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Possible Worlds and Many Truth Values*

Some interesting relationships among propositions (or, if you prefer, among statements or sentences) are clearly not truth-functional — for example, the implication relation. And, at least according to some views, some of these relationships make sense only when propositions are understood as capable of being neither true nor false — for example, the presupposition relation, as 'The King of France is wise' presupposes 'There is a King of France'. It is convenient for formal analysis to regard "implies" and "presupposes" not as relations but as connectives; thus 'P implies Q' is the proposition which is true when P does imply Q and false when P does not imply Q. One interesting formal representation of implication is provided by the possible-worlds semantics of modal logic, in which 'P implies Q' is represented by the formula $\Box(p \Rightarrow q)$ An analogous approach to presupposition would be based on a many-valued logic: 'P presupposes Q', or 'if P is either true or false, then Q is true', would be represented by $\Box\big((\tau p \vee \varphi p) \Rightarrow \tau q\big)$. Of course, the problem of presupposition is only one of several reasons for considering many-valued modal logic; Morgan [1] mentions others.

Several authors have proved completeness and decidability theorems for particular many-valued modal systems analogous to familiar two-valued modal systems. Schotch et al. [2] consider two three-valued analogues of $\boldsymbol{K}$, which differ in the interpretation given to $\Box$. Morgan studies a class of many-valued analogues of $\boldsymbol{T}$. Segerberg [3] examines three-valued analogues of $\boldsymbol{K}$, $\boldsymbol{T}$, $\mathbf{S4}$, $\boldsymbol{B}$, and $\mathbf{S5}$ having two $\Box$-operators, one stronger than the other. The general flavour of all this work is: given a two-valued modal system S which is canonical (the Lemmon-Scott canonical model for the system is based on a frame for the system, so in particular the system is complete), and given a many-valued truth-functional logic $\mathscr{M}_0$ of sufficient expressive power (enough connectives are definable in it), and given a reasonable way Ψ of evaluating $\Box p$ at a possible world in terms of the values of p at alternative possible worlds, one can find axioms and rules of inference for a system $S^{\mathscr{M}}$ analogous to S, but based on $\mathscr{M}_0$ and Ψ, and prove completeness of $S^{\mathscr{M}}$ by a canonical-model construction. Naturally, one is led to inquire whether a general theorem to this effect can be proved. Morgan gives precise meanings to the terms "sufficient", "reasonable", and "analogous" as used above, but (the last paragraph of [1] notwithstanding) he does not prove a general

* This work was supported in part by the National Research Council of Canada.

theorem of the sort I have in mind — he does not suggest a general proof that if S is canonical then so is $S^{\mathcal{M}}$.

But the main problem, I think, is not to construct many-valued analogues of two-valued systems; it is rather to understand many-valued systems generally. In §1 I shall show that any formula of any many-valued modal logic is semantically equivalent (in the sense of being valid on the same frames) to a formula of two-valued modal logic; the latter formula is called a $\mathcal{K}$-reduct of the former In §2 I define, for any many-valued modal logic $\mathcal{M}$ (based upon a many-valued truth-functional logic of sufficient expressive power and a reasonable way of evaluating the necessity operator) a set of $\mathcal{M}$-axioms and $\mathcal{M}$-rules, and show that any formula of $\mathcal{M}$ is, in the presence of these axioms and using just these rules, syntactically equivalent to any of its $\mathcal{K}$-reducts Thus any formal system in the language of $\mathcal{M}$ (whose rules are the $\mathcal{M}$-rules and whose axioms include the $\mathcal{M}$-axioms) is essentially just an ordinary two-valued system, augmented by the $\mathcal{M}$-axioms and $\mathcal{M}$-rules In §3 I return to the original problem of many-valued analogues of two-valued systems.

