Chapter 2

A generalized data envelopment analysis model: A unification and extension of existing methods for efficiency analysis of decision making units

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In this paper, we introduce a generalized Data Envelopment Analysis (DEA) model which unifies and extends most of the well-known DEA models developed over the past fifteen years and points the way to new models. By setting three binary parameters of this model to different values, we obtain subclasses of the DEA models with general K cone and W cone descriptions to represent the evaluator's preferences for the Decision Making Units (DMU) and the input/output categories. We also show relationships among the various different subclasses of the generalized DEA model and give special attention to efficiency definitions and solutions. Furthermore, we state and rigorously prove the equivalence between DEA efficiency and the nondominated solutions of a corresponding multi-objective program. This latter result is especially important for understanding and interpreting the concept of efficiency. Detailed examples are also presented to demonstrate the functions of K cone and W cone, as well as their characteristics.

Keywords: Data envelopment analysis, efficient production frontier, multiple input/output decision making, production possibility set.

1 Introduction

Since its original development in 1978, Data Envelopment Analysis (DEA) has become one of the core tools available to management scientists for the analysis of organizational performance. Indeed, a recent survey of the DEA literature by Seiford (1990) indicates that approximately 400 articles have been written on the subject. Although different distinct forms of DEA have been used to address different managerial/production problems, all the DEA models are oriented toward frontier concepts associated with locating efficiencies and inefficiencies of the decision making units (DMU) responsible for converting multiple inputs into multiple outputs. Unlike statistical regressions, productivity indices, etc., which (via least squares methodologies) are oriented toward *average* production or *mean* output for given input, the DEA method is oriented toward *individual* productivity and the identification of *extremal* relations between input and output for different decision making units. According to Seiford (1990), some of the hundreds of applications of DEA methods include identification of efficient production frontiers to aid in allocation of U.S. Army recruiting efforts (Charnes and Thomas (1990)), evaluation of the efficiency of savings institutions (Charnes et al. (1990)), evaluation of the efficiency of schools (Charnes et al. (1978)), and evaluation of managerial efficiency in not-for-profit organizations (Charnes and Cooper (1985)).

In many instances, a particular problem under consideration necessitated the development of a new or modified DEA model different from the original Charnes et al. (1978) model. We call particular attention to the DEA models of Charnes, Cooper, Wei and Huang (CCWH) (1989); Banker, Charnes and Cooper (BCC) (1984); Charnes, Cooper, Wei and Yue (CCWY) (1988); Färe and Grosskopf (FG) (1985); and Seiford and Thrall (ST) (1990). One purpose of this paper is to demonstrate that most of the preceding models are special cases of a new model (which we call the generalized DEA model). In addition to subsuming all the previous models, our generalized DEA model points to several new hitherto undeveloped DEA models and their solutions. This extension and unification of some well-known models is fairly simply achieved by using only a three binary parameters vector (δ_1, δ_2, δ_3) and incorporating it into the DEA generalized formulation. Moreover, we show the equivalence between DEA efficiency and the *nondominated solutions* of a corresponding multi-objective program, thereby giving rise to an important interpretation and understanding of DEA efficiency.

In section 2, we introduce our generalized unifying and extending DEA model, define efficiency for the model, and show explicitly that most known DEA models fall out as simple special cases of the generalized DEA model with various binary parameters δ_1, δ_2 and δ_3 set at fixed values. A guide to the exact properties of the generalized DEA model examined in this paper is also provided. In section 3, we examine in detail the relationships among different subclasses of the generalized DEA model (i.e., those obtained by setting (δ_1, δ_2, δ_3) equal to different values) with particular attention paid to relations between the corresponding efficiency definitions. In section 4, we prove the previously mentioned relationship between DEA efficiency and nondominated solutions of a corresponding multi-objective program. The generalized DEA model with polyhedra cones W and K (as defined in section 2) is investigated in section 5. We present two detailed examples for illustrative purposes in section 6, and the final section includes the summary, conclusions and remarks on possible extensions.

2 The generalized DEA model

The original DEA model (and its extensions) is conceptualized as a ratio of "virtual" output to "virtual" input, subject to constraints. After applying the Charnes–Cooper (1962) transformation, this information may be put into a linear programming formulation and solved for each decision making unit to obtain efficiency scores and other managerially useful information (such as the production frontier). We begin our generalized DEA model formulation from the linear programming equivalent. Accordingly, we shall study the following generalized data envelopment analysis model:

$$
\text{(P)} \quad \begin{cases} \text{maximize} & (\mu^T y_0 - \delta_1 \mu_0) \\ \text{subject to} & \omega^T X - \mu^T Y + \mu_0 \delta_1 e^T \in K, \\ & \omega^T x_0 = 1, \\ & \begin{pmatrix} \omega \\ \mu \end{pmatrix} \in W, \delta_1 \delta_2 (-1)^{\delta_3} \mu_0 \geq 0, \end{cases}
$$

and its dual:

$$
\text{(D)} \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \begin{pmatrix} X\lambda - \theta x_0 \\ -Y\lambda + y_0 \end{pmatrix} \in W^*, \\ & \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \\ & \lambda \in -K^*, \lambda_{n+1} \geq 0, \theta \in E^1, \end{cases}
$$

where:

$X = (x_1, x_2, \ldots, x_n)$ is an $m \times n$ matrix,

$Y = (y_1, y_2, \ldots, y_n)$ is an $s \times n$ matrix,

x_j is the input vector for the jth decision making unit, $j = 1, \ldots, n$,

y_j is the output vector for the jth decision making unit, $j = 1, \ldots, n$,

δ_1, δ_2, δ_3 are binary parameters assuming only the values 0 and 1,

$W \subseteq E_+^{m+s}$ is a closed convex cone, Int $W \neq \varnothing$, and for $\binom{\omega}{\mu} \in W \backslash \{0\}$, we have:

$$\omega^T x_j > 0, \quad \mu^T y_j > 0, \quad j = 1, \ldots, n,$$

$K \subseteq E_+^n$ is a closed convex cone, with Int $K \neq \varnothing$,

$W^* \subseteq E^{m+s}$ is the negative polar cone of the set W,

$K^* \subseteq E^n$ is the negative polar cone of the set K,

$e = (1, 1, \ldots, 1)^T \in E^n$.

For convenience, we denote $x_0 = x_{j_0}$, $y_0 = y_{j_0}$, for $1 \le j_0 \le n$. The cone W is used to describe the relative importance of different input/output categories, as viewed by the evaluator (one who evaluates the DMUs). We call cone K the *predilection cone* and use it to represent the evaluator's preferences for different DMUs. We shall show that our generalized DEA model substantially generalizes the (CCR) model by Charnes, Cooper and Rhodes (1978); the (BCC) model by Banker, Charnes and Cooper (1984); the (FG) model by Färe and Grosskopf (1985); the (CCWH) model by Charnes, Cooper, Wei and Huang (1989); the (CCWY) model by Charnes, Cooper, Wei and Yue (1989) for finite number of DMUs, and the (ST) model by Seiford and Thrall (1990).

To begin our analysis, we note that since $K \subseteq E^n_+$, for any feasible solution of (P) we have:

$$\omega^T X - \mu^T Y + \delta_1 \mu_0 e^T \ge 0.$$

In particular, for the j_0th component, we have:

$$\omega^T x_0 - \mu^T y_0 + \delta_1 \mu_0 \ge 0,$$

and hence the objective function of (P) is

$$\mu^T y_0 - \delta_1 \mu_0 \le \omega^T x_0 = 1.$$

These equations lead to definition 1, which may be compared to information provided by Charnes et al. (1978).

Definition 1

If there exists an optimal solution ω^0, μ^0, μ^0_0 of (P) with $\binom{\omega^0}{\mu^0} \in \text{Int}\, W$ and $\mu^{0^T} y_0 - \delta_1 \mu^0_0 = 1$, then DMU_{j_0} is called DEA efficient.

The production possibility set for the generalized DEA model (P) is defined as:

$$T = \left\{ (x, y) \middle| \begin{pmatrix} X\lambda - x \\ -Y\lambda + y \end{pmatrix} \in W^*,\ \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \lambda \in -K^*, \lambda_{n+1} \ge 0 \right\}.$$

We now examine the various important special cases of the generalized models (P) and (D) which are obtained by setting parameters δ_1, δ_2, δ_3 to different binary values.

Case 1: $\delta_1 = 0$, $W = E^{m+s}_+$ and $K = E^n_+$. In this case, the model described by (P) and (D) reduces to the (CCR) model (Charnes et al. (1978)):

$$(\mathrm{P1})\quad \begin{cases} \text{maximize} & \mu^T y_0 \\ \text{subject to} & \omega^T x_j - \mu^T y_j \geq 0 \qquad j = 1, \ldots, n, \\ & \omega^T x_0 = 1, \\ & \omega \geq 0,\ \mu \geq 0, \end{cases}$$

and

$$(\mathrm{D1})\quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \sum_{j=1}^n x_j \lambda_j \leq \theta x_0, \\ & \sum_{j=1}^n y_j \lambda_j \geq y_0, \\ & \theta \in E^1, \lambda_j \geq 0 \qquad j = 1, \ldots, n. \end{cases}$$

In case 1, the corresponding production possibility set is:

$$T_{\mathrm{CCR}} = \left\{ (x, y) \,\middle|\, \sum_{j=1}^n x_j \lambda_j \leq x, \sum_{j=1}^n y_j \lambda_j \geq y, \lambda_j \geq 0,\ \ j = 1, \ldots, n \right\}.$$

Case 2: $\delta_1 = 1$, $\delta_2 = 0$, $W = E_+^{m+s}$ and $K = E_+^n$. In this case, the model described by (P) and (D) reduces to the (BCC) model (Banker et al. (1984)):

$$(\mathrm{P2})\quad \begin{cases} \text{maximize} & (\mu^T y_0 - \mu_0) \\ \text{subject to} & \omega^T x_j - \mu^T y_j + \mu_0 \geq 0 \qquad j = 1, \ldots, n, \\ & \omega^T x_0 = 1, \\ & \omega \geq 0,\ \mu \geq 0,\ \mu_0 \in E^1, \end{cases}$$

and

$$(\mathrm{D2})\quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \sum_{j=1}^n x_j \lambda_j \leq \theta x_0, \\ & \sum_{j=1}^n y_j \lambda_j \geq y_0, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \theta \in E^1, \lambda_j \geq 0 \qquad j = 1, \ldots, n. \end{cases}$$

The corresponding production possibility set for case 2 is:

$$T_{\mathrm{BCC}} = \left\{ (x, y) \,\middle|\, \sum_{j=1}^n x_j \lambda_j \leq x, \sum_{j=1}^n y_j \lambda_j \geq y, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0,\ \ j = 1, \ldots, n \right\}.$$

Case 3: $\delta_1 = \delta_2 = 1$, $\delta_3 = 0$, $W = E_+^{m+s}$ and $K = E_+^n$. In this situation, the model described by (P) and (D) reduces to the (FG) model (Färe and Grosskopf (1985)):

$$\text{(P3)} \quad \begin{cases} \text{maximize} & (\mu^T y_0 - \mu_0) \\ \text{subject to} & \omega^T x_j - \mu^T y_j + \mu_0 \geq 0 \quad j = 1, \dots, n, \\ & \omega^T x_0 = 1, \\ & \omega \geq 0, \ \mu \geq 0, \ \mu_0 \geq 0, \end{cases}$$

and

$$\text{(D3)} \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \sum_{j=1}^n x_j \lambda_j \leq \theta x_0, \\ & \sum_{j=1}^n y_j \lambda_j \geq y_0, \\ & \sum_{j=1}^n \lambda_j \leq 1, \\ & \theta \in E^1, \lambda_j \geq 0 \qquad j = 1, \dots, n. \end{cases}$$

In case 3, the corresponding production possibility set is:

$$T_{\text{FG}} = \left\{ (x, y) \,\middle|\, \sum_{j=1}^n x_j \lambda_j \leq x, \sum_{j=1}^n y_j \lambda_j \geq y, \sum_{j=1}^n \lambda_j \leq 1, \lambda_j \geq 0, \ j = 1, \dots, n \right\}.$$

Case 4: $\delta_1 = \delta_2 = \delta_3 = 1$, $W = E_+^{m+s}$ and $K = E_+^n$. In this case, the model given by (P) and (D) reduces to the ST model (Seiford and Thrall (1990)):

$$\text{(P4)} \quad \begin{cases} \text{maximize} & \mu^T y_0 - \mu_0 \\ \text{subject to} & \omega^T x_j - \mu^T y_j + \mu_0 \geq 0 \quad j = 1, \dots, n, \\ & \omega^T x_0 = 1, \\ & \omega \geq 0, \ \mu \geq 0, \ \mu_0 \leq 0, \end{cases}$$

and

$$\text{(D4)} \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \sum_{j=1}^n x_j \lambda_j \leq \theta x_0, \\ & \sum_{j=1}^n y_j \lambda_j \geq y_0, \\ & \sum_{j=1}^n \lambda_j \geq 1, \\ & \theta \in E^1, \lambda_j \geq 0 \qquad j = 1, \dots, n. \end{cases}$$

The corresponding production possibility set for case 4 is:

$$T_{ST} = \left\{(x, y) \middle| \sum_{j=1}^{n} x_j\lambda_j \le x, \sum_{j=1}^{n} y_j\lambda_j \ge y, \sum_{j=1}^{n} \lambda_j \ge 1, \lambda_j \ge 0, \quad j = 1, \dots, n\right\}.$$

Case 5: $\delta_1 = 0$ and $W = V \times U$, where $V \subseteq E_+^m$ and $U \subseteq E_+^s$ are both closed convex cones. Thus, the model described by (P) and (D) reduces to the (CCWH) model (Charnes et al. (1989)):

$$\text{(P5)} \quad \begin{cases} \text{maximize} & \mu^T y_0 \\ \text{subject to} & \omega^T X - \mu^T Y \in K, \\ & \omega^T x_0 = 1, \\ & \omega \in V, \ \mu \in U, \end{cases}$$

and

$$\text{(D5)} \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & X\lambda - \theta x_0 \in V^*, \\ & -Y\lambda + y_0 \in U^*, \\ & \lambda \in -K^*, \ \theta \in E^1. \end{cases}$$

In case 5, the corresponding production possibility set is:

$$T_{CCWH} = \{(x, y) \mid X\lambda - x \in V^*, -Y\lambda + y \in U^*, \lambda \in -K^*\}.$$

