliary fundamental nodes $\Gamma_1, \ldots, \Gamma_m$ in such a way that

$$\Phi(\Gamma_1) = \xi^{(k_1)}_{N^1_1}, \ldots\ldots, \Phi(\Gamma_{m_1}) = \xi^{(k_1)}_{N^1_{m_1}}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\Phi(\Gamma_{m-m_r+1}) = \xi^{(k_r)}_{N^r_1}, \ldots\ldots, \Phi(\Gamma_m) = \xi^{(k_r)}_{N^r_{m_r}}$$

(4) $\Phi(p) = \overset{**}{\eta}$, where p is the vertex of Γ.



Now we state: the printed graph $\langle \Gamma, \Phi \rangle$ is a *compound expression* belonging to $\mathcal{L}(\mathfrak{V})$ and its index is t_k if the printed graphs $\langle \Gamma_{m+1}, \Phi | \Gamma_{m+1} \rangle, \ldots\ldots,$ $\langle \Gamma_{m+n}, \Phi | \Gamma_{m+n} \rangle$ are such expressions (simple or compound) belonging to $\mathcal{L}(\mathfrak{V})$ that

(1) their indices are $t_{j_1}, \ldots\ldots, t_{j_n}$ respectively and

(2) for every $l = 1, \ldots, n$ and for every $s = 1, \ldots, r$ it occurs free in $\langle \Gamma_{m+l}, \Phi | \Gamma_{m+l} \rangle$ exactly $m_{l,s}$ variable-types among $\xi^{(k_s)}_{N^s_1}, \ldots\ldots, \xi^{(k_s)}_{N^s_{m_s}}$.

Note. If Γ_0 is such a subgraph of Γ that Γ_0 and Γ have the common vertex p and the direct subgraphs of Γ_0 are identical to the auxiliary nodes $\Gamma_1, \ldots,$ Γ_m and if $\Phi_0 = \Phi | \Gamma_0$ then the printed graph $\langle \Gamma_0, \Phi_0 \rangle$ is called the *prefix* of the expression $\langle \Gamma, \Phi \rangle$.

We have seen that they are the m variable-types $\xi^{(k_1)}_{N^1_1}, \ldots\ldots, \xi^{(k_r)}_{N^r_{m_r}}$ that are printed on the m auxiliary fundamental nodes $\Gamma_1, \ldots, \Gamma_m$ of the prefix of $\langle \Gamma, \Phi \rangle$. Now let us consider in a more detailed way the index $\mathfrak{i}_l$ and the printed graphs $\langle \Gamma_{m+l}, \Phi | \Gamma_{ml+} \rangle$ for $l = 1, \ldots, n$. Let $i = m_{l,1} + \ldots, + m_{l,r}$.

If $i = 0$ then $\mathfrak{i}_l = \mathfrak{t}_{j_l}$ and no variable-type among $\xi^{(k_1)}_{N^1_1}, \ldots\ldots, \xi^{(k_r)}_{N^r_{m_r}}$ occur free in $\langle \Gamma_{m+l}, \Phi | \Gamma_{m+l} \rangle$. Therefore, any variable-type occurring free in $\langle \Gamma_{m+l}, \Phi | \Gamma_{m+l} \rangle$ occurs free also in $\langle \Gamma, \Phi \rangle$.

If $i \neq 0$ then $\mathfrak{i}_l = \dfrac{\mathfrak{t}_{j_l}}{\mathfrak{t}_{l_1} \cdot \cdot \cdot \cdot \cdot \mathfrak{t}_{l_i}}$ and there are exactly i such variable-types $\xi^{(l_1)}_{N_1}, \ldots, \xi^{(l_i)}_{N_i}$ among $\xi^{(k_1)}_{N^1_1}, \ldots\ldots, \xi^{(k_r)}_{N^r_{m_r}}$ that $\xi^{(l_1)}_{N_1}, \ldots, \xi^{(l_i)}_{N_i}$ are enumerated according to the enumeration of variable-types $\xi^{(k_1)}_{N^1_1}, \ldots\ldots, \xi^{(k_r)}_{N^r_{m_r}}$ and the variable-types $\xi^{(l_1)}_{N_1}, \ldots\ldots, \xi^{(l_i)}_{N_i}$ occur free in $\langle \Gamma_{m+l}, \Phi \,|\, \Gamma_{m+l} \rangle$ but they do not occur free in $\langle \Gamma, \Phi \rangle$.

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Thus we have determined which printed graphs belong to the set $\mathfrak{L}(\mathfrak{V})$. In other words we have defined, for every syntactically marked vocabulary $\mathfrak{V}$, the set $\mathfrak{L}(\mathfrak{V})$ of all expressions which are constructible on the ground of $\mathfrak{V}$. At the same time we have defined an assignement according to which to every expression $\langle \Gamma, \Phi \rangle$ belonging to $\mathfrak{L}(\mathfrak{V})$ there is assigned an index

$$\mathfrak{J}(\mathfrak{V}, \langle \Gamma, \Phi \rangle)$$

namely the *index of the expression* $\langle \Gamma, \Phi \rangle$ *belonging to* $\mathfrak{L}(\mathfrak{V})$.