§1. Let us now be more specific. A *many-valued modal logic* (or, briefly, a *logic*) $\mathcal{M}$ consists of a *many-valued truth-functional logic* $t(\mathcal{M})$ and a *necessity* operator $\Psi_{\mathcal{M}}$. In turn, $t(\mathcal{M})$ consists of a finite set $T_{\mathcal{M}}$ (of "truth values"), a non-empty proper subset $D_{\mathcal{M}}$ (of "designated" truth values), and finitely many finitary operations $*$ on $T_{\mathcal{M}}$ (corresponding to the *truth-functional connectives*, denoted by the same symbols as the operations). Intuitively, a necessity operator is to provide a way of evaluating 'necessarily-P' in a possible world, in terms of the truth values realized by 'P' in alternative possible worlds — in terms, that is, of the *set* of such truth values, independently of where, or how often, the truth values be realized. Moreover, 'necessarily-P' should have a "truth-like" truth value in a given world if and only if 'P' has "truth-like" truth values in all alternative worlds. Formally, then, $\Psi_{\mathcal{M}}\colon P(T_{\mathcal{M}})\to T_{\mathcal{M}}$ and, for all $S \subseteq T_{\mathcal{M}}$, $\Psi_{\mathcal{M}}(S) \in D_{\mathcal{M}} \Leftrightarrow S \subseteq D_{\mathcal{M}}$.

There is just one interesting two-valued modal logic, which we shall call $\mathcal{K}$: $T_{\mathcal{K}} = \{0, 1\}$, $D_{\mathcal{K}} = \{1\}$, the connectives of $t(\mathcal{K})$ are $\neg$ and $\vee$, and $\Psi_{\mathcal{K}}(S)$ is 0 or 1 according as $0 \in S$ or not.

Let *Var* be a countably infinite set of (propositional) variables. The set $Fla_{\mathcal{M}}$ of formulas of $\mathcal{M}$ is formed in the usual way, beginning with the variables and using the connectives $*$ of $t(\mathcal{M})$ and a unary connective $\Box$.

A *frame* is a pair (W, R), where W is a non-empty set (of "possible worlds") and R is a binary ("possible alternative") relation on W. If (W, R) is a frame and $\mathcal{M}$ is a logic, then an *$\mathcal{M}$-valuation* on (W, R) is a function $V\colon Var \times W \to T_{\mathcal{M}}$. An $\mathcal{M}$-valuation V has a unique extension $V\colon Fla_{\mathcal{M}} \times W \to T_{\mathcal{M}}$ satisfying $V(*a_1 \ldots a_m, w) = *(V(a_1, w), \ldots, V(a_m, w))$ and $V(\Box a, w) = \Psi_{\mathcal{M}}(\{V(a, v) \mid wRv\})$. A formula a of $\mathcal{M}$ is *valid* on (W, R), or $(W, R) \vDash \alpha$, if $V(\alpha, w) \in D_{\mathscr{M}}$ for every $w \in W$ and every $\mathscr{M}$-valuation V on (W, R).

Two formulas, possibly of different logics, are called *frame-equivalent* if they are valid on exactly the same frames. Two formulas α and β of the same logic $\mathscr{M}$ are called *designation-equivalent* if, for every frame (W, R), every $w \in W$, and every $\mathscr{M}$-valuation V on (W, R), $V(\alpha, w) \in D_{\mathscr{M}} \Leftrightarrow V(\beta, w) \in D_{\mathscr{M}}$.