Case 6: $\delta_1 = 1$, $\delta_2 = 0$, $K = E_+^n$ and $W = V \times U$, where $V \subseteq E_+^m$ and $U \subseteq E_+^s$ are both closed convex cones. In this case, the model given by (P) and (D) reduces to the (CCWY) model with a finite number of DMUs (Charnes et al. (1988)):

$$\text{(P6)} \quad \begin{cases} \text{maximize} & (\mu^T y_0 - \mu_0) \\ \text{subject to} & \omega^T X - \mu^T Y + \mu_0 e^T \ge 0, \\ & \omega^T x_0 = 1, \\ & \omega \in V, \ \mu \in U, \ \mu_0 \in E^1, \end{cases}$$

and

$$\text{(D6)} \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & X\lambda - \theta x_0 \in V^*, \\ & -Y\lambda + y_0 \in U^*, \\ & e^T\lambda = 1, \\ & \lambda \ge 0. \end{cases}$$

The corresponding production possibility set for case 6 is:

$$T_{\text{CCWY}} = \{(x, y) \mid X\lambda - x \in V^*, -Y\lambda + y \in U^*, e^T\lambda = 1, \lambda \geq 0\}.$$

To summarize the preceding enumeration, the generalized DEA model presented in (P) and (D) constitutes the most general model to date and includes all the previous specialized DEA models as its special cases. Table 1 lists all the DEA models that are subclasses of the generalized DEA model, with the symbol (*) indicating a hitherto nonexistent model. Although all the previously known models are included in the generalized DEA model described in this paper, we address the following relevant relationships in particular:

Table 1

		$\delta_1 = 0$	$\delta_1 = 1, \delta_2 = 0$	$\delta_1 = \delta_2 = 1, \delta_3 = 0$	$\delta_1 = \delta_2 = \delta_3 = 1$
K	W	$\sum_j \lambda_j$ free	$\sum_j \lambda_j = 1$	$\sum_j \lambda_j \leq 1$	$\sum_j \lambda_j \geq 1$
E_+^n	E_+^{n+s}	(CCR)	(BCC)	(FG)	(ST)
K	$V \times U$	(CCWH)	(*)	(*)	(*)
E_+^n	$V \times U$	(*)	(CCWY)	(*)	(*)
K	W	Generalized (CCR) model ($\hat{P}_1$)	Generalized (BCC) model ($\hat{P}_2$)	Generalized (FG) model ($\hat{P}_3$)	Generalized (ST) model ($\hat{P}_4$)

(i) The relationships among DEA efficient solutions obtained by using the different special subclasses of (P) and (D) resulting from setting (δ_1, δ_2, δ_3) to different values.

(ii) The equivalence between DEA efficient solutions from the generalized DEA model and the nondominated solutions in the following multi-objective program:

$$\text{(VP)} \quad \begin{cases} \text{V-min } (x, -y), \\ (x, y) \in T, \end{cases}$$

where V-min denotes vector minimization.

(iii) The equivalence (in the sense of DEA efficiencies) between the generalized DEA model and its corresponding additive DEA model.

(iv) The characteristics and functions of input/output preference cone W and the predilection cone K, especially in the polyhedra case.

(v) The use of concrete examples to illustrate the implications and uses of the generalized DEA model in many different situations.

We note that Wei and Yu (1993) have studied the characteristics, properties, economic interpretations, and possible applications of the predilection cone K in the generalized DEA model. Additionally, Wei et al. (1993) provide detailed analysis and rigorous proofs of the necessary and sufficient conditions for return to scale properties in the generalized DEA model.

3 The relationships among special subclasses of the generalized DEA model

In the generalized DEA model (P) and (D), there are four different subclasses of DEA models which can be generated by setting $(\delta_1, \delta_2, \delta_3)$ to different values. For clarity of exposition, we restate these as:

Case 1: $(\delta_1, \delta_2, \delta_3) = (0, *, *)$, where the symbol $*$ allows the corresponding parameter to take values of either 0 or 1. In this case, (P) and (D) reduce to the generalized (CCR) model:

$$(\hat{\mathrm{P}}_1) \quad \begin{cases} \text{maximize} & \mu^T y_0 \\ \text{subject to} & \omega^T X - \mu^T Y \in K, \\ & \omega^T x_0 = 1, \\ & \begin{pmatrix} \omega \\ \mu \end{pmatrix} \in W, \end{cases}$$

and

$$(\hat{\mathrm{D}}_1) \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \begin{pmatrix} X\lambda - \theta x_0 \\ -Y\lambda + y_0 \end{pmatrix} \in W^*, \\ & \theta \in E^1, \ \lambda \in -K^*. \end{cases}$$

Case 2: $(\delta_1, \delta_2, \delta_3) = (1, 0, *)$. Thus, (P) and (D) reduce to the generalized (BCC) model:

$$(\hat{\mathrm{P}}_2) \quad \begin{cases} \text{maximize} & (\mu^T y_0 - \mu_0) \\ \text{subject to} & \omega^T X - \mu^T Y + \mu_0 e^T \in K, \\ & \omega^T x_0 = 1, \\ & \begin{pmatrix} \omega \\ \mu \end{pmatrix} \in W, \ \mu_0 \in E^1, \end{cases}$$

and

$$(\hat{\mathrm{D}}_2) \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \begin{pmatrix} X\lambda - \theta x_0 \\ -Y\lambda + y_0 \end{pmatrix} \in W^*, \\ & \theta \in E^1, \ \lambda \in -K^*. \end{cases}$$

Case 3: $(\delta_1, \delta_2, \delta_3) = (1, 1, 0)$. In this case, (P) and (D) reduce to the generalized (FG) model:

$$(\hat{P}_3)\quad \begin{cases} \text{maximize} & (\mu^T y_0 - \mu_0) \\ \text{subject to} & \omega^T X - \mu^T Y + \mu_0 e^T \in K, \\ & \omega^T x_0 = 1, \\ & \begin{pmatrix} \omega \\ \mu \end{pmatrix} \in W,\ \mu_0 \geq 0, \end{cases}$$

and

$$(\hat{D}_3)\quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \begin{pmatrix} X\lambda - \theta x_0 \\ -Y\lambda + y_0 \end{pmatrix} \in W^*, \\ & e^T\lambda \leq 1, \\ & \theta \in E^1,\ \lambda \in -K^*. \end{cases}$$

Case 4: $(\delta_1, \delta_2, \delta_3) = (1, 1, 1)$. Thus, (P) and (D) reduce to the generalized (ST) model:

$$(\hat{P}_4)\quad \begin{cases} \text{maximize} & \mu^T y_0 - \mu_0 \\ \text{subject to} & \omega^T X - \mu^T Y + \mu_0 e^T \in K, \\ & \omega^T x_0 = 1, \\ & \begin{pmatrix} \omega \\ \mu \end{pmatrix} \in W,\ \mu_0 \leq 0, \end{cases}$$

and

$$(\hat{D}_4)\quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & \begin{pmatrix} X\lambda - \theta x_0 \\ -Y\lambda + y_0 \end{pmatrix} \in W^*, \\ & \sum_{j=1}^n y_j\lambda_j \geq y_0, \\ & e^T\lambda \geq 1, \\ & \theta \in E^1,\ \lambda \in -K^*. \end{cases}$$

For convenience, throughout this paper we use the terminology "(CCR)-DEA efficient" to indicate that the decision making unit DMU_{j_0} is DEA efficient under the generalized (CCR) model. In a similar manner, we will also use the terms "(BCC)-DEA efficient", "(FG)-DEA efficient", etc.

In the remainder of this section, we will present five theorems which show relationships between the different generalized DEA models.

Theorem 1

$$\text{(CCR)-DEA efficient} \begin{array}{c} \nearrow \text{(FG)-DEA efficient} \searrow \\ \searrow \text{(ST)-DEA efficient} \nearrow \end{array} \text{(BCC)-DEA efficient,}$$

where arrows denote implications.

Proof

Consider the programs $(\hat{P}_1), (\hat{P}_2), (\hat{P}_3)$ and $(\hat{P}_4)$. The result of the theorem follows immediately once we notice that $(\hat{P}_1)$ does not contain variable μ_0 (equivalent to setting $\mu_0 = 0$), $(\hat{P}_3)$ requires $\mu_0 \geq 0$, $(\hat{P}_4)$ requires $\mu_0 \leq 0$, and $(\hat{P}_2)$ does not place any restriction on μ_0. With these facts, the theorem can be easily derived. □

Theorem 2

Assume DMU_{j_0} is (BCC)-DEA efficient. If $(\hat{P}_2)$ has an optimal solution ω^0, μ^0, μ_0^0, then:

(i) if $\mu_0^0 \geq 0$, then DMU_{j_0} is (FG)-DEA efficient,

(ii) if $\mu_0^0 \leq 0$, then DMU_{j_0} is (ST)-DEA efficient,

(iii) if $\mu_0^0 = 0$, then DMU_{j_0} is (CCR)-DEA efficient.

Proof

In this paper, we only prove theorem 2(i), since theorem 2(ii) and (iii) may be proved in a similar manner. Say ω^0, μ^0, μ_0^0 is an optimal solution of $(\hat{P}_2)$ with

$$\mu^{0^T} y_0 - \mu_0^0 = 1,$$

$$\omega^{0^T} X - \mu^{0^T} Y + \mu_0^0 e^T \in K,$$

$$\omega^{0^T} x_0 = 1,$$

$$\begin{pmatrix} \omega^0 \\ \mu^0 \end{pmatrix} \in \text{Int } W.$$

Now $\mu_0^0 \geq 0$; thus, ω^0, μ^0, μ_0^0 is also an optimal solution to $(\hat{P}_3)$, so that by definition 1, DMU_{j_0} is (FG)-DEA efficient. □

By a proof similar to that of theorem 2, we also have theorems 3 and 4.

Theorem 3

Assume DMU_{j_0} is (FG)-DEA efficient. If $(\hat{P}_3)$ has an optimal solution ω^0, μ^0, μ_0^0 with $\mu_0^0 = 0$, then DMU_{j_0} is (CCR)-DEA efficient.

Theorem 4
Assume DMU_{j_0} is (ST)-DEA efficient. If $(\hat{\mathrm{P}}_4)$ has an optimal solution ω^0, μ^0, μ_0^0 with $\mu_0^0 = 0$, then DMU_{j_0} is (CCR)-DEA efficient.

Based on the preceding information, we can now develop theorem 5.

Theorem 5
If DMU_{j_0} is both (FG)-DEA efficient and (ST)-DEA efficient, then it is (CCR)-DEA efficient.

Proof
Let ω^*, μ^*, μ_0^* be an optimal solution of $(\hat{\mathrm{P}}_3)$. We then have:

$$\mu^{*T} y_0 - \mu_0^* = 1,$$
$$\omega^{*T} X - \mu^{*T} Y + \mu_0^* e^T \in K,$$
$$\omega^{*T} x_0 = 1,$$
$$\begin{pmatrix} \omega^{*T} \\ \mu^{*T} \end{pmatrix} \in \operatorname{Int} W, \quad \mu_0^* \geq 0.$$

Now assume that ω^{**}, μ^{**}, μ_0^{**} is an optimal solution of $(\hat{\mathrm{P}}_4)$. We then have:

$$\mu^{**T} y_0 - \mu_0^{**} = 1,$$
$$\omega^{**T} X - \mu^{**T} Y + \mu_0^{**} e^T \in K,$$
$$\omega^{**T} x_0 = 1,$$
$$\begin{pmatrix} \omega^{**T} \\ \mu^{**T} \end{pmatrix} \in \operatorname{Int} W, \quad \mu_0^{**} \leq 0.$$

Without loss of generality, we may assume $\mu_0^* > 0$ and $\mu_0^{**} < 0$ (since otherwise, by theorems 3 and 4, we know that DMU_{j_0} is (CCR)-DEA efficient). Let

$$\omega^0 = \alpha\omega^* + (1-\alpha)\omega^{**},$$
$$\mu^0 = \alpha\mu^* + (1-\alpha)\mu^{**},$$
$$\mu_0^0 = \alpha\mu_0^* + (1-\alpha)\mu_0^{**},$$

where

$$\alpha = \frac{-\mu_0^{**}}{\mu_0^* - \mu_0^{**}}.$$

Thus, $0 < \alpha < 1$, $\mu_0^0 = 0$ and

$$\mu^{0^T} y_0 = \mu^{0^T} y_0 - \mu_0^0 = 1,$$

$$\omega^{0^T} X - \mu^{0^T} Y + \mu_0^0 e^T \in K,$$

$$\omega^{0^T} x_0 = 1,$$

$$\begin{pmatrix} \omega^0 \\ \mu^0 \end{pmatrix} \in \text{Int } W.$$

Thus, by definition 1, we know that DMU_{j_0} is (CCR)-DEA efficient. □

From the previous analysis, we can state corollary 1.

Corollary 1

Decision making unit DMU_{j_0} is (CCR)-DEA efficient if and only if it is (FG)-DEA efficient and (ST)-DEA efficient.

4 The equivalence between DEA efficiency and the nondominated solutions of multi-objective programs

In this section, we present several theorems and assumptions regarding DEA efficiency and nondominated solutions. To begin, we consider the following *generalized additive* DEA model (see Charnes et al. (1985) and by setting $K = E_+^n$ and $W = E_+^{m+s}$):

$$(\text{P}_0) \quad \begin{cases} \text{maximize} & (\tau^T s^- + \hat{\tau}^T s^+) \\ \text{subject to} & \sum_{j=1}^n x_j \lambda_j + s^- = x_0, \\ & \sum_{j=1}^n y_j \lambda_j - s^+ = y_0, \\ & \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \\ & \lambda \in -K^*, \ \lambda_{n+1} \geq 0, \begin{pmatrix} s^- \\ s^+ \end{pmatrix} \in -W^*, \end{cases}$$

and its dual:

$$(\text{D}_0) \quad \begin{cases} \text{minimize} & (\omega^T x_0 - \mu^T y_0 + \delta_1 \mu^0) \\ \text{subject to} & \omega^T X - \mu^T Y + \delta_1 \mu_0 e^T \in K, \\ & \begin{pmatrix} \omega - \tau \\ \mu - \hat{\tau} \end{pmatrix} \in W, \\ & \delta_1 \delta_2 (-1)^{\delta_3} \mu_0 \geq 0, \end{cases}$$

where $\binom{\tau}{\hat{\tau}} \in \text{Int } W$.