There are always simple expressions in $\mathfrak{L}(\mathfrak{V})$. But there exist, of course, such syntactically marked vocabularies $\mathfrak{V}$ that the corresponding sets $\mathfrak{L}(\mathfrak{V})$ contain no compound expression.

The index $\mathfrak{J}(\mathfrak{V}, \langle \Gamma, \Phi \rangle)$ of a simple expression $\langle \Gamma, \Phi \rangle$ in $\mathfrak{L}(\mathfrak{V})$ may be of rank 0 or 1 or 2. But the index of any compound expression in $\mathfrak{L}(\mathfrak{V})$ is of rank 0. It is easy to see that in any compound expression $\langle \Gamma, \Phi \rangle$ if p is the vertex of the graph Γ then the word-type $\Phi(p)$ assigned to it, is an operator-type and the index of this compound expression $\langle \Gamma, \Phi \rangle$ i.e. $\mathfrak{J}(\mathfrak{V}, \langle \Gamma, \Phi \rangle)$ equals to the numerator of the index $\mathfrak{J}(\Phi(p))$ assigned to the vertex p.

Every expression $\langle \Gamma, \Phi \rangle$ in $\mathfrak{L}(\mathfrak{V})$ such that its index is of rank 0 is called a *fundamental expression*. Each nonfundamental expression belonging to $\mathfrak{L}(\mathfrak{V})$ is a simple expression, namely an operator.

Let $\langle \Gamma, \Phi \rangle$ be a fundamental expression in $\mathfrak{L}(\mathfrak{V})$ and let Γ_0 be a branch of Γ. If p is a fundamental node situated on Γ_0 then the word-type $\Phi(p)$ is a fundamental word-type and the index $\mathfrak{J}(\Phi(p))$ is of rank 0. On the contrary, if p is a non-fundamental node situated on Γ_0 then the word-type $\Phi(p)$ is an operator-type and the index $\mathfrak{J}(\Phi(p))$ is of rank 1 or 2. Therefore, for every node p in Γ: the word-type $\Phi(p)$ is a fundamental word-type and the index $\mathfrak{J}(\Phi(p))$ is of rank 0 if and only if the node p is fundamental in Γ.

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We introduce some relations between expressions belonging to $\mathcal{L}(\mathcal{V})$.

Let $\langle \Gamma, \Phi \rangle$ be an expression. The expression $\langle \Gamma_0, \Phi_0 \rangle$ is a *subexpression* of $\langle \Gamma, \Phi \rangle$ or is *contained* (or *included*) in $\langle \Gamma, \Phi \rangle$ if and only if Γ_0 is a subgraph of Γ and $\Phi_0 = \Phi \mid \Gamma_0$.

It follows that if $\langle \Gamma_0, \Phi_0 \rangle$ is a simple expression then $\langle \Gamma_0, \Phi_0 \rangle$ is a subexpression of $\langle \Gamma, \Phi \rangle$ if and only if the only one (fundamental proper) node p_0 of Γ_0 is identical to some nonauxiliary node p in Γ and $\Phi_0(p_0) = \Phi(p)$.

Similarly, if $\langle \Gamma_0 \Phi_0 \rangle$ is a compound expression then $\langle \Gamma_0, \Phi_0 \rangle$ is a subexpression of $\langle \Gamma, \Phi \rangle$ if and only if there is a nonfundamental node q in Γ such that Γ_0 is identical to the fundamental subgraph determined in Γ by the mode q and $\Phi_0(p) = \Phi(p)$ for each node p in Γ_0.

The expression $\langle \Gamma_0, \Phi_0 \rangle$ is said to be a *direct subexpression* of the expression $\langle \Gamma, \Phi \rangle$ if Γ_0 is a direct subgraph of Γ and $\Phi_0 = \Phi \mid \Gamma_0$. Every direct subexpression is an expression. It follows that the printed graph $\langle \Gamma_0, \Phi_0 \rangle$ is a direct subexpression of the expression $\langle \Gamma, \Phi \rangle$ if and only if there exists in Γ a nonfundamental node q which is directly subordinate to the vertex of Γ and such that Γ_0 is identical to the fundamental subgraph determined in Γ by q and $\Phi_0(p) = \Phi(p)$ for each node p in Γ_0. There are, of course, no direct subexpression in any simple expression.

The direct subexpressions of a compound expression $\langle \Gamma, \Phi \rangle$ are called its *members* or *arguments*. The simple expression $\langle \Gamma_0, \Phi_0 \rangle$ being an operator such, that the vertex p_0 of $\langle \Gamma, \Phi \rangle$ is the only one proper node of Γ_0 and $\Phi_0(p_0) = \Phi(p)$ is called the *head* of the compound expression $\langle \Gamma, \Phi \rangle$. The head $\langle \Gamma_0, \Phi_0 \rangle$ being a subexpression, is not a direct subexpression in $\langle \Gamma, \Phi \rangle$. It is clear that there are exactly n members in the compound expression $\langle \Gamma, \Phi \rangle$ if and only if the denominator of the index of the head $\langle \Gamma_0, \Phi_0 \rangle$ is a sequence of exactly n indices.