A logic $\mathscr{M}$ is called *standard* if the connectives of $t(\mathscr{M})$ include $\neg$, $\vee$, and τ_a $(a \in T_{\mathscr{M}})$ satisfying

$$\neg b \in D_{\mathscr{M}} \Leftrightarrow b \notin D_{\mathscr{M}},$$
$$b \vee c \in D_{\mathscr{M}} \Leftrightarrow b \in D_{\mathscr{M}} \text{ or } c \in D_{\mathscr{M}},$$
$$\tau_a b \in D_{\mathscr{M}} \Leftrightarrow b = a,$$

for all $b, c \in T_{\mathscr{M}}$. The symbols $\neg$, $\vee$, τ_a will always denote connectives satisfying the above conditions; $\wedge$, $\Rightarrow$, and $\Leftrightarrow$ will abbreviate combinations of $\neg$ and $\vee$ in the usual way. A *standard connective* is any of $\neg$, $\vee$, τ_a, or $\Box$. A *standard formula* is one whose connectives are all standard. Note that if $\mathscr{M}$ is standard then $Fla_{\mathscr{K}} \subseteq Fla_{\mathscr{M}}$.

We now embark upon a series of technical definitions, needed to establish the connections between $\mathscr{K}$ and an arbitrary logic $\mathscr{M}$.

Define $d(q, \beta)$, where $q \in Var$ and $\beta \in Fla_{\mathscr{M}}$, by:

$$d(q, p) = 0 \text{ if } p \in Var,$$
$$d(q, *\alpha_1 \ldots \alpha_m) = \max\{d(q, \alpha_i) \mid 1 \leqslant i \leqslant m\},$$
$$d(q, \Box\alpha) = \begin{cases} 1 + d(q, \alpha) & \text{if } q \text{ occurs in } \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Thus $d(q, \beta)$ is the depth of q in β — the maximum number of occurrences of $\Box$ having the same occurrence of q in their scope.

If $q_1, \ldots, q_m$ are distinct variables and $\beta, \alpha_1, \ldots, \alpha_m$ are formulas, let $\beta(\alpha_1, \ldots, \alpha_m/q_1, \ldots, q_m)$ be the result of simultaneously substituting α_i for all occurrences of q_i in $\beta (1 \leqslant i \leqslant m)$.

If $*$ is an m-ary non-standard connective of a standard logic $\mathscr{M}$, let $\delta_*(\alpha_1, \ldots, \alpha_m, \beta)$ be the disjunction, over all $(a_1, \ldots, a_m, b) \in T_{\mathscr{M}}^{m+1}$ such that $b = *(a_1, \ldots, a_m)$, of the formulas $\tau_{a_1}\alpha_1 \wedge \ldots \wedge \tau_{a_m}\alpha_m \wedge \tau_b \beta$.

PROPOSITION 1. *If $\alpha_1, \ldots, \alpha_m$ are standard formulas, then $\delta_*(\alpha_1, \ldots, \alpha_m)$ is a standard formula, and for any (W, R), V, and w,*

$$V\big(\delta_*(\alpha_1, \ldots, \alpha_m, \beta), w\big) \in D_{\mathscr{M}} \Leftrightarrow V(\beta, w) = V(*\alpha_1 \ldots \alpha_m, w).$$

{Exactly how $\delta_*(\alpha_1, \ldots, \alpha_m, \beta)$ would be written in terms of $\neg$ and $\vee$ is of no concern; all that matters is that Proposition 1 should hold, as will be the case under any reasonable unabbreviation conventions. In what follows, observations similar to this one will be left to the reader.}

Define the *height* $h(\gamma)$ of γ in the usual way: $h(p) = 0$, $h(*a_1 \ldots a_m) = 1+\max\{h(a_i) \mid 1 \leqslant i \leqslant m\}$, and $h(\Box a) = 1+h(a)$. Define the *rank* $r(\gamma)$ of γ to be the maximum (for all $a \in T_{\mathscr{M}}$) height of scopes of occurrences of τ_a in γ.