Lemma 1 (Weak Duality Theorem)
Let λ, λ_{n+1}, s^-, s^+ be a feasible solution of (P_0) and let ω, μ, μ_0 be a feasible solution of (D_0). Then,

$$\omega^T x_0 - \mu^T y_0 + \delta_1 \mu_0 \geq \tau^T s^- + \hat{\tau}^T s^+.$$

Proof

By $\binom{\omega - \tau}{\mu - \hat{\tau}} \in W$ and $\binom{s^-}{s^+} \in -W^*$, we have

$$(\omega - \tau)^T s^- + (\mu - \hat{\tau})^T s^+ \geq 0.$$

Thus,

$$\begin{aligned} \tau^T s^- + \hat{\tau} s^+ &\leq \omega^T s^- + \mu^T s^+ \\ &= \omega^T \left(x_0 - \sum_{j=1}^n x_j \lambda_j \right) + \mu^T \left(\sum_{j=1}^n y_j \lambda_j - y_0 \right) \\ &= (\omega^T x_0 - \mu^T y_0) - \sum_{j=1}^n (\omega^T x_j - \mu^T y_j) \lambda_j . \end{aligned} \tag{1}$$

From $\lambda \in -K^*$ and $\omega^T X - \mu^T Y + \delta_1 \mu_0 e^T \in K$, we have

$$\sum_{j=1}^n (\omega^T x_j - \mu^T y_j) \lambda_j + \delta_1 \mu_0 \sum_{j=1}^n \lambda_j \geq 0. \tag{2}$$

We also have

$$\begin{aligned} \delta_1 \mu_0 \sum_{j=1}^n \lambda_j - \delta_1 \mu_0 &= \delta_1 \mu_0 \left(\sum_{j=1}^n \lambda_j - 1 \right) \\ &= -\delta_1 \delta_2 (-1)^{\delta_3} \mu_0 \lambda_{n+1} \\ &\leq 0, \end{aligned}$$

i.e.,

$$\delta_1 \mu_0 \sum_{j=1}^n \lambda_j \leq \delta_1 \mu_0 . \tag{3}$$

The theorem is now a direct result of (1), (2) and (3). □

Before we proceed, we will state assumption 1, which will naturally hold in the case when cones K and W are polyhedral (i.e., the program (P_0) is linear).

Assumption 1
Let λ^0, λ^0_{n+1}, s^{-0}, s^{+0} be a feasible solution of (P_0) and let $\overline{D}(\lambda^0, \lambda^0_{n+1}, s^{-0}, s^{+0})$ be a closed set, where

$$\overline{D}(\lambda^0, \lambda_{n+1}^0, s^{-0}, s^{+0})$$

$$= \left\{ \begin{bmatrix} X^T\omega - Y^T\mu + \mu_0\delta_1 e + y_1, \\ \omega + y_2, \\ \mu + y_3, \\ \delta_1\delta_2(-1)^{\delta_3}\mu_0 + y_4, \end{bmatrix} \middle| \begin{matrix} y_1^T \in K, \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \in W, y_4 \geq 0, \\ y_1^T\lambda^0 = 0,\ y_2^T s^{-0} + y_3^T s^{+0} = 0 \\ y_4\lambda_{n+1}^0 = 0. \end{matrix} \right\}.$$

In the discussions in the sequel, we shall always assume that assumption 1 holds.

Lemma 2 (Duality Theorem)
Let λ^0, λ_{n+1}^0, s^{-0}, s^{+0} be an optimal solution of (P_0). Accordingly, there must exist an optimal solution ω^0, μ_0, μ_0^0 of (D_0) such that

$$\tau^T s^{-0} + \hat{\tau} s^{+0} = \omega^{0^T} x_0 - \mu^{0^T} y_0 + \delta_1\mu_0^0.$$

Proof
Since the dual of (P_0) is (D_0) and assumption 1 holds, by applying theorem A2 from Charnes et al. (1989), we obtain the result of this lemma. □

Theorem 6
If the optimal objective value of (P_0) is 0, then DMU_{j_0} is DEA efficient.

Proof
By lemma 2, the optimal objective value of (D_0) is also 0. Let ω^*, μ^*, μ_0^* be an optimal solution of (D_0). Then we have

$$\omega^{*T} X - \mu^{*T} Y + \delta_1\mu_0^* e^T \in K,$$

$$\begin{pmatrix} \omega^* - \tau \\ \mu^* - \hat{\tau} \end{pmatrix} \in W,$$

$$\delta_1\delta_2(-1)^{\delta_3}\mu_0^* \geq 0$$

and

$$\omega^{*T} x_0 - \mu^{*T} y_0 + \delta_1\mu_0^* = 0.$$

Now let

$$\omega^0 = \frac{\omega^*}{\omega^{*T} x_0}, \quad \mu^0 = \frac{\mu^*}{\omega^{*T} x_0}, \quad \mu_0^0 = \frac{\mu_0^*}{\omega^{*T} x_0}.$$

Since $\binom{\tau}{\hat{\tau}} \in \text{Int}\, W$, we have

$$\begin{pmatrix} \omega^* \\ \mu^* \end{pmatrix} \in \begin{pmatrix} \tau \\ \hat{\tau} \end{pmatrix} + W \subseteq \text{Int}\, W.$$

Thus, $\omega^{*T}x_0 > 0$, and

$$\begin{pmatrix} \omega^0 \\ \mu^0 \end{pmatrix} = \frac{1}{\omega^{*T}x_0} \begin{pmatrix} \omega^* \\ \mu^* \end{pmatrix} \in \text{Int } W,$$

while

$$\delta_1\delta_2(-1)^{\delta_3}\mu_0^0 = \delta_1\delta_2(-1)^{\delta_3}\frac{\mu_0^*}{\mu^{*T}x_0} \geq 0$$

and

$$\mu^{0^T}x_0 - \mu^{0^T}y_0 + \delta_1\mu_0^0 = \frac{1}{\omega^{*T}x_0}(\mu^{*T}x_0 - \mu^{*T}y_0 + \delta_1\mu_0^*) = 0.$$

However, since

$$\omega^{0^T}x_0 = \frac{\mu^{*T}}{\mu^{*T}x_0}x_0 = 1,$$

we conclude by definition 1 that DMU_{j_0} is DEA efficient. □

Lemma 3
If ω^0, μ^0, μ_0^0 is an optimal solution of (P) and $\mu^{0^T}y_0 - \delta_1\mu_0^0 = 1$, then for any $(x, y) \in T$ we have

$$\omega^{0^T}x - \mu^{0^T}y \geq \omega^{0^T}x_0 - \mu^{0^T}y_0,$$

where T is the production possibility set of the generalized DEA model.

Proof
Assume $(x, y) \in T$. In this case there must exist $\binom{s^-}{s^+} \in -W^*$, $\lambda \in -K^*$, $\lambda_{n+1} \geq 0$ such that

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n x_j\lambda_j \\ -\sum_{j=1}^n y_j\lambda_j \end{pmatrix} + \begin{pmatrix} s^- \\ s^+ \end{pmatrix}.$$

Thus,

$$\begin{aligned} \omega^{0^T}x - \mu^{0^T}y &= \sum_{j=1}^n (\omega^{0^T}x_j - \mu^{0^T}y_j)\lambda_j + (\omega^{0^T}s^- + \mu^{0^T}s^+) \\ &\geq \sum_{j=1}^n (\omega^{0^T}x_j - \mu^{0^T}y_j)\lambda_j, \end{aligned} \tag{4}$$

while by $\lambda \in -K^*$ and $\omega^{0^T}X - \mu^{0^T}Y + \delta_1\mu_0^0 e^T \in K$, we have

$$\sum_{j=1}^n (\omega^{0^T}x_j - \mu^{0^T}y_j)\lambda_j + \delta_1\mu_0^0\sum_{j=1}^n \lambda_j \geq 0. \tag{5}$$

Also, we have

$$\omega^{0^T} x_0 - \mu^{0^T} y_0 + \delta_1 \mu_0^0 = 0. \tag{6}$$

From (4), (5) and (6), for any $(x, y) \in T$ we have

$$\begin{aligned}\omega^{0^T} x - \mu^{0^T} y + \delta_1 \mu_0^0 \sum_{j=1}^{n} \lambda_j &\geq \sum_{j=1}^{n} (\omega^{0^T} x_j - \mu^{0^T} y_j)\lambda_j + \delta_1 \mu_0^0 \sum_{j=1}^{n} \lambda_j \\ &\geq 0 \\ &= \omega^{0^T} x_0 - \mu^{0^T} y_0 + \delta_1 \mu_0^0,\end{aligned}$$

and thus

$$\begin{aligned}\omega^{0^T} x - \mu^{0^T} y &\geq \omega^{0^T} x_0 - \mu^{0^T} y_0 + \delta_1 \mu_0^0 \left(1 - \sum_{j=1}^{n} \lambda_j\right) \\ &= \omega^{0^T} x_0 - \mu^{0^T} y_0 + \delta_1 \delta_2 (-1)^{\delta_3} \mu_0^0 \lambda_{n+1} \\ &\geq \omega^{0^T} x_0 - \mu^{0^T} y_0.\end{aligned}$$

□

Theorem 7
If DMU_{j_0} is DEA efficient, then the optimal objective value of (P_0) is 0.

Proof
Let an optimal solution of (P_0) be λ^*, s^{-*}, s^{+*}, λ^*_{n+1} and $\tau^T s^{-*} + \hat{\tau}^T s^{+*} \neq 0$. Then,

$$\begin{pmatrix} s^{-*} \\ s^{+*} \end{pmatrix} \neq 0; \quad \text{i.e.,} \begin{pmatrix} s^{-*} \\ s^{+*} \end{pmatrix} \in -W^* \backslash \{0\}.$$

Now, since DMU_{j_0} is DEA efficient, there exists an optimal solution ω^0, μ^0, μ_0^0 of (P) which satisfies

$$\omega^{0^T} x_0 - \mu^{0^T} y_0 + \delta_1 \mu_0^0 = 1$$

and $\binom{\omega^0}{\mu^0} \in \mathrm{Int}\, W$. Let

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x_0 - s^{-*} \\ y_0 + s^{+*} \end{pmatrix}.$$

Then,

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} x_j \lambda_j^* \\ \sum_{j=1}^{n} y_j \lambda_j^* \end{pmatrix} \in T,$$

and from lemma 3, we have

$$\omega^{0^T} x^* - \mu^{0^T} y^* \geq \omega^{0^T} x_0 - \mu^{0^T} y_0.$$

On the other hand, since

$$\begin{pmatrix} \omega^0 \\ \mu^0 \end{pmatrix} \in \operatorname{Int} W, \quad \begin{pmatrix} s^{-*} \\ s^{+*} \end{pmatrix} \in -W^*\backslash\{0\},$$

and Int $W \neq \emptyset$, we know that W is an acute cone (see Charnes et al. (1989, 1990)), and hence $(\omega^{0^T} s^{-*} + \mu^{0^T} s^{+*}) < 0$. Thus, we have

$$\omega^{0^T} x^* - \mu^{0^T} y^* = (\omega^{0^T} x_0 - \mu^{0^T} y_0) - (\omega^{0^T} s^{-*} + \mu^{0^T} s^{+*}) < \omega^{0^T} x_0 - \mu^{0^T} y_0,$$

which contradicts our assumption. □

Corollary 2

The necessary and sufficient condition for DMU_{j_0} to be DEA efficient is that there is no feasible solution to the following set of inequalities:

$$\begin{cases} \sum_{j=1}^n x_j \lambda_j + s^- = x_0, \\ -\sum_{j=1}^n y_j \lambda_j + s^+ = y_0, \\ \delta_1 \sum_{j=1}^n + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \\ \lambda \in -K^*, \lambda_{n+1} \geq 0, \\ \begin{pmatrix} s^- \\ s^+ \end{pmatrix} \in -W^*\backslash\{0\}. \end{cases}$$

Proof

This corollary is a straightforward conclusion from theorems 6 and 7. □

We will now turn our attention to the notion of nondominated vector extremals, as we consider the following multi-objective mathematical programming:

$$\text{(VP)} \quad \begin{cases} \text{V-min } (f_1(x, y), \ldots, f_{m+s}(x, y)) \\ (x, y) \in T, \end{cases}$$

where

$$f_k(x, y) = \begin{cases} x_k & 1 \leq k \leq m, \\ -y_{k-m} & m+1 \leq k \leq m+s, \end{cases}$$

and $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_s)$, where V-min denotes vector extremization. Note that

$$T = \{(x, y) | \begin{pmatrix} X\lambda - x \\ -Y\lambda + y \end{pmatrix} \in W^*, \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \lambda \in -K^*, \lambda_{n+1} \geq 0\}.$$

Denote

$$F(x, y) = (f_1(x, y), \ldots, f_{m+s}(x, y))^T = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

With the preceding preparation, we can state definition 2.

Definition 2

(x_0, y_0) is called a nondominated solution of the multi-objective program (VP) associated with W^* if there exists no $(x, y) \in T$ such that

$$F(x, y) \in F(x_0, y_0) + W^*, \quad F(x, y) \neq F(x_0, y_0);$$

i.e., if there exists no $(x, y) \in T$ such that

$$\begin{pmatrix} x \\ -y \end{pmatrix} \in \begin{pmatrix} x_0 \\ -y_0 \end{pmatrix} + W^*, \begin{pmatrix} x \\ -y \end{pmatrix} \neq \begin{pmatrix} x_0 \\ -y_0 \end{pmatrix}.$$

Based on the preceding information, we have theorem 8.

Theorem 8

DMU_{j_0} is DEA efficient if and only if (x_0, y_0) is a nondominated solution of (VP) associated with W^*.

Proof

By definition of nondominated solutions, the necessary and sufficient condition for (x_0, y_0) to be a nondominated solution is that there is no feasible solution to the following inequalities:

$$\begin{cases} \sum_{j=1}^{n} x_j \lambda_j + s^- = x_0, \\ -\sum_{j=1}^{n} y_j \lambda_j + s^+ = y_0, \\ \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \\ \lambda \in -K^*, \lambda_{n+1} \geq 0, \\ \begin{pmatrix} s^- \\ s^+ \end{pmatrix} \in -W^* \backslash \{0\}. \end{cases}$$

The theorem then holds by corollary 2. □

We now consider the impact of efficient frontiers in DEA analysis. Definition 3 begins our discussion of this topic.

Definition 3

Assume

$$\begin{pmatrix} \omega^0 \\ \mu^0 \end{pmatrix} \in \operatorname{Int} W, \delta_1 \delta_2 (-1)^{\delta_3} \mu_0^0 \geq 0, \text{ and } \omega^{0^T} X - \mu^{0^T} T + \delta_1 \mu_0^0 e^T \in K,$$

and let $L = \{(x, y) | \omega^{0^T} x - \mu^{0^T} y + \delta_1 \mu_0^0 = 0\}$. Then, L is called an efficient frontier surface of T, or simply an efficient frontier, if $L \cap T \neq \varnothing$.

From this definition, we can now provide theorem 9.