Let $\langle \Gamma_0, \Phi_0 \rangle$ be such a printed graph that Γ_0 is a simple graph and $\Phi_0(p_0) = \xi_N^{(k)}$ where p_0 is the only one node of Γ_0. If p_0 is an auxiliary node in Γ_0 then $\langle \Gamma_0, \Phi_0 \rangle$ is not an expression and, therefore, $\langle \Gamma_0, \Phi_0 \rangle$ cannot be a subexpression of any expression. If p_0 is a proper node in Γ_0 then $\langle \Gamma_0, \Phi_0 \rangle$ is a variable and it may be a subexpression of some expression $\langle \Gamma, \Phi \rangle$. Then, p_0 is identical to some fundamental proper node p in Γ and $\Phi_0(p_0) = \Phi(p) = \xi_N^{(k)}$. Consider the branch in Γ on which the node p is situated. If there is such a node q on this branch that $\Phi(q)$ is an operator-type of rank 2 and $\Phi(q^\star) = \xi_N^{(k)}$ for some auxiliary node $q^\star$ directly subordinate to the node q, then we say the variable $\langle \Gamma_0, \Phi_0 \rangle$ to be *bound* in $\langle \Gamma, \Phi \rangle$. If on the branch on which the node p is situated there does not exist such a node q that $\Phi(q)$ is an operator-type of rank 2 and $\Phi(q^\star) = \xi_N^{(k)}$ for some auxiliary node $q^\star$ directly subordinate to the node q, then we say that the variable $\langle \Gamma_0, \Phi_0 \rangle$ is *free* in $\langle \Gamma, \Phi \rangle$.

An expression is called *closed* if no variable is free in it. An expression is called *open* if no variable is bound in it.

Let $\langle \Gamma_1, \Phi_1 \rangle$ and $\langle \Gamma_2, \Phi_2 \rangle$ be two expressions and let φ be such one-to-one correspondence between the nodes in Γ_1 and those in Γ_2 that:

(1) a node p is directly subordinate in Γ_1 to the node q if and only if the node $\varphi(p)$ is directly subordinate in Γ_2 to the node $\varphi(q)$,

(2) a node p is auxiliary fundamental node of Γ_1 if and only if the node $\varphi(p)$ is auxiliary fundamental node of Γ_2,

(3) if the graph Γ_1 is compound then if p^* is the distinguished directly subordinate node to the vertex of Γ_1 then $\varphi(p^*)$ is the distinguished directly subordinate node to the vertex of Γ_2.

Every such correspondence we call an *isomorphism* of Γ_1 and Γ_2, and we introduce some notions of isomorphism of the expressions $\langle \Gamma_1, \Phi_1 \rangle$ and $\langle \Gamma_2, \Phi_2 \rangle$. Suppose that there is an isomorphism φ of Γ_1 and Γ_2.

If for each node p in Γ_1:

$$\Phi_1(p) = \Phi_2(\varphi(p))$$

then the expressions $\langle \Gamma_1, \Phi_1 \rangle, \langle \Gamma_2, \Phi_2 \rangle$ are said to be *isomorphic* (or *equiform*), symbolically

$$\langle \Gamma_1, \Phi_1 \rangle \equiv \langle \Gamma_2, \Phi_2 \rangle$$

If for each node p in Γ_1 such that $\Phi_1(\mathrm{p})$ is a constant word-type:

$$\Phi_1(p) = \Phi_2(\varphi(p))$$

then the expressions $\langle \Gamma_1, \Phi_1 \rangle, \langle \Gamma_2, \Phi_2 \rangle$ are said to be *weakly isomorphic* (or *weakly equiform*), symbolically

$$\langle \Gamma_1, \Phi_1 \rangle \stackrel{\cdot}{\equiv} \langle \Gamma_2, \Phi_2 \rangle$$

If for each node p in Γ_1:

$$\mathfrak{I}(\Phi_1(p)) = \mathfrak{I}(\Phi_2(\varphi(p)))$$

then the expressions $\langle \Gamma_1, \Phi_1 \rangle, \langle \Gamma_2, \Phi_2 \rangle$ are said to be *syntactically isomorphic* (or *equal with regard to the syntactic structure*), symbolically

$$\langle \Gamma_1, \Phi_1 \rangle \stackrel{*}{\equiv} \langle \Gamma_2, \Phi_2 \rangle$$

The relations $\equiv$, $\stackrel{\cdot}{\equiv}$, $\stackrel{*}{\equiv}$ are *equality relations* (i. e. reflexive, symmetric and transitive relations) in the set $\Omega(\mathfrak{V})$. They fulfil the following conditions:

$$\text{if } \langle \Gamma_1, \Phi_1 \rangle \equiv \langle \Gamma_2, \Phi_2 \rangle \text{ then } \langle \Gamma_1, \Phi_1 \rangle \stackrel{\cdot}{\equiv} \langle \Gamma_2, \Phi_2 \rangle$$

and

$$\text{if } \langle \Gamma_1, \Phi_1 \rangle \stackrel{\cdot}{\equiv} \langle \Gamma_2, \Phi_2 \rangle \text{ then } \langle \Gamma_1, \Phi_1 \rangle \stackrel{*}{\equiv} \langle \Gamma_2, \Phi_2 \rangle .$$

Making use of the relation $\equiv$ we can generalize the relation of being-a-subexpression. Namely, we say that the expression $\langle \Gamma_0,\Phi_0 \rangle$ is a *subexpression in a generalized sense* of the expression $\langle \Gamma,\Phi \rangle$ if and only if there is a subexpression in the former sense $\langle \overset{*}{\Gamma},\overset{*}{\Phi} \rangle$ of the expression $\langle \Gamma,\Phi \rangle$ such that

$$\langle \Gamma_0,\Phi_0 \rangle \equiv \langle \overset{*}{\Gamma},\overset{*}{\Phi} \rangle$$

In an analogous way one can generalize the relation of being-a-direct-subexpression.