Define δ_a^a, where $a \in T_{\mathscr{M}}$ and a is a standard formula of $\mathscr{M}$ with $h(a) \geqslant 1$, as follows: if $a = \tau_b \gamma$ then δ_a^a is the disjunction of all the formulas $\tau_c \gamma$ such that $\tau_b(c) = a$; if $a = \neg\gamma$ then δ_a^a is the disjunction of all formulas $\tau_c \gamma$ such that $\neg c = a$; if $a = \gamma \vee \delta$ then δ_a^a is the disjunction of all formulas $\tau_c \gamma \wedge \tau_d \delta$ such that $c \vee d = a$; and if $a = \Box\gamma$ than δ_a^a is the disjunction of all formulas $\bigwedge\{\neg\Box\neg\tau_c\gamma \mid c \in S\} \wedge \bigwedge\{\Box\neg\tau_c\gamma \mid c \notin S\}$ such that $\Psi_{\mathscr{M}}(S) = a$.

PROPOSITION 2. *The formula δ_a^a has rank one less than the rank of $\tau_a a$, and δ_a^a is designation-equivalent to $\tau_a a$.*

A formula is *closed* if every occurrence in it of a variable is within the scope of some τ_a. For any γ, let $\tau(\gamma) = \bigvee\{\tau_a\gamma \mid a \in D_{\mathscr{M}}\}$; then $\tau(\gamma)$ is designation-equivalent to γ.

If $a_1, \ldots, a_m$ is a listing without repititions of $T_{\mathscr{M}}$, and $a_1, \ldots, a_m \in Fla_{\mathscr{M}}$, let $\delta(a_1, \ldots, a_m)$ be the disjunction, for $i = 1, \ldots, m$, of the formulas $a_i \wedge \neg\bigvee\{a_j \mid 1 \leqslant j \leqslant m,\ j \neq i\}$. Then $V(\delta(a_1, \ldots, a_m), w) \in D_{\mathscr{M}}$ if and only if exactly one of $V(a_i, w) \in D_{\mathscr{M}}$; consequently $\delta(\tau_{a_1} p, \ldots, \tau_{a_m} p)$ is valid on all frames.

Now we put all the previous definitions to use, to define a relation $\succ$ between formulas of a standard logic $\mathscr{M}$.

(A) If η is non-standard, then $\eta \succ \eta'$ if and only if

$$\eta' = \Big[\bigwedge_{j=0}^{n} \Box^j \delta_*(a_1, \ldots, a_m, q)\Big] \Rightarrow \beta, \text{ where } \eta = \beta(a/q),\ a = *a_1 \ldots a_m,$$

each a_k is standard, q does not occur in a, and $n = d(q, \beta)$.

PROPOSITION 3. *If η is non-standard then an η' such that $\eta \succ \eta'$ can be found effectively, and any such η' has one fewer occurrence of non--standard connectives than η has.*

(B) If η is standard and $r(\eta) \geqslant 1$, then $\eta \succ \eta'$ if and only if $\eta' = \beta(\delta_a^a, a_1, \ldots, a_m/q, q_1, \ldots, q_m)$, where $\eta = \beta(\tau_a a, a_1, \ldots, a_m/q, q_1, \ldots, q_m)$, $h(a) = r(\eta)$, β has only $\neg$, $\vee$, and $\Box$ as connectives, and no q_i occurs in a.

PROPOSITION 4. *If η is standard and $r(\eta) \geqslant 1$ then an η' such that $\eta \succ \eta'$ can be found effectively; and any such η' is standard, and either $r(\eta') < r(\eta)$ or else $r(\eta') = r(\eta)$ and η' has one fewer occurrence of τ_a's with scope of height $r(\eta)$ than η has.*

(C) If η is standard, $r(\eta) = 0$, η is not closed, and η is not a formula of $\mathscr{K}$, then $\eta \succ \eta'$ if and only if η' is obtained from η by replacing some occurrence of some q, not within the scope of any τ_a, by $\tau(q)$.