Theorem 9
The decision making unit DMU_{j_0} is DEA efficient if and only if (x_0, y_0) is on one of the efficient frontiers of T.

Proof
If DMU_{j_0} is DEA efficient, by definition 1 there exist ω^0, μ^0, μ_0^0 such that

$$\omega^{0^T} X - \mu^{0^T} Y + \delta_1 \mu_0^0 e^T \in K,$$

$$\omega^{0^T} x_0 = 1,$$

$$\begin{pmatrix} \omega^0 \\ \mu^0 \end{pmatrix} \in \operatorname{Int} W,$$

and

$$\mu^{0^T} y_0 - \delta_1 \mu_0^0 = 1.$$

Now let $L = \{(x, y) | \omega^{0^T} x - \mu^{0^T} y + \delta_1 \mu_0^0 = 0\}$. Then clearly $(x_0, y_0) \in L \cap T$.

On the other hand, if (x_0, y_0) lies on an efficiemt frontier $\overline{L}$, where $\overline{L} = \{(x, y) | \overline{\omega}^T x - \overline{\mu}^T y + \delta_1 \overline{\mu}_0 = 0\}$, then we have

$$\overline{\omega}^T X - \overline{\mu}^T Y + \delta_1 \overline{\mu}_0 e^T \in K,$$

$$\begin{pmatrix} \overline{\omega} \\ \overline{\mu} \end{pmatrix} \in \operatorname{Int} W, \delta_1 \delta_2 (-1)^{\delta_3} \overline{\mu}_0 \geq 0,$$

$$\overline{\omega}^T x_0 - \overline{\mu}^T y_0 + \delta_1 \overline{\mu}_0 = 0.$$

Let

$$(\omega^{0^T}, \mu^{0^T}, \mu_0^0) = \frac{1}{\overline{\omega}^T x_0} (\overline{\omega}^T, \overline{\mu}^T, \overline{\mu}_0).$$

Since $\overline{\omega}^T x_0 > 0$, we have

$$\omega^{0^T} X - \mu^{0^T} Y + \delta_1 \mu_0^0 e^T = \frac{1}{\overline{\omega}^T x_0} (\overline{\omega}^T X - \overline{\mu}^T Y + \delta_1 \overline{\mu}_0 e^T) \in K,$$

$$\begin{pmatrix} \omega^0 \\ \mu^0 \end{pmatrix} = \frac{1}{\overline{\omega}^T x_0} \begin{pmatrix} \overline{\omega} \\ \overline{\mu} \end{pmatrix} \in \operatorname{Int} W,$$

$$\delta_1 \delta_2 (-1)^{\delta_3} \mu_0^0 = \frac{1}{\overline{\omega}^T x_0} \delta_1 \delta_2 (-1)^{\delta_3} \overline{\mu}_0 \geq 0,$$

and furthermore,

$$1 = \omega^{0^T} x_0 = \mu^{0^T} y_0 - \delta_1 \mu_0^0.$$

Thus, ω^0, μ^0, μ_0^0 is an optimal solution of (P). From definition 1, we conclude that DMU_{j_0} is DEA efficient, thereby proving theorem 9. □

We now define the projection of DMU in terms of the additive DEA model. Let an optimal solution to the generalized additive DEA model (P_0) be λ_j^0, $j = 1,\ldots,n, n+1$; s^{-0} and s^{+0}. Define

$$\hat{x}_0 = x_0 - s^{-0} = \sum_{j=1}^{n} x_j \lambda_j^0$$

and

$$\hat{y}_0 = x_0 + s^{+0} = \sum_{j=1}^{n} y_j \lambda_j^0.$$

We call $(\hat{x}_0, \hat{y}_0)$ the projection of DMU_{j_0}. Theorem 10 follows.

Theorem 10

The projection of DMU_{j_0} is on one of the efficient frontiers of T.

Proof

By theorems 8 and 9, we need only show that $(\hat{x}_0, \hat{y}_0)$ is a nondominated solution to the multi-objective program (VP) associated with W^*.

If $(\hat{x}_0, \hat{y}_0)$ is not a nondominated solution of (VP), then there must exist $(\bar{x}, \bar{y}) \in T$ and $\binom{\hat{\omega}}{\hat{\mu}} \in W^*$ such that

$$\begin{pmatrix} \bar{x} \\ -\bar{y} \end{pmatrix} = \begin{pmatrix} \hat{x}_0 \\ -\hat{y}_0 \end{pmatrix} + \begin{pmatrix} \hat{\omega} \\ \hat{\mu} \end{pmatrix}, \quad \begin{pmatrix} \hat{\omega} \\ \hat{\mu} \end{pmatrix} \neq 0.$$

Since $\binom{\bar{x}}{-\bar{y}} \in T$, we have

$$\begin{pmatrix} \bar{x} \\ -\bar{y} \end{pmatrix} = \begin{pmatrix} X\bar{\lambda} \\ -Y\bar{\lambda} \end{pmatrix} - \begin{pmatrix} \bar{\omega} \\ \bar{\mu} \end{pmatrix},$$

where $\binom{\bar{\omega}}{\bar{\mu}} \in W^*, \delta_1 e^T \bar{\lambda} + \delta_1 \delta_2 (-1)^{\delta_3} \bar{\lambda}_{n+1} = \delta_1, \bar{\lambda}_{n+1} \geq 0, \lambda \in -K^*$.

Thus, we have

$$\begin{aligned} \begin{pmatrix} X\bar{\lambda} \\ -Y\bar{\lambda} \end{pmatrix} &= \begin{pmatrix} \bar{x} \\ -\bar{y} \end{pmatrix} + \begin{pmatrix} \bar{\omega} \\ \bar{\mu} \end{pmatrix} \\ &= \begin{pmatrix} \hat{x}_0 \\ -\hat{y}_0 \end{pmatrix} + \begin{pmatrix} \hat{\omega} + \bar{\omega} \\ \hat{\mu} + \bar{\mu} \end{pmatrix} \\ &= \begin{pmatrix} x_0 \\ -y_0 \end{pmatrix} - \begin{pmatrix} s^{-0} \\ s^{+0} \end{pmatrix} + \begin{pmatrix} \hat{\omega} + \bar{\omega} \\ \hat{\mu} + \bar{\mu} \end{pmatrix}. \end{aligned}$$

Since W is an acute cone, $W^* \cap (-W^*) = \{0\}$ (see Charnes et al. (1989, 1990)). By

$$\begin{pmatrix} \hat{\omega} \\ \hat{\mu} \end{pmatrix} \in W^* \backslash \{0\}, \ \begin{pmatrix} \overline{\omega} \\ \overline{\mu} \end{pmatrix} \in W^*,$$

if

$$\begin{pmatrix} \hat{\omega} + \overline{\omega} \\ \hat{\mu} + \overline{\mu} \end{pmatrix} = 0,$$

then

$$\begin{pmatrix} \hat{\omega} \\ \hat{\mu} \end{pmatrix} = - \begin{pmatrix} \overline{\omega} \\ \overline{\mu} \end{pmatrix} \in - W^*, \ \begin{pmatrix} \hat{\omega} \\ \hat{\mu} \end{pmatrix} \in W^* \cap (- W^*).$$

So $\binom{\hat{\omega}}{\hat{\mu}} = 0$, which contradicts our assumption.

We have

$$\begin{pmatrix} \hat{\omega} + \overline{\omega} \\ \hat{\mu} + \overline{\mu} \end{pmatrix} \in W^* \backslash \{0\}, \tag{7}$$

and hence,

$$\begin{cases} \begin{pmatrix} X\overline{\lambda} \\ -Y\overline{\lambda} \end{pmatrix} + \begin{pmatrix} s^{-0} - \hat{\omega} - \overline{\omega} \\ s^{+0} - \hat{\mu} - \overline{\mu} \end{pmatrix} = \begin{pmatrix} x_0 \\ -y_0 \end{pmatrix}, \\ \delta_1 e^T \overline{\lambda} + \delta_1 \delta_2 (-1)^{\delta_3} \overline{\lambda}_{n+1} = \delta_1, \\ \overline{\lambda} \in -K^*, \overline{\lambda}_{n+1} \geq 0, \\ \begin{pmatrix} s^{-0} - \hat{\omega} - \overline{\omega} \\ s^{+0} - \hat{\mu} - \overline{\mu} \end{pmatrix} \in -W^*. \end{cases}$$

Since $\binom{\tau}{\hat{\tau}} \in \text{Int}\, W$ and (7) holds, we have $\tau^T(\hat{\omega} + \overline{\omega}) + \hat{\tau}^T(\hat{\mu} + \overline{\mu}) < 0$.Then,

$$\tau^T(s^{-0} - \hat{\omega} - \overline{\omega}) + \hat{\tau}^T(s^{+0} - \hat{\mu} - \overline{\mu}) = (\tau^T s^{-0} + \hat{\tau}^T s^{+0}) - [\tau^T(\hat{\omega} + \overline{\omega}) + \hat{\tau}^T(\hat{\mu} + \overline{\mu})]$$
$$> (\tau^T s^{-0} + \hat{\tau}^T s^{+0}).$$

The preceding information contradicts the assumption that λ_j^0, $j = 1, \ldots, n, n+1$; s^{-0}, s^{+0} is an optimal solution to the generalized additive DEA model (P_0), thus, theorem 10 holds. □

5 The generalized DEA model with polyhedral W and K

We consider now the generalized DEA model (P) and (D) where the cones W and K are polyhedra; i.e., are described as

$$W = \{(\beta^T C)^T \mid \beta \geq 0\} \subseteq E_+^{m+s},$$
$$K = \{\alpha^T \Gamma \mid \alpha \geq 0\} \subseteq E_+^n,$$

where

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_{m'+s'} \end{pmatrix}$$

is an $(m' + s') \times (m + s)$ matrix and

$$\Gamma = \begin{pmatrix} r_1 \\ r_2 \\ \cdots \\ r_{n'} \end{pmatrix}$$

is an $n' \times n$ matrix.

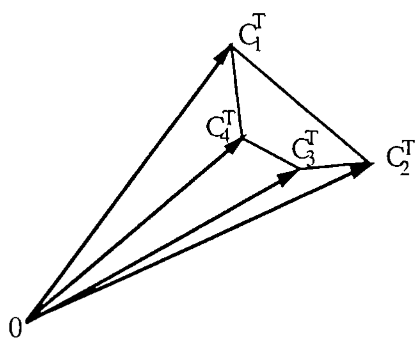

Figure 1.

In figure 1, W is a polyhedron formed by vectors $c_1^T, c_2^T, c_3^T, c_4^T$. Although the polyhedron defined previously is generated by finitely many vectors, polyhedra may also be represented by the intersection of finitely many half spaces. The two representations have the following relations.

Lemma 4
If $K = \{\alpha^T\Gamma \mid \alpha \geq 0\}$, then $K = \{k \mid k\Gamma^+ \geq 0\}$, where Γ^+ is the generalized inverse of Γ (see Rao and Mitra (1971) for a definition of the Moore–Penrose generalized inverse).

Proof
Assume $k \in \{\alpha^T\Gamma \mid \alpha \geq 0\}$, which means there exists $\alpha^0 \geq 0$ with $k = \alpha^{0^T}\Gamma$. Thus, $\alpha^{0^T} = k\Gamma^+ \geq 0$; i.e. $k \in \{k \mid k\Gamma^+ \geq 0\}$.

Conversely, assume $k \in \{k \mid k\Gamma^+ \geq 0\}$ and let $\alpha^T = k\Gamma^+$. We then have $\alpha \geq 0$ and $k = \alpha^T\Gamma^{++} = \alpha^T\Gamma$, i.e., $k \in \{\alpha^T\Gamma \mid \alpha \geq 0\}$. □

For some special cases, the generalized inverse has an explicit expression, as in the following examples:

(a) If $\Gamma = \Gamma_{n' \times n}$, $\operatorname{rank}(\Gamma) = n'$, then

$$\Gamma^+ = \Gamma^T (\Gamma \Gamma^T)^{-1}.$$

(b) If $\Gamma = \Gamma_{n' \times n}$, $\operatorname{rank}(\Gamma) = n$, then

$$\Gamma^+ = (\Gamma^T \Gamma)^{-1} \Gamma^T.$$

(c) If $\Gamma = \Gamma_{n \times n}$, $\operatorname{rank}(\Gamma) = n$, then

$$\Gamma^+ = \Gamma^{-1}.$$

By lemma 4, we have

$$K = \{k \mid k\Gamma^+ \geq 0\},$$
$$-K^* = \{\Gamma^+ k' \mid k' \geq 0\},$$
$$W^* = \{\beta' \mid C\beta' \leq 0\}.$$

In this case, (P) and (D) can be rewritten as

$$(\mathrm{P}') \quad \begin{cases} \text{maximize} & (\mu^T y_0 - \delta_1 \mu_0) \\ \text{subject to} & \omega^T X \Gamma^+ - \mu^T Y \Gamma^+ + \mu_0 \delta_1 e^T \Gamma^+ \geq 0, \\ & \omega^T x_0 = 1, \\ & (\omega^T, \mu^T) = (\omega'^T, \mu'^T) C, \\ & \omega' \geq 0,\ \mu' \geq 0,\ \delta_1 \delta_2 (-1)^{\delta_3} \mu_0 \geq 0, \end{cases}$$

and its dual:

$$(\mathrm{D}') \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & C \begin{pmatrix} X \\ -Y \end{pmatrix} \Gamma^+ \lambda' - C \begin{pmatrix} \theta x_0 \\ -y_0 \end{pmatrix} \leq 0, \\ & \delta_1 e^T \Gamma^+ \lambda' + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \\ & \lambda' \geq 0,\ \lambda_{n+1} \geq 0,\ \theta \in E^1, \end{cases}$$

where

$$\begin{pmatrix} \omega' \\ \mu' \end{pmatrix} \in E^{m'+s'}, \mu_0 \in E^1,\ \lambda' \in E^{n'},\ \lambda_{n+1} \in E^1.$$

The production possibility set is

$$T = \left\{ (x, y) \mid C \begin{pmatrix} X \\ -Y \end{pmatrix} \Gamma^+ \lambda' + C \begin{pmatrix} x \\ -y \end{pmatrix} \leq 0, \delta_1 e^T \Gamma^+ \lambda' + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \right.$$
$$\left. \lambda' \geq 0, \lambda_{n+1} \geq 0 \right\}.$$

Clearly, (P′) and (D′) are linear programs and T is a polyhedron. If we choose $W = V \times U$, $V \subseteq E_+^m$, $U \subseteq E_+^s$, and