8. *The standard formalized languages*

To every expression η in the set $\mathfrak{L}(\mathfrak{V})$ there is assigned an index $\mathfrak{J}(\mathfrak{V},\eta)$, namely the index of the expression η. Therefore, the set $\mathfrak{L}(\mathfrak{V})$ may be divided into classes of expressions to which the same index is assigned.

For every index $\mathfrak{j}$ the class $\mathfrak{L}(\mathfrak{V},\mathfrak{j})$ of expressions in $\mathfrak{L}(\mathfrak{V})$ such that any expression η belongs to it if and only if

$$\mathfrak{J}(\mathfrak{V},\eta) = \mathfrak{j}$$

is called the *semantical category* $\mathfrak{j}$ of expressions in $\mathfrak{L}(\mathfrak{V})$.

Every category $\mathfrak{L}(\mathfrak{V},\mathfrak{t}_k)$ where $k = 0,1, \ldots,$ is called a *fundamental category* or category of rank 0. Besides the fundamental categories there are *categories of higher rank*. The category $\mathfrak{L}(\mathfrak{V},\mathfrak{j})$ is of rank 1 or 2, if the index $\mathfrak{j}$ is of rank 1 or 2. Thus every expression belonging to some category of higher rank is simple expression, namely an operator. It follows that every compound expression belongs some fundamental category. On the other hand, every variable is a simple expression belonging to some fundamental category.

One fundamental semantical category of expressions shall be distinguished as the *category of sentences*. Namely, we determine that the category

$$\mathfrak{L}(\mathfrak{V}, \mathfrak{t}_0)$$

is the category of sentences. In other words we say that an expression η is a sentence if and only if

$$\mathfrak{J}(\mathfrak{V},\eta) = \mathfrak{t}_0$$

The distinction of the category of sentences is the only to be made from purely syntactical point of view among the fundamental semantical categories of expressions. In other words there is no syntactical difference between two non-sentential fundamental semantical categories.

If there are sentences in the set $\mathfrak{L}(\mathfrak{V})$ and if every simple expression is a subexpression of some sentence belonging to $\mathfrak{L}(\mathfrak{V})$ then we say that the syntactically marked vocabulary $\mathfrak{V}$ is *syntactically coherent*. Of course, there are syntactically incoherent vocabularies, particularly there are vocabularies such that the corresponding sets of all constructible expressions contain no sentences.

Now we state that the set $\mathcal{L}(\mathcal{V})$ of all expressions which are constructible on basis of the syntactically marked vocabulary $\mathcal{V}$ is a *standard formalized language generated by the vocabulary* $\mathcal{V}$ if and only if the vocabulary is syntactically coherent.

The set $\mathcal{L}(\mathcal{V}^\star)$ where $\mathcal{V}^\star$ is the vocabulary considered in the section 3 is a standard formalized language. We give some other examples of standard formalized languages.

The simplest formalized languages considered in formal logic are the so called *sentential languages*. These languages can be represented as standard formalized languages generated by vocabularies containing infinite series of constant word-types (for example the following graphic symbols: $p_1, p_2, \ldots, p_n, \ldots$) marked with the principal index $\mathfrak{s}$ and a set of operator-types of rank 1 marked with some of the indices

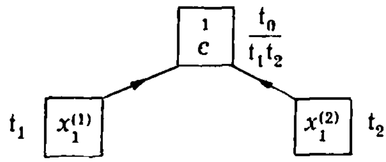
$$\frac{\mathfrak{s}}{\mathfrak{s}}, \quad \frac{\mathfrak{s}}{\mathfrak{s}\mathfrak{s}}, \quad \frac{\mathfrak{s}}{\mathfrak{s}\mathfrak{s}\mathfrak{s}}, \quad \ldots$$

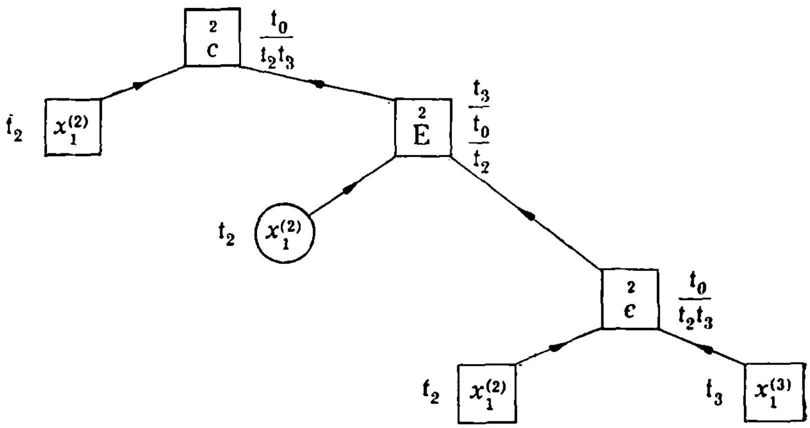
On the other side we consider a modified form of the language of the so called *general theory of classes* as in TARSKI's sense [6]. This example shows expressly the difference between the index-method applied here and the index-method employed formerly by K. AJDUKIEWICZ [1], [2].