PROPOSITION 5. *If η is standard, $r(\eta) = 0$, η is not closed, and η is not a formula of $\mathscr{K}$, then an η' such that $\eta \succ \eta'$ can be found effectively; and any such η' is standard, and $r(\eta') = 0$, and η' has one fewer occurrence of variables not within the scope of any τ_a than η has.*

(D) If η is a closed standard formula of rank zero, then $\eta \succ \eta'$ if and only if η' is

$$\{\bigwedge_{j=0}^{n} \Box^j [\bigwedge_{i=1}^{k} \delta(q_i^1, \ldots, q_i^m)]\} \Rightarrow \beta,$$

where $p_1, \ldots, p_k$ are all the variables in η, $q_i^r (1 \leqslant i \leqslant k,\ 1 \leqslant r \leqslant m)$ are distinct variables, $n = \max\{d(p_i, \eta) \mid 1 \leqslant i \leqslant k\}$, and β is the result of replacing every occurrence of $\tau_{a_r} p_i$ in η by q_i^r.

PROPOSITION 6. *If η is a closed standard formula of rank zero, than an η' such that $\eta \succ \eta'$ can be found effectively; and any such η' is a formula of $\mathscr{K}$.*

(E) If η is a formula of $\mathscr{K}$, then $\eta \succ \eta'$ if and only if $\eta = \eta'$.

Finally, if a is a formula of a standard logic $\mathscr{M}$, then β is a *$\mathscr{K}$-reduct* of a (in $\mathscr{M}$) if β is a formula of $\mathscr{K}$ and there is a finite sequence $a \succ \eta_1 \succ \eta_2 \succ \ldots \succ \eta_p \succ \beta$ (of formulas of $\mathscr{M}$).

THEOREM 7. *Given a formula a of any logic $\mathscr{M}$, one can effectively find a standard logic $\mathscr{M}'$ extending $\mathscr{M}$ and a $\mathscr{K}$-reduct β of a in $\mathscr{M}'$.*

PROOF. Pick elements 1 and 0 of $D_{\mathscr{M}}$ and $T_{\mathscr{M}} - D_{\mathscr{M}}$ respectively, define new connectives by

$$\neg a = \begin{cases} 0 & \text{if } a \in D_{\mathscr{M}} \\ 1 & \text{if } a \notin D_{\mathscr{M}}, \end{cases}$$

$$a \vee b = \begin{cases} 1 & \text{if } a \in D_{\mathscr{M}} \text{ or } b \in D_{\mathscr{M}} \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_a(d) = \begin{cases} 1 & \text{if } d = a \\ 0 & \text{if } d \neq a, \end{cases}$$

and form $\mathscr{M}'$ by adding these to $\mathscr{M}$. Then a is a formula of $\mathscr{M}'$, and $\mathscr{M}'$ is standard. From Propositions 3 – 6 it follows that in any infinite sequence $a \succ \eta_1 \succ \eta_2 \succ \ldots$ of formulas of $\mathscr{M}'$, there must occur a formula of $\mathscr{K}$. [Formally: (A) – (E) implicitly define a partial ordering $\preccurlyeq$ (the reflexive, transitive closure of the converse of $\succ$) and Propositions 3 – 6 state, in effect, that $\preccurlyeq$ is well-founded and has exactly the members of $Fla_{\mathscr{K}}$ as minimal elements. Informally: One begins with a, eliminates non-standard connectives one at a time (working from the inside out) via (A), until a standard formula is obtained. Then one eliminates connectives within the scopes of τ_a' s (working from the outside in, beginning with τ_a' s with longest scopes) via (B) until a standard formula of rank zero is obtained. If this is not a formula of $\mathscr{K}$, then it is made closed by applications of (C), and then replaced, via (D), by a formula of $\mathscr{K}$.] So to find a $\mathscr{K}$-reduct of α in $\mathscr{M}'$ it suffices to produce $\alpha \succ \eta_1 \succ \eta_2 \succ \ldots$, halting when a formula of $\mathscr{K}$ appears. By Propositions 3–6, this can be done effectively.

The following theorem shows that any formula of any logic is semantically equivalent to a formula of $\mathscr{K}$.

THEOREM 