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

$$V = \{(\omega'^T A)^T \mid \omega' \geq 0\},$$

$$U = \{(\mu'^T B)^T \mid \mu' \geq 0\}.$$

The corresponding (P′) and (D′) are

$$(\mathrm{P}'') \quad \begin{cases} \text{maximize} & (\mu'^T(By_0) - \delta_1\mu_0) \\ \text{subject to} & \omega'^T(AX)\Gamma^+ - \mu'^T(BY)\Gamma^+ + \mu_0\delta_1 e^T\Gamma^+ \geq 0, \\ & \omega'^T(Ax_0) = 1, \\ & \omega' \geq 0,\ \mu' \geq 0,\ \delta_1\delta_2(-1)^{\delta_3}\mu_0 \geq 0, \end{cases}$$

and

$$(\mathrm{D}'') \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & (AX)\Gamma^+\lambda' \leq \theta(Ax_0), \\ & (BY)\Gamma^+\lambda' \geq (By_0), \\ & \delta_1 e^T\Gamma^+\lambda' + \delta_1\delta_2(-1)^{\delta_3}\lambda_{n+1} = \delta_1, \\ & \lambda' \geq 0,\ \lambda_{n+1} \geq 0,\ \theta \in E^1. \end{cases}$$

From the preceding analysis, we can see that if $K = E_+^n$, then $\Gamma = I^{(n)} = \Gamma^+$. Now the corresponding (P″) and (D″) are

$$(\mathrm{P}_0'') \quad \begin{cases} \text{maximize} & (\mu'^T(By_0) - \delta_1\mu_0) \\ \text{subject to} & \omega'^T(AX) - \mu'^T(BY) + \mu_0\delta_1 e^T \geq 0, \\ & \omega'^T(Ax_0) = 1, \\ & \omega' \geq 0,\ \mu' \geq 0,\ \delta_1\delta_2(-1)^{\delta_3}\mu_0 \geq 0, \end{cases}$$

and

$$(\mathrm{D}_0'') \quad \begin{cases} \text{minimize} & \theta \\ \text{subject to} & (AX)\lambda' \leq \theta(Ax_0), \\ & (BY)\lambda' \geq (By_0), \\ & \delta_1 e^T\lambda' + \delta_1\delta_2(-1)^{\delta_3}\lambda_{n+1} = \delta_1, \\ & \lambda' \geq 0,\ \lambda_{n+1} \geq 0,\ \theta \in E^1. \end{cases}$$

We see that when polyhedra V and U are used, it is a simple transformation of the original input/output data:

$$\begin{array}{cc|ccccc|cc}
 & & 1 & 2 & \cdots & j & \cdots & n & \\
1 & \rightarrow & & & & & & & \\
\vdots & \vdots & x_1 & x_2 & \cdots & x_j & \cdots & x_n & \\
m & \rightarrow & & & & & & & \\
\hline
 & & & & & & & & \rightarrow\ 1 \\
 & & y_1 & y_2 & \cdots & y_j & \cdots & y_n & \vdots\ \ \vdots \\
 & & & & & & & & \rightarrow\ s
\end{array}$$

to the following modified data:

$$\begin{array}{cc|ccccc|cc}
 & & 1 & 2 & \cdots & j & \cdots & n & \\
1 & \rightarrow & & & & & & & \\
\vdots & \vdots & Ax_1 & Ax_2 & \cdots & Ax_j & \cdots & Ax_n & \\
m' & \rightarrow & & & & & & & \\
\hline
 & & & & & & & & \rightarrow\ 1 \\
 & & By_1 & By_2 & \cdots & By_j & \cdots & By_n & \vdots\ \ \vdots \\
 & & & & & & & & \rightarrow\ s'
\end{array}$$

and (P″) and (D″) correspond to (P_0'') and (D_0'') with the addition of the predilection cone – polyhedron K.

6 An illustrative example

We now provide an example to illustrate the functions and implications of the input/output preference cone W and the predilection cone K with various settings of parameters $(\delta_1, \delta_2, \delta_3)$. By theorem 8, a necessary and sufficient condition for DMU_{j_0} to be DEA efficient is that (x_0, y_0) is a nondominated solution of the multi-objective program (VP) associated with W^*. Accordingly, define $\overline{W}^* = \{(x, y) | (x, -y) \in W^*\}$. Then we have

$$T = \left\{(x, y) \middle| \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X\lambda \\ Y\lambda \end{pmatrix} + (-\overline{W}^*), \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \right.$$
$$\left. \lambda \in -K^*, \lambda_{n+1} \geq 0 \right\}.$$

Thus, the necessary and sufficient condition for (x_0, y_0) to be a nondominated solution of the multi-objective program (VP) associated with W^* is that (x_0, y_0) is a nondominated solution of ($\overline{\mathrm{VP}}$) associated with $\overline{W}^*$ as defined below:

$$(\overline{\mathrm{VP}}) \qquad \begin{cases} V - \{(\min x, \max y)\} \\ (x, y) \in T. \end{cases}$$

Example 1

We now provide an example with 6 DMUs, one input and one output as follows:

	1	2	3	4	5	6	
$m = 1 \rightarrow$	5	6	10	13	17	25	
	1	5	10	13	16	18	$\rightarrow s = 1$

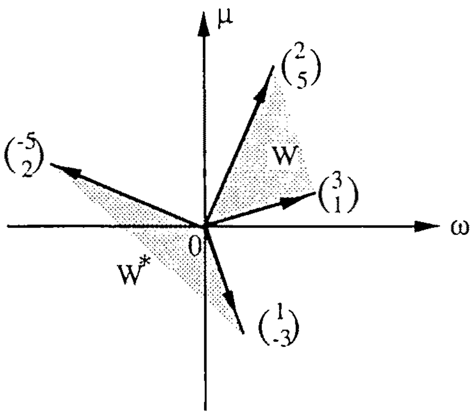

Figure 2

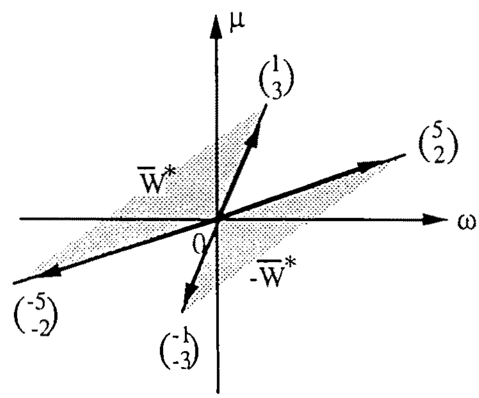

Figure 3.

As shown in figures 2 and 3, we have

$$W_1 = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \beta_1 + \begin{pmatrix} 2 \\ 5 \end{pmatrix} \beta_2 \mid \beta_1 \geq 0, \beta_2 \geq 0 \right\},$$

$$W_1^* = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \beta_1 + \begin{pmatrix} -5 \\ 2 \end{pmatrix} \beta_2 \mid \beta_1 \geq 0, \beta_2 \geq 0 \right\},$$

$$\overline{W}^* = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \beta_1 + \begin{pmatrix} -5 \\ -2 \end{pmatrix} \beta_2 \mid \beta_1 \geq 0, \beta_2 \geq 0 \right\},$$

$$-\overline{W}^* = \left\{ \begin{pmatrix} -1 \\ -3 \end{pmatrix} \beta_1 + \begin{pmatrix} 5 \\ 2 \end{pmatrix} \beta_2 \mid \beta_1 \geq 0, \beta_2 \geq 0 \right\}.$$

Choose $K_1 = \{\alpha \Gamma_5 \mid \alpha \geq 0\}$, where

$$\Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

then,

$$-K_1^* = \{(k_1, k_2, \dots, k_6)^T | k_j \ge 0, j = 1, 2, 3, 4, 6, \sum_{j=1}^{6} k_j \ge 0\},$$

$$T_1 = \{(x, y) | \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{6} x_j \lambda_j \\ \sum_{j=1}^{6} y_j \lambda_j \end{pmatrix} + (-\overline{W}^*), \delta_1 \sum_{j=1}^{6} \lambda_j + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_7 = \delta_1,$$

$$\sum_{j=1}^{6} \lambda_j \ge 0,\ \lambda_j \ge 0,\ 1 \le j \le 7,\ j \ne 5\}.$$

Note that λ_5 does not have a nonnegativity restriction in the production possibility set T_1, which is shown in figure 4. Clearly, K_1 favors DMU_5, since in the production possibility set T_1, the efficient frontiers L_4 and L_5 remain the same with or without K_1 (for rigorous proof, see Wei and Yu (1993)).

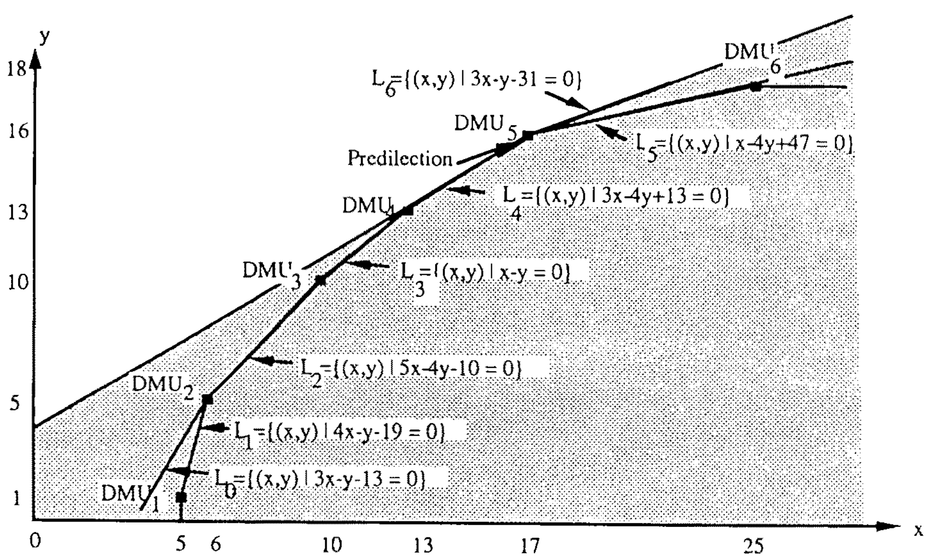

Figure 4.

Similarly, choose

$$K_2 = \{\alpha \Gamma_2 | \alpha \ge 0\},$$

$$K_3 = \{\alpha \Gamma_4 \Gamma_5 | \alpha \ge 0\},$$

where

$$\Gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\Gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For this example, K_2 is the predilection cone for DMU_2 and K_3 is the predilection cone for both DMU_4 and DMU_5. Similar to K_1^*, we have

$$-K_2^* = \{(k_1, \dots, k_6)^T \mid k_j \geq 0, j \neq 2, \sum_{j=1}^{6} k_j \geq 0\}.$$

Since

$$\Gamma_4\Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{3}{2} & \frac{1}{2} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we have

$$-K_3^* = \{(k_1, \dots, k_6)^T \mid k_j \geq 0, j \neq 4, 5, \sum_{j=1}^{6} k_j \geq 0, k_1 + k_2 + k_3 + \tfrac{3}{2} k_4 + \tfrac{1}{2} k_5 + k_6 \geq 0\}.$$

By definition of production possibility set T_3, notice that there are no nonnegativity requirements on k_3 and k_4, and the efficient frontier where DMU_4 and DMU_5 lie will not change (see figure 9).

Overall, we therefore have the following situations:

(1) The generalized (BCC) model ($\delta_1 = 1$, $\delta_2 = 0$).

(1a) Choose $W = E_+^2$, $K = E_+^6$. As figure 5 illustrates, the efficient frontiers are L_1, L_2, L_3, L_4 and L_5. $\overline{W}^* = \overline{E}_-^2 = \{(x, y) \mid x \leq 0, y \geq 0\}$ is a nondominated cone and T is the production possibility set. Since in this case there is no predilection, all input and output categories are equally important. To examine the efficiency of a DMU, we need to place W with its origin coincident with the DMU. Only if W and T intersect at a single point where a DMU resides is the DMU a nondominated solution and DEA efficient. Thus, in this example, DMUs 1, 2, 3, 4, 5 and 6 are all DEA efficient.

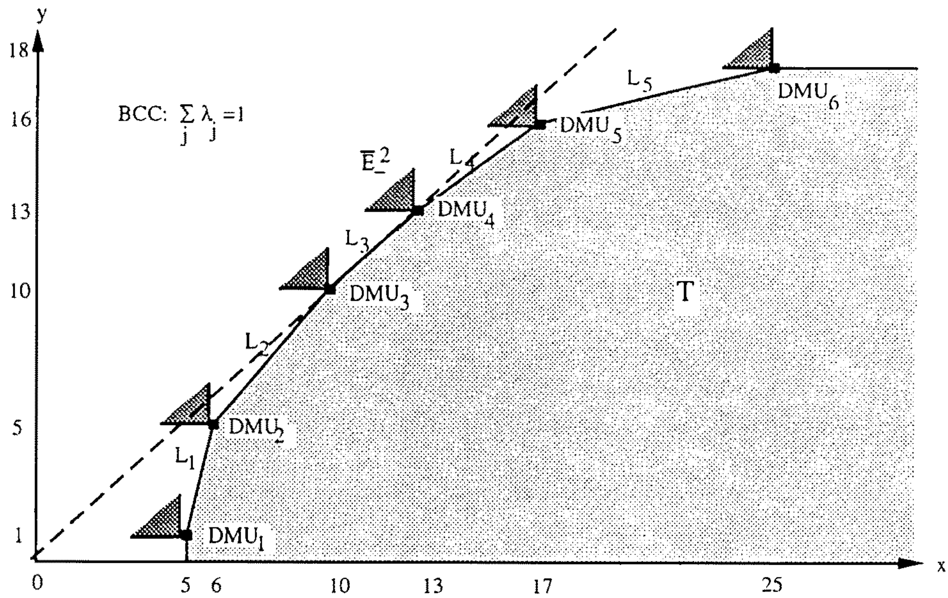

Figure 5. $W = E_+^2$, $K = E_+^6$.