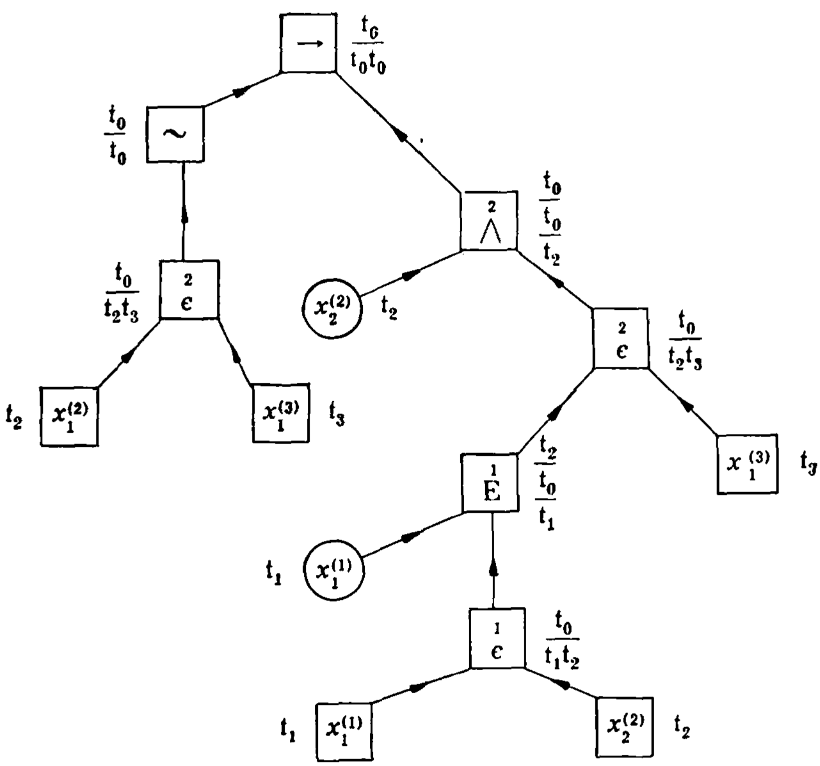
We present the vocabulary $\mathcal{V}_0$ of the language of the general theory of classes in the manner applied before in the case of the vocabulary $\mathcal{V}^\star$ considered in the section 3. We assume that $k = 1, 2, \ldots$.

t_k	$x_1^{(k)}, \; x_2^{(k)}, \; \ldots, \; x_n^{(k)}, \; \ldots,$	(variables of the k-th kind)
$\frac{t_0}{t_k\, t_{k+1}}$	$\overset{k}{\epsilon}$	(predicates of being-an-element-of-a-class)
$\frac{t_0}{\frac{t_0}{t_k}}$	$\overset{k}{\bigwedge}$	(the general quantifiers)
$\frac{t_{k+1}}{\frac{t_0}{t_k}}$	$\overset{k}{\mathrm{E}}$	(the abstraction-operators)
$\frac{t_0}{t_0}$	$\sim$	(the sign of negation)
$\frac{t_0}{t_0 t_0}$	$\rightarrow$	(the sign of implication)

We present some examples of diagrams of expressions belonging to $\mathcal{L}(\mathcal{V}_0)$;







These expressions may be written in the ususal calligraphy as follows:

$$x_1^{(1)} \overset{1}{\epsilon}\, x_1^{(2)} \qquad x_1^{(2)} \overset{2}{\epsilon} \underset{x_1^{(2)}}{\overset{2}{\mathrm{E}}} x_1^{(2)} \overset{2}{\epsilon}\, x_1^{(3)}$$

$$\sim (x_1^{(2)} \epsilon\, x_1^{(3)}) \rightarrow \bigwedge_{x_2^{(2)}} ([\underset{x_1^{(1)}}{\overset{1}{\mathrm{E}}}\; x_1^{(1)} \overset{1}{\epsilon}\, x_2^{(2)}] \overset{2}{\epsilon}\, x_1^{(3)})$$

Customarily some special principles concerning calligraphy are adopted and instead of expressions given above one writes the following simplified graphic expressions:

$$x_1^{(1)} \epsilon\, x_1^{(2)} \qquad \text{or} \qquad x_1^{(2)}(x_1^{(1)})$$

$$x_1^{(2)} \epsilon \underset{x_1^{(2)}}{\mathrm{E}}\, x_1^{(2)} \epsilon\, x_1^{(3)}$$

$$\sim (x_1^{(2)} \epsilon\, x_1^{(3)}) \rightarrow \bigwedge_{x_2^{(2)}} ([\underset{x_1^{(1)}}{\mathrm{E}}\; x_1^{(1)} \epsilon\, x_2^{(2)}] \epsilon\, x_1^{(3)})$$

In such cases the graphic symbols of the form ϵ, $\bigwedge$, E are *syntactically ambiguous*, i. e. their occurences in different expressions may belong to different semantical categories, and they may have the following ambiguous indices:

$$\frac{t_0}{t_k\, t_{k+1}} \qquad \frac{t_0}{\dfrac{t_0}{t_k}} \qquad \frac{t_{k+1}}{\dfrac{t_0}{t_k}}$$

Further, we note that if we adopt some suitable calligraphical conventions then the graphic symbol ϵ may be quite eliminated and replaced by the ordering of remaining graphic symbols.

The example just considered is an illustration of the fact that in artificial symbolic languages which are considered in formal logic, some conventions of calligraphy are adopted and that these conventions may conceal in some degree the purely syntactical structure of expressions in these languages. In order to see this structure of expressions we have to disregard the calligraphy of expressions and to take into account the general ideas of standard formalization as presented in section 2. Sometimes the application of these ideas to the given artificial symbolic language mey be done easily by introducing suitable modification in its calligraphy. In this manner we can reach the conclusion that any artificial symbolic language considered in formal logic falls under the scheme of standard formalization i.e. it is a standard formalized language generated by suitable vocabulary.[8]

* *
*

Every standard formalized language is generated by suitable marked and coherent vocabulary. Therefore, some relations between standard formalized languages can be reduced to the relations between corresponding vocabularies. The most important of such relations are the relation of *being-a-sublanguage* (or of containing or of including) and the relation of *syntactic-structure-preserving-translation.*

In the following definitions we make use of the notion of a permutation of secondary indices. This is strictly connected with the fact of the non-existence of any syntactic difference among the non-sentential fundamental semantical categories of expressions.

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be marked and coherent vocabularies and let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be corresponding syntactically-marking-functions in these vocabularies.

If every constant-type in $\mathcal{V}_1$ is a constant-type in $\mathcal{V}_2$ and every variable-type in $\mathcal{V}_1$ is a variable-type in $\mathcal{V}_2$ and if there is a permutation f of indices such that for each word-type ζ in $\mathcal{V}_1$:

$$\mathfrak{I}_2(\zeta) = f(\mathfrak{I}_1(\zeta))$$

[8] It should be noted that in the case of CHURCH's *simple theory of types* [4] for instance, it will be convenient to use another set of secondary indices of rank 0. Namely we define the set of all indices of rank 0 in the following inductive way: (1) $\mathfrak{t}_0$ is the principal index of rank 0, (2) $\mathfrak{t}_1$ is the first secondary index of rank 0 and (3) if $\mathfrak{t}_\alpha$ and $\mathfrak{t}_\beta$ are indices of rank 0 then $\mathfrak{t}_{(\alpha\beta)}$ is also an index of rank 0. Thus, in the vocabulary which generates the language of Church's simple theory of types we encounter besides fundamental word types an infiinite set of operator-types of rank 1 which are called *application-operator*-types and marked with the indices of rank 1

$$\frac{\mathfrak{t}_\alpha}{\mathfrak{t}_{(\alpha\beta)}\ \mathfrak{t}_\beta}$$

and an infinite set of operator-types of rank 2 that are called *λ-operator*-types and marked with the indices of rank 2

$$\cfrac{\mathfrak{t}_{(\alpha\beta)}}{\cfrac{\mathfrak{t}_\alpha}{\mathfrak{t}_\beta}}$$

The customary conventional calligraphy adopted in the language of Church's simple theory of types implies that in the expressions of this language the λ -operator-types are printed by means of graphic symbols of the same form

$$\lambda$$

and the application-operator-types are not printed at all because they are concealed in the ordering of other inscriptions.

It is clear that any change in the choice of the infinite set of secondary indices of rank 0 does not violate in any way the essential principles of the scheme of standard formalization.

then we say that the language $\mathfrak{L}(\mathfrak{V}_1)$ is *contained* (or *included*) in the language $\mathfrak{L}(\mathfrak{V}_2)$ or that $\mathfrak{L}(\mathfrak{V}_1)$ is a *sublanguage* of $\mathfrak{L}(\mathfrak{V}_2)$.

If $\mathfrak{L}(\mathfrak{V}_1)$ is a sublanguage of $\mathfrak{L}(\mathfrak{V}_1)$ and if f is the corresponding permutation of indices then every expression η in $\mathfrak{L}(\mathfrak{V}_1)$ is also an expression in $\mathfrak{L}(\mathfrak{V}_2)$ and

$$\mathfrak{I}(\mathfrak{V}_2, \eta) = f(\mathfrak{I}(\mathfrak{V}_1, \eta))$$

i. e. every expression belonging to the category $\mathfrak{L}(\mathfrak{V}_1, \mathfrak{j})$ belongs to the category $\mathfrak{L}(\mathfrak{V}_2, f(\mathfrak{j}))$. Therefore, every sentence in $\mathfrak{L}(\mathfrak{V}_1)$ is a sentence in $\mathfrak{L}(\mathfrak{V}_2)$.