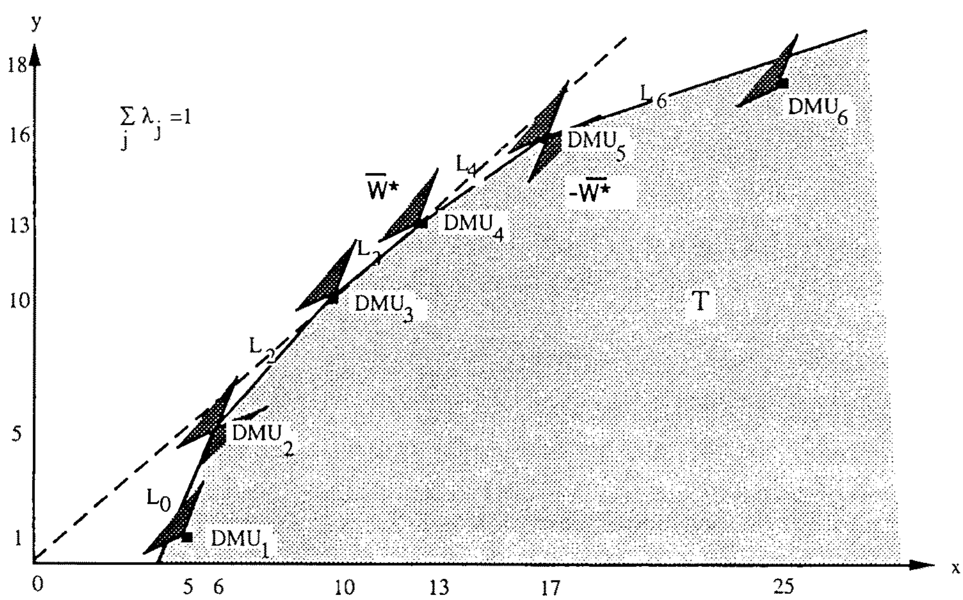

Figure 6. $W = W_1$, $K = E_+^6$.

(1b) Choose $W = W_1$, $K = E_+^6$. See figure 6 for illustration. Due to the existence of predilection, the production possibility set T in figure 5 is extended by $-\overline{W}^*$. Hence, $\overline{W}^*$ is the nondominated cone. Since DMUs 2, 3, 4 and 5 are the unique intersecting points of $\overline{W}^*$ and T at places where the corresponding DMUs reside, they are nondominated Pareto solutions and thus are DEA efficient. The efficient frontiers are L_0, L_2, L_3, L_4 and L_6.

(1c) Choose $W = E_+^2$, $K = K_1$, where K_1 favors DMU_5. As shown in figure 7, DMUs 4, 5 and 6 are DEA efficient and the efficient frontiers are L_4 and L_5. Since K_1 is the predilection cone favoring DMU_5 with no restriction to λ_5 in the definition of T_1, the production frontier extends from DMU_5 along the direction of DMU_6 (i.e., L_5) and along the direction of DMU_4 (i.e., L_4). Figure 7 highlights the fact that T_1 is the expansion of T of figure 5 by showing the expanded domain in light shade.

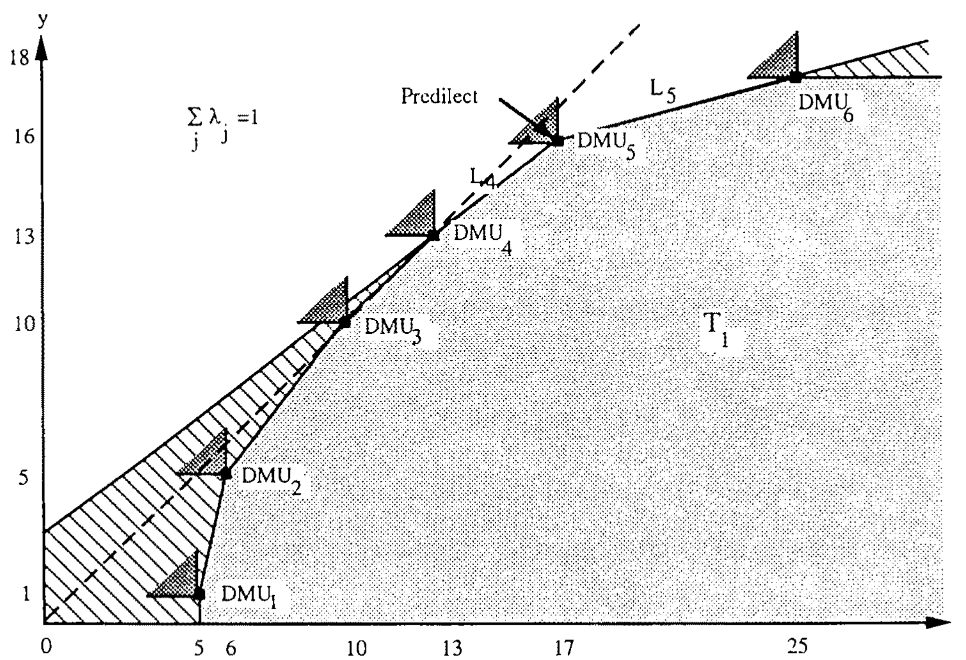

Figure 7. $W = E_+^2$, $K = K_1$.

(1d) Choose $W = W_1$, $K = K_1$. As figure 8 demonstrates, the efficient frontiers are L_4 and L_6, and DMUs 4 and 5 are DEA efficient. The lightly shaded part of the figure is due to the presence of K_1.

(1e) Choose $W = E_+^2$, $K = K_3$, where K_3 favors both DMU_4 and DMU_5. As shown in figure 9, T_3 is the enlarged production possibility set from T of figure 6, DMUs 4 and 5 are DEA efficient and the only efficient frontier is L_4. The lightly shaded part in the figure is due to the presence of K_3.

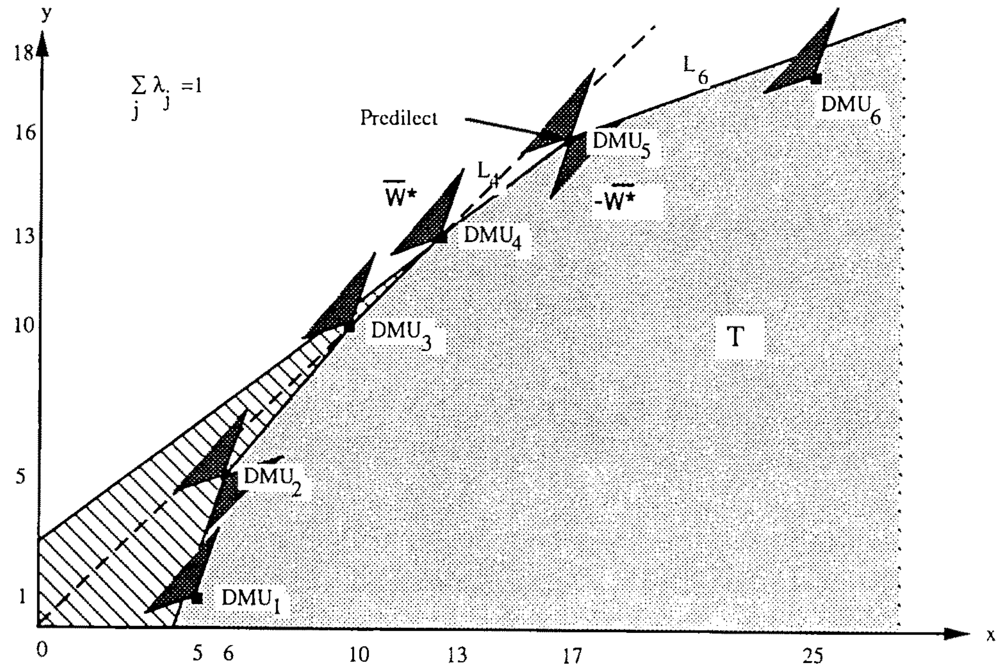

Figure 8. $W = W_1$, $K = K_1$.

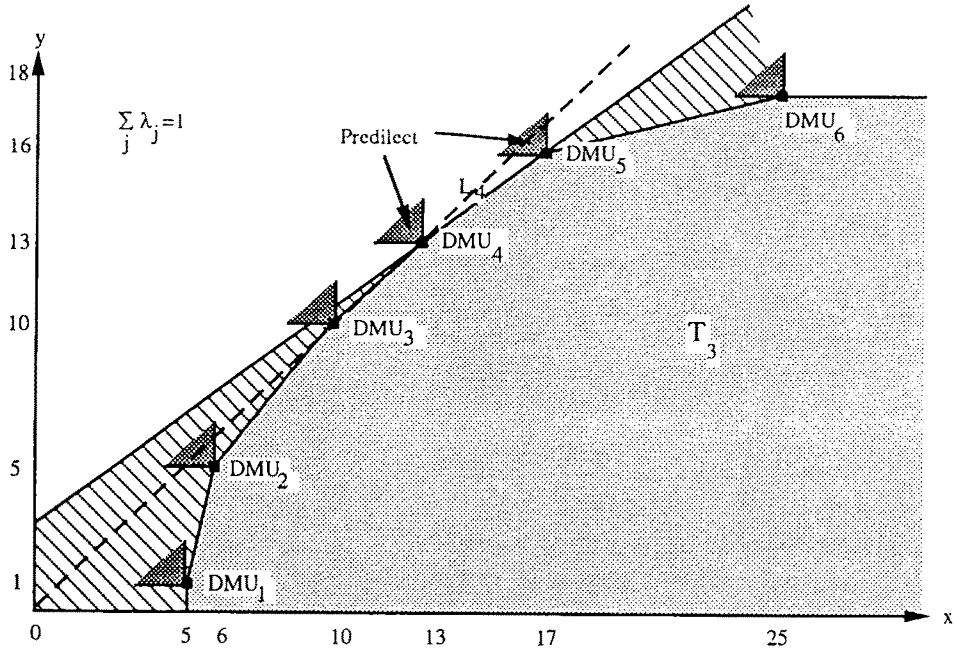

Figure 9. $W = E_+^2$, $K = K_3$.

Since the explanations for the rest of the figures are similar to the above, we omit the details.

(2) The generalized (FG) model ($\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 0$).

(2a) Choose $W = E_+^2$, $K = E_+^6$. As illustrated in figure 10, DMUs 3, 4, 5 and 6 are DEA efficient, and the efficient frontiers are L_3, L_4 and L_5.

(2b) Choose $W = W_1$, $K = E_+^6$. As shown in figure 11, DMUs 3, 4 and 5 are DEA efficient, and the efficient frontiers are L_3, L_4 and L_6.

(2c) Choose $W = E_+^2$, $K = K_1$. As figure 12 demonstrates, DMUs 4, 5 and 6 are DEA efficient, and the efficient frontiers are L_4 and L_5.

(2d) Choose $W = W_1$, $K = K_1$. As shown in figure 13, DMUs 4 and 5 are DEA efficient, and the efficient frontiers are L_4 and L_6.

(3) The generalized (ST) model ($\delta_1 = 1$, $\delta_2 = 1$, $\delta_3 = 1$).

(3a) Choose $W = E_+^2$, $K = E_+^6$. As figure 14 illustrates, DMUs 1, 2, 3 and 4 are DEA efficient, and the efficient frontiers are L_1, L_2 and L_3.

(3b) Choose $W = W_1$, $K = E_+^6$. As shown in figure 15, DMUs 2, 3 and 4 are DEA efficient, and the efficient frontiers are L_0, L_2 and L_3.

(3c) Choose $W = E_+^2$, $K = K_2$. As figure 16 demonstrates, DMUs 1, 2 and 3 are DEA efficient, and the efficient frontiers are L_1 and L_2.

(3d) Choose $W = W_1$, $K = K_2$. As shown in figure 17, DMUs 2 and 3 are DEA efficient, and the efficient frontiers are L_0 and L_2.

Example 2
To illustrate the generalized (BCC) model, consider the following case with two inputs and one output:

	1	2	3	4	5	6	
1 →	2	3	5	6	9	14	
2 →	14	9	6	5	3	2	
	1	1	1	1	1	1	→ 1

Choose $W = V_1 \times U_1$, where

$$V_1 = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \alpha_1 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \alpha_2 \middle| \alpha_1 \geq 0, \alpha_2 \geq 0 \right\}, \quad U = E_+^1.$$

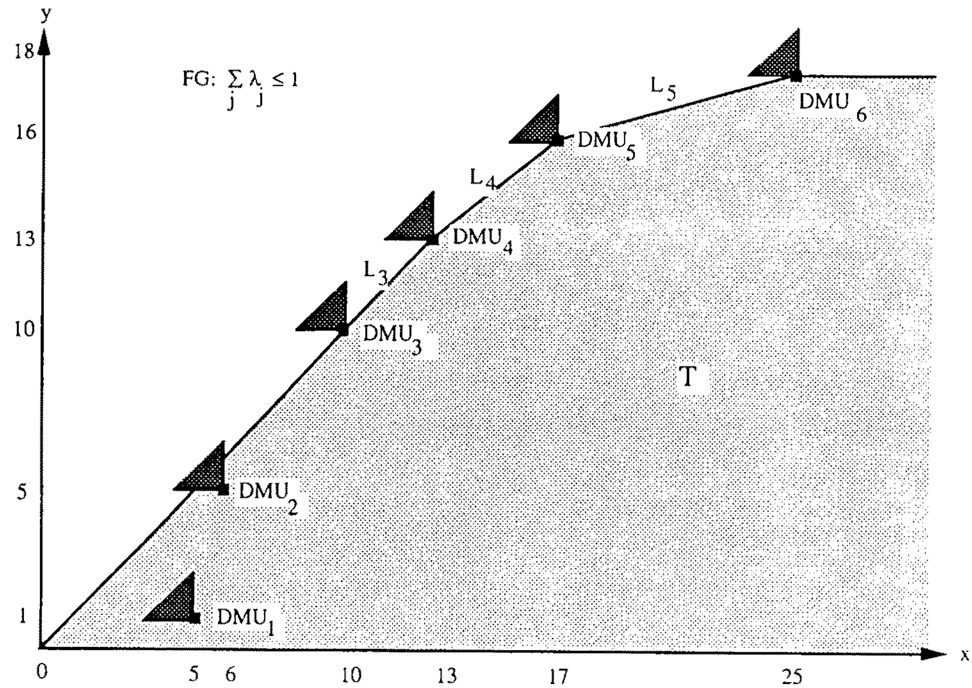

Figure 10. $W = E_+^2$, $K = E_+^6$.