Now let f be a permutation of indices and let $\overset{\sim}{\rightarrow}$ be a relation holding between the word-types in the vocabulary $\mathfrak{V}_1$ and the word-types in the vocabulary $\mathfrak{V}_2$. If f and $\overset{\sim}{\rightarrow}$ fulfil the following conditions:

(1) for every word-type ζ_1 in $\mathfrak{V}_1$ there exists a word-type ζ_2 in $\mathfrak{V}_2$ such that $\zeta_1 \overset{\sim}{\rightarrow} \zeta_2$.

(2) if $\zeta_1 \overset{\sim}{\rightarrow} \zeta_2$ then ζ_1, ζ_2 are constant-types in $\mathfrak{V}_1$ and $\mathfrak{V}_2$ respectively or ζ_1, ζ_2 are variable-types in $\mathfrak{V}_1$ and $\mathfrak{V}_2$ respectively,

(3) there does not exist two different variable-types ξ', ξ'' in $\mathfrak{V}_1$ such that $\xi' \overset{\sim}{\rightarrow} \xi$ and $\xi'' \overset{\sim}{\rightarrow} \xi$ for some variable-type ξ in $\mathfrak{V}_2$,

(4) if $\zeta_1 \overset{\sim}{\rightarrow} \zeta_2$ then $\mathfrak{I}_2(\zeta_2) = f(\mathfrak{I}_1(\zeta_1))$

then we say that the relation $\overset{\sim}{\rightarrow}$ is a *translation of* $\mathfrak{V}_1$ *into* $\mathfrak{V}_2$ with respect to the permutation f.

Every translation of vocabularies can be extended to a translation of the corresponding languages. Namely, if the relation $\overset{\sim}{\rightarrow}$ is a translation of $\mathfrak{V}_1$ into $\mathfrak{V}_2$ with respect to f, then we can define a relation that we denote by the same symbol $\overset{\sim}{\rightarrow}$ and which holds between the expressions in $\mathfrak{L}(\mathfrak{V}_1)$ and the expressions in $\mathfrak{L}(\mathfrak{V}_2)$. This relation can be called a *syntactic-structure-preserving-translation of* $\mathfrak{L}(\mathfrak{V}_1)$ *into* $\mathfrak{L}(\mathfrak{V}_2)$ generated by the given translation of $\mathfrak{V}_1$ into $\mathfrak{V}_2$ with respect to the permutation f. In defining of it we procede as follows.

If $\langle \Gamma^{(1)}, \Phi^{(1)} \rangle$ is a simple expression in $\mathfrak{L}(\mathfrak{V}_1)$ such that p_1 is the only node of $\Gamma^{(1)}$, then we state that

$$\langle \Gamma^{(1)}, \Phi^{(1)} \rangle \overset{\sim}{\rightarrow} \langle \Gamma^{(2)}, \Phi^{(2)} \rangle$$

if and only if $\langle \Gamma^{(2)}, \Phi^{(2)} \rangle$ is a simple expression in $\mathfrak{L}(\mathfrak{V}_2)$ and $\Phi^{(1)}(p_1) \overset{\sim}{\rightarrow} \Phi^{(2)}(p_2)$ where p_2 is the only node of $\Gamma^{(2)}$.

Now let $\langle \Gamma^{(1)}, \Phi^{(1)} \rangle$ be a compound expression in $\mathfrak{L}(\mathfrak{V}_1)$. We consider two cases corresponding strictly to these of building the compound expressions considered in section 7. Taking into account these two building-cases we procede as follows.

In the first case if p_1 is the vertex of Γ_1 then $\Phi^{(1)}(p_1)$ is an operator-type of rank 1 in $\mathfrak{V}_1$ and there are exactly n direct subexpressions

$$\langle \Gamma_1^{(1)}, \Phi^{(1)} | \Gamma_1^{(1)} \rangle, \ldots\ldots, \langle \Gamma_n^{(1)}, \Phi^{(1)} | \Gamma_n^{(1)} \rangle$$

in $\langle \Gamma^{(1)}, \Phi^{(1)} \rangle$. We state in this case that

$$\langle \Gamma^{(1)}, \Phi^{(1)} \rangle \overset{\sim}{\rightarrow} \langle \Gamma^{(2)}, \Phi^{(2)} \rangle$$

if and only if $\langle \Gamma^{(2)}, \Phi^{(2)} \rangle$ is a compound expression in $\mathcal{L}(\mathcal{V}_2)$ such that, firstly, there are exactly n direct subexpressions

$$\langle \Gamma_1^{(2)}, \Phi^{(2)} | \Gamma_1^{(2)} \rangle, \ldots\ldots, \langle \Gamma_n^{(2)}, \Phi^{(2)} | \Gamma_n^{(2)} \rangle$$

in $\langle \Gamma^{(2)}, \Phi^{(2)} \rangle$ and

$$\langle \Gamma_1^{(1)}, \Phi^{(1)} | \Gamma_1^{(1)} \rangle \underset{\rightarrow}{\simeq} \langle \Gamma_2^{(2)}, \Phi^{(2} | \Gamma_2^{(2)} \rangle$$

$$\vdots$$

$$\langle \Gamma_n^{(1)}, \Phi^{(1)} | \Gamma_n^{(1)} \rangle \underset{\rightarrow}{\simeq} \langle \Gamma_n^{(2)}, \Phi^{(2)} | \Gamma_n^{(2)} \rangle$$

and, secondly if p_2 is the vertex in $\Gamma^{(2)}$ then $\Phi^{(1)}(p_1) \underset{\rightarrow}{\simeq} \Phi^{(2)}(p_2)$.