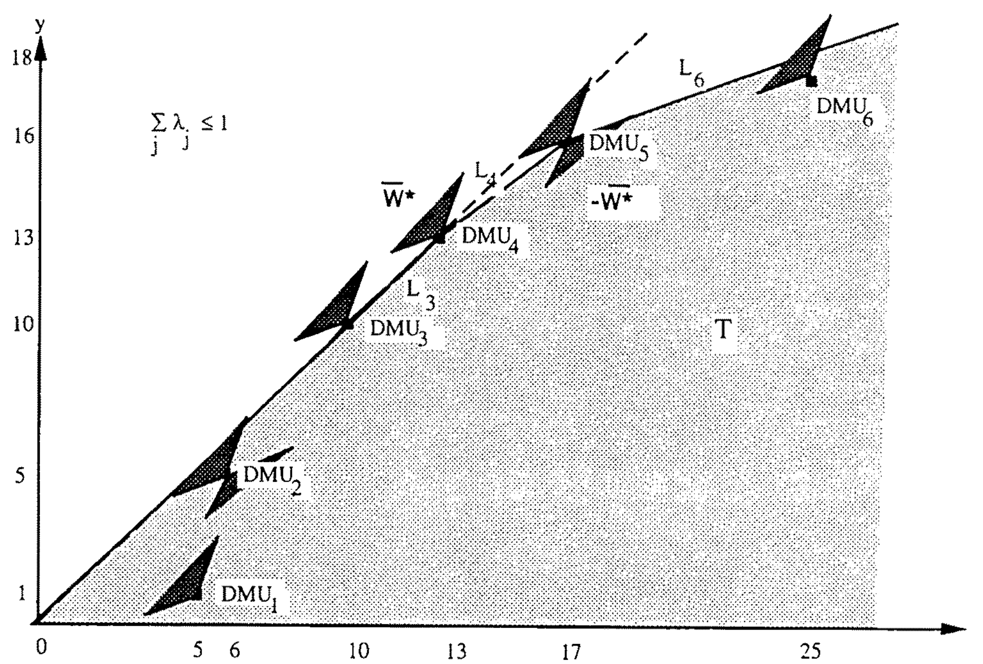

Figure 11. $W = W_1$, $K = E_+^6$.

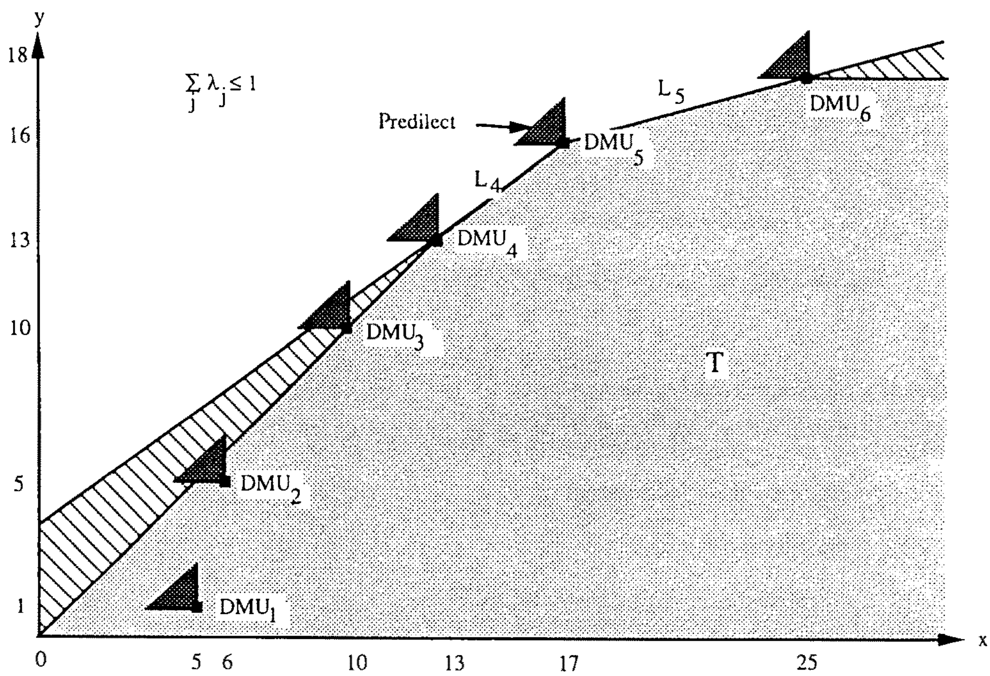

Figure 12. $W = E_+^2$, $K = K_1$.

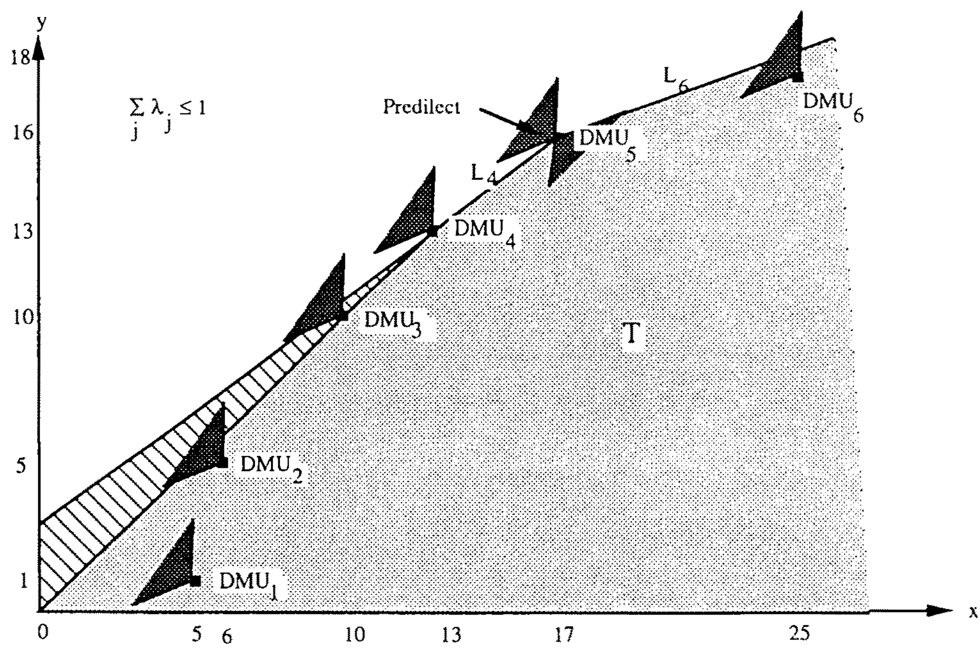

Figure 13. $W = W_1$, $K = K_1$.

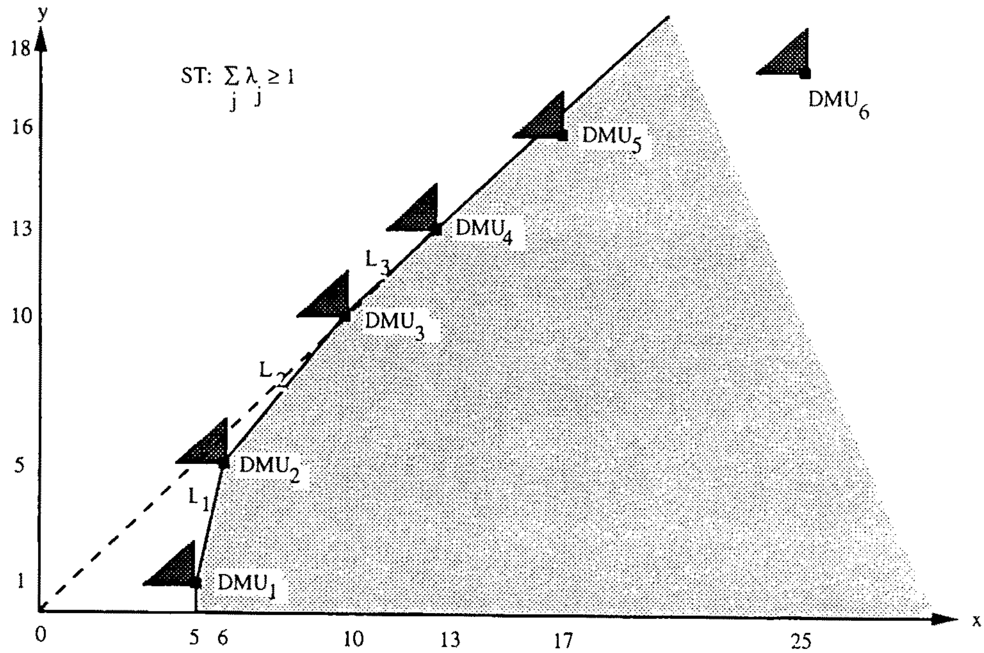

Figure 14. $W = E_+^2$, $K = E_+^6$.

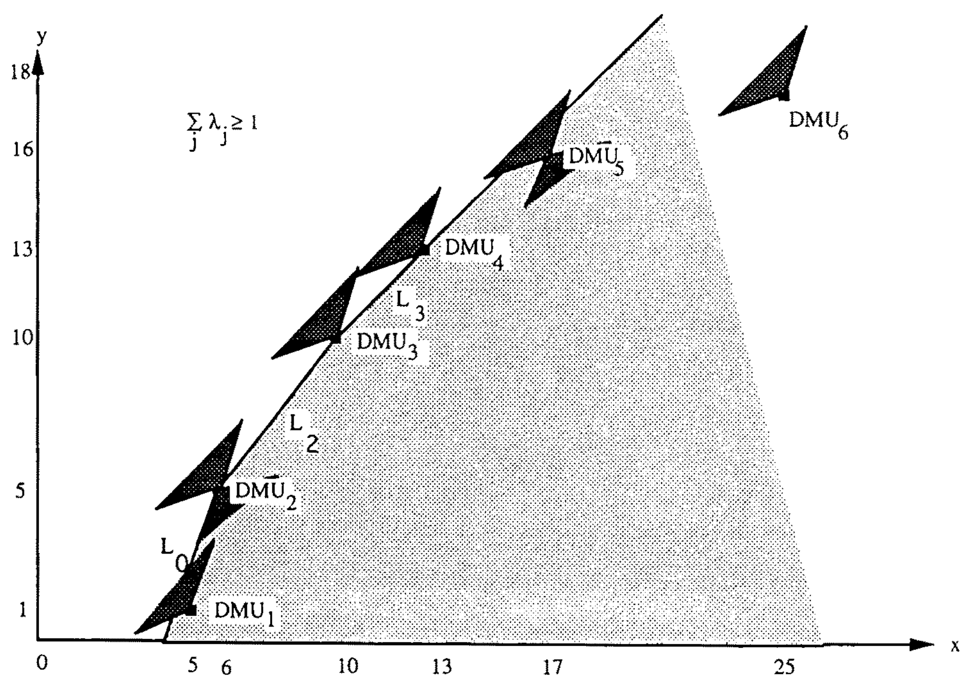

Figure 15. $W = W_1$, $K = E_+^6$.

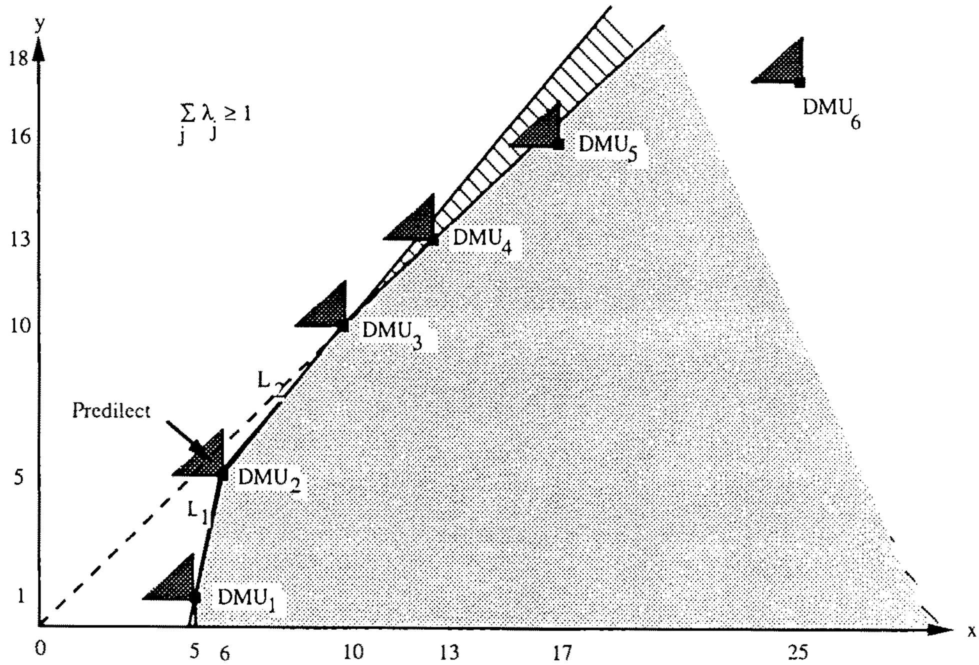

Figure 16. $W = E_+^2$, $K = K_2$.

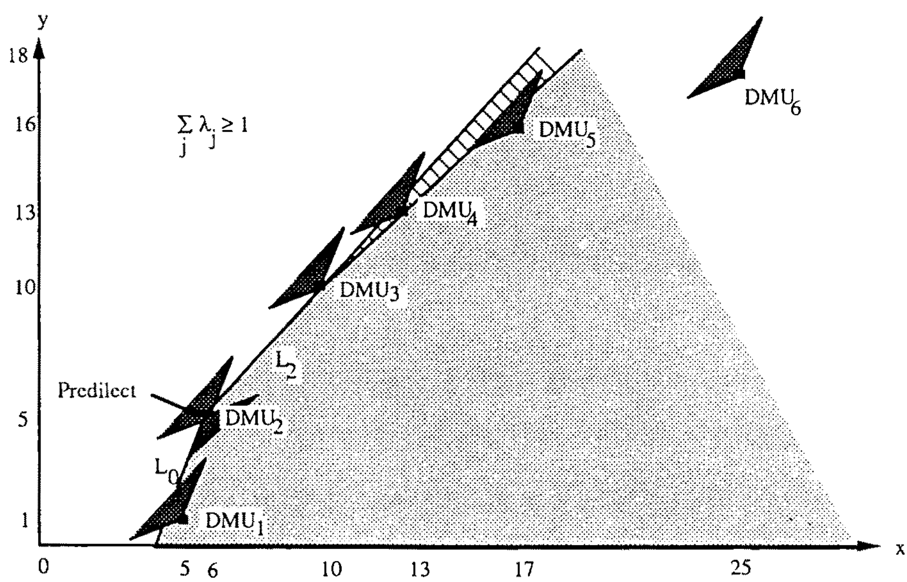

Figure 17. $W = W_1$, $K = K_2$.

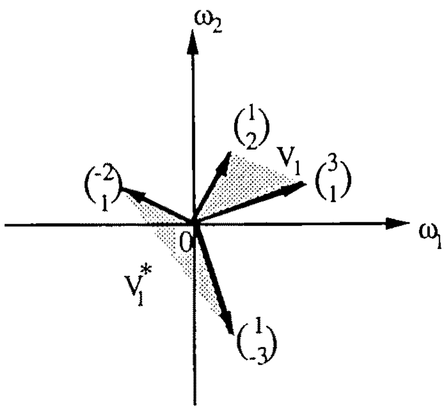

Figure 18.