In the second case if p_1 is the vertex of $\Gamma^{(1)}$ then $\Phi^{(1)}(p_1)$ is an operator-type of rank 2 in $\mathcal{V}_1$ and there are exactly $m+n$ direct subgraphs $\Gamma_1^{(1)}, \ldots, \Gamma_m^{(1)}, \Gamma_{m+1}^{(1)}, \ldots,$ $\Gamma_{m+n}^{(1)}$ in $\Gamma^{(1)}$ such that $\Gamma_1^{(1)}, \ldots, \Gamma_m^{(1)}$ are auxiliary fundamental nodes in $\Gamma^{(1)}$ and

$$\langle \Gamma_{m+1}^{(1)}, \Phi^{(1)} | \Gamma_{m+1}^{(1)} \rangle, \ldots\ldots, \langle \Gamma_{m+n}^{(1)}, \Phi^{(1)} | \Gamma_{m+n}^{(1)} \rangle$$

are all direct subexpressions in $\langle \Gamma^{(1)}, \Phi^{(1)} \rangle$. We state in this case that

$$\langle \Gamma^{(1)}, \Phi^{(1)} \rangle \underset{\rightarrow}{\simeq} \langle \Gamma^{(2)}, \Phi^{(2)} \rangle$$

if and only if $\langle \Gamma^{(2)}, \Phi^{(2)} \rangle$ is a compound expression in $\mathcal{L}(\mathcal{V}_1)$ such that there are exactly $m+n$ direct subgraphs $\Gamma_1^{(2)}, \ldots, \Gamma_m^{(2)}, \Gamma_{m+1}^{(2)}, \ldots, \Gamma_{m+n}^{(2)}$ in $\Gamma^{(2)}$ and, firstly

$$\langle \Gamma_{m+1}^{(1)}, \Phi^{(1)} | \Gamma_{m+1}^{(1)} \rangle \underset{\rightarrow}{\simeq} \langle \Gamma_{m+1}^{(2)}, \Phi^{(2)} | \Gamma_{m+1}^{(2)} \rangle$$

$$\vdots$$

$$\langle \Gamma_{m+n}^{(1)}, \Phi^{(1)} | \Gamma_{m+n}^{(1)} \rangle \underset{\rightarrow}{\simeq} \langle \Gamma_{m+n}^{(2)}, \Phi^{(2)} | \Gamma_{m+n}^{(2)} \rangle$$

secondly

$$\Phi^{(1)}(\Gamma_1^{(1)}) \underset{\rightarrow}{\simeq} \Phi^{(2)}(\Gamma_1^{(2)})$$

$$\vdots$$

$$\Phi^{(1)}(\Gamma_m^{(1)}) \underset{\rightarrow}{\simeq} \Phi^{(2)}(\Gamma_m^{(2)})$$

thirdly, if p_2 is the vertex in $\Gamma^{(2)}$ then $\Phi^{(1)}(p_1) \underset{\rightarrow}{\simeq} \Phi^{(2)}(p_2)$.

Thus we have defined the syntactic-structure-preserving-translation $\underset{\rightarrow}{\simeq}$ of $\mathcal{L}(\mathcal{V}_1)$ into $\mathcal{L}(\mathcal{V}_2)$ generated by the given translation of $\mathcal{V}_1$ into $\mathcal{V}_2$ with respect to permutation of indices f. It is not difficult to see this the translation has the following properties.

For every expression η_1 in $\mathcal{L}(\mathcal{V}_1)$ there exists an expression η_2 in $\mathcal{L}(\mathcal{V}_2)$ such that $\eta_1 \xrightarrow{\simeq} \eta_2$ and $\mathfrak{I}(\mathcal{V}_2, \eta_2) = f(\mathfrak{I}(\mathcal{V}_1, \eta_1))$.

If $\langle \Gamma^{(1)}, \Phi^{(1)} \rangle \xrightarrow{\simeq} \langle \Gamma^{(2)}, \Phi^{(2)} \rangle$ then there exists an isomorphism φ of the graphs $\Gamma^{(1)}$ and $\Gamma^{(2)}$, and for every node p of $\Gamma^{(1)}$:

$$\Phi^{(1)}(p) \xrightarrow{\simeq} \Phi^{(2)}(\varphi(p))$$
$$\mathfrak{I}_1(\Phi^{(1)}(p)) = f(\mathfrak{I}_2(\Phi^{(2)}(\varphi(p)))$$

* *
*

The relations of being-a-sublanguage and of translation just introduced, may be applied in the cases of extending a language (for instance, in the process of introducing new words by means of definitions or postulates and in the process of introducing a new kind of variables). But we do not consider here these special problems. Similarly, we do not consider here some problems connected with the equality relations $\equiv$, $\dot{\equiv}$, $\overset{*}{\equiv}$.

We pass to the second part of our paper where we will consider the fundamental properties of semantical reference.[9]

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[9] The second part will be published in the next volume of „Studia Logica".

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