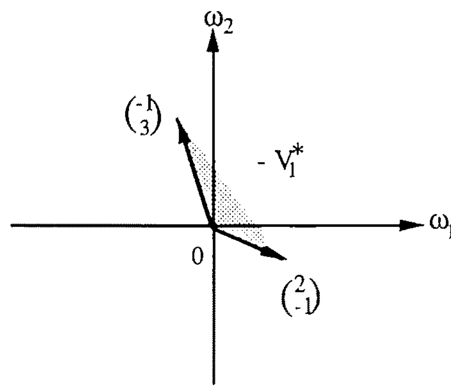

Figure 19.

Figures 18 and 19 illustrate this information. Then we have $W^* = V_1^* \times U_1^*$, where

$$V_1^* = \left\{ \begin{pmatrix} 1 \\ -3 \end{pmatrix} \alpha_1 + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \alpha_2 \,|\, \alpha_1 \geq 0, \alpha_2 \geq 0 \right\}, \qquad U^* = E_-^1.$$

Now select $K_1 = \{\alpha\Gamma_5 | \alpha \geq 0\}$ and $K_3 = \{\alpha\Gamma_4\Gamma_5 | \alpha \geq 0\}$, where Γ_4 and Γ_5 are the same as in example 1. For our current example, since all DMUs have the same output, DEA efficiency is determined only by the input data. Accordingly, we need only consider

$$T|_{y=1} = \{x | T \cap \{(x, y) | y = 1\}\}$$
$$= \{x | x = \sum_{j=1}^{n} x_j\lambda_j - V^*, \delta_1 e^T\lambda + \delta_1\delta_2(-1)^{\delta_3}\lambda_{n+1} = \delta_1, \lambda_{n+1} \geq 0, \lambda \in -K^*\}.$$

When all the DMUs have the same output, the various ((CCR), (BCC), (FG) and (ST)) notions of DEA efficiency are the same as the (CCR)-DEA efficiency. Hence, we need only discuss the case with $\delta_1 = 0$. Now,

$$T|_{y=1} = \{x | x = \sum_{j=1}^{n} x_j\lambda_j - V^*, \lambda \in -K^*\}.$$

The following alternatives exhaust the possible situations:

(a) Choose $V = E_+^2$, $K = E_+^6$. As figure 20 illustrates, DMUs 1, 2, 3, 4, 5 and 6 are all DEA efficient, and the efficient frontiers are L_1, L_2, L_3, L_4 and L_5.

(b) Choose $V = V_1$, $K = E_+^6$. As shown in figure 21, DMUs 2, 3, 4 and 5 are DEA efficient, and the efficient frontiers are L_0, L_2, L_3, L_4 and L_6.

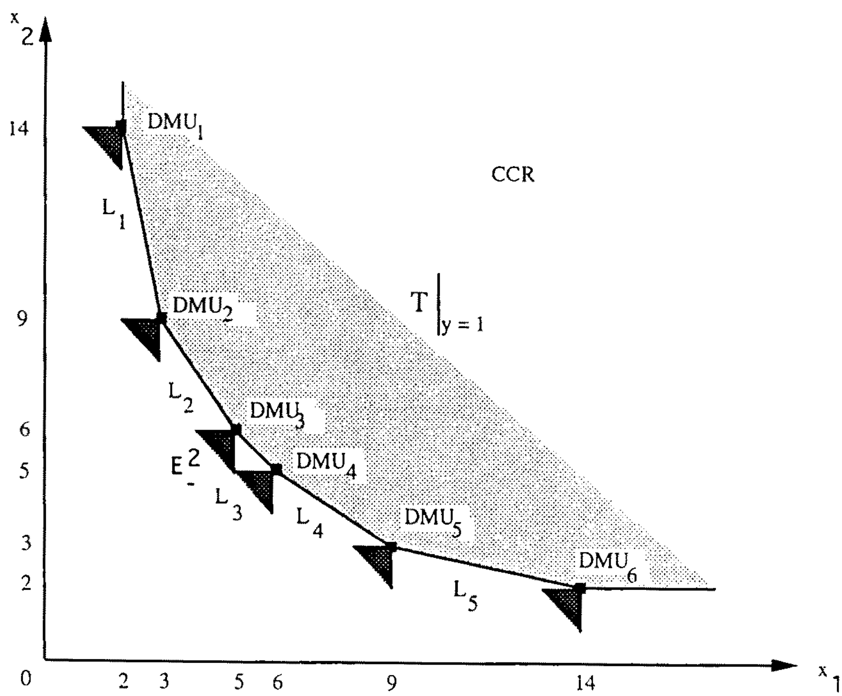

Figure 20. $V = E_+^2$, $K = E_+^6$.

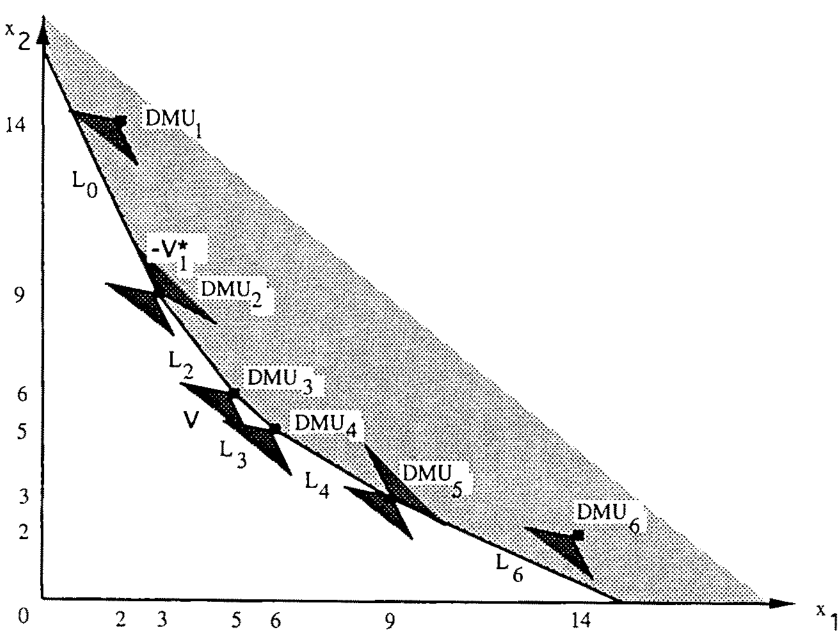

Figure 21. $V = V_1$, $K = E_+^6$.

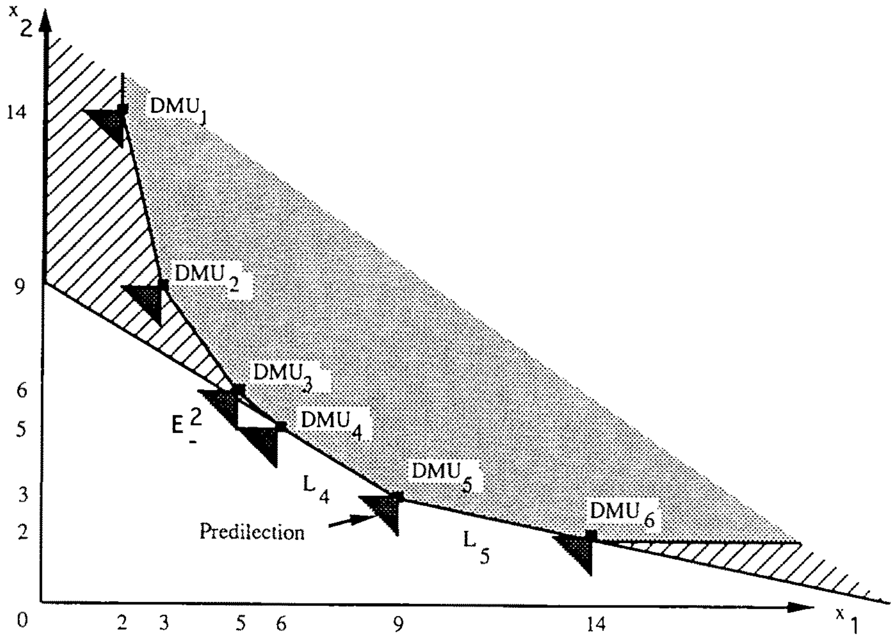

Figure 22. $V = E^2_+$, $K = K_1$.

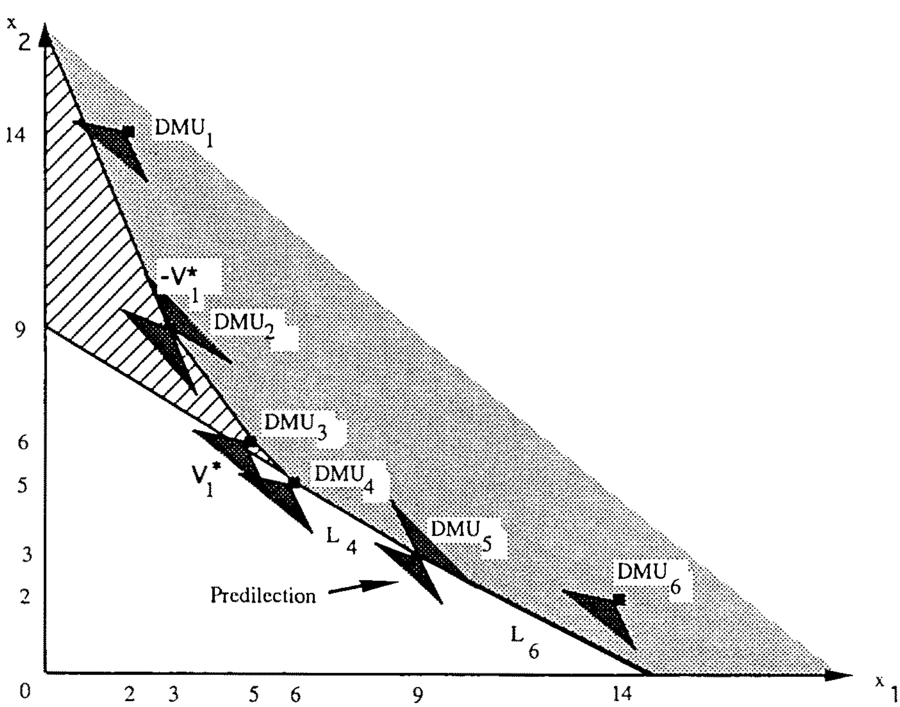

Figure 23. $V = V_1$, $K = K_1$.

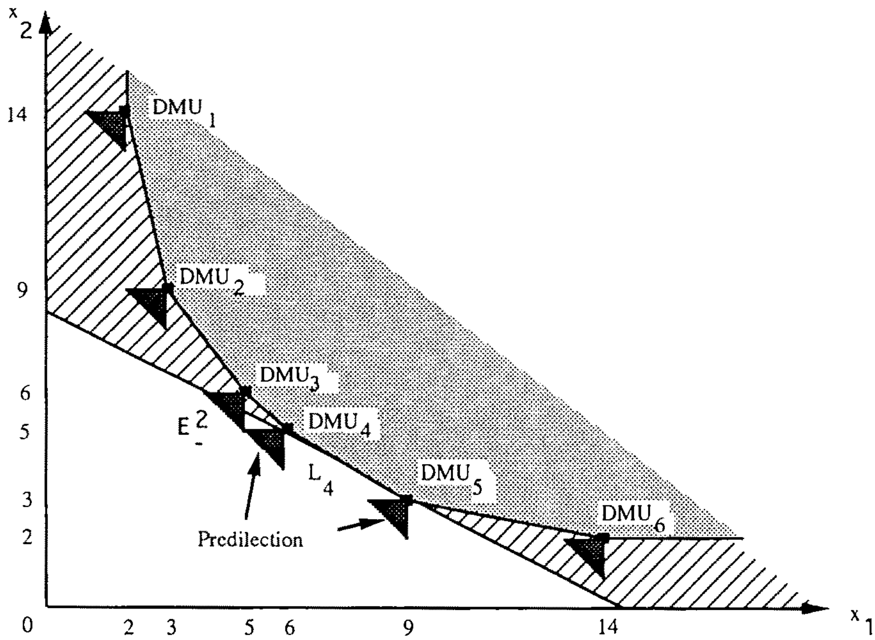

Figure 24. $V = E^2_+$, $K = K_3$.

(c) Choose $V = E^2_+$, $K = K_1$. As demonstrated in figure 22, DMUs 4, 5 and 6 are DEA efficient, and the efficient frontiers are L_4 and L_5.

(d) Choose $V = V_1$, $K = K_1$. As shown in figure 23, DMUs 4 and 5 are DEA efficient, and the efficient frontiers are L_4 and L_6.

(e) Choose $V = E_+{}^2$, $K = K_3$. As illustrated in figure 24, DMUs 4 and 5 are DEA efficient, and the only efficient frontier is L_4.

7 Summary

In this paper, we have introduced a new generalized DEA model which unifies and extends the well-known DEA models which have been developed over the past fifteen years. By setting three binary parameters (δ_1, δ_2, δ_3) to different values, this model reduces to subclasses of DEA models which have general K cone and W cone descriptions and may represent the evaluator's preferences for the DMUs and the input/output categories. We have shown relationships among several well-known subclasses of the generalized DEA model, focusing especially on efficiency definitions and notions of solutions. We have also stated and rigorously proved the equivalence between the notion of DEA efficiency and the notion of nondominated solutions of multi-objective (vector extremal) programs, which will aid the understanding and interpretation of the concept of DEA efficiency. We also provided detailed examples to demonstrate the functions of K cone and W cone, as well as their characteristics.

Finally, we remark that the model discussed in this paper can be called the generalized *input-oriented* model. For the following generalized *output-oriented* model,

$$(\bar{\mathrm{P}}) \quad \begin{cases} \text{minimize} & (\mu^T x_0 + \delta_1 \mu_0) \\ \text{subject to} & \omega^T X - \mu^T Y + \mu_0 \delta_1 e^T \in K, \\ & \mu^T y_0 = 1, \\ & \begin{pmatrix} \omega \\ \mu \end{pmatrix} \in W, \ \delta_1 \delta_2 (-1)^{\delta_3} \mu_0 \geq 0, \end{cases}$$

$$(\bar{\mathrm{D}}) \quad \begin{cases} \text{maximize} & z \\ \text{subject to} & \begin{pmatrix} X\lambda - x_0 \\ -Y\lambda + z y_0 \end{pmatrix} \in W^*, \\ & \delta_1 e^T \lambda + \delta_1 \delta_2 (-1)^{\delta_3} \lambda_{n+1} = \delta_1, \\ & \lambda \in -K^*, \ \lambda_{n+1} \geq 0, \end{cases}$$

the results are very similar and can be derived easily by using the methods described here.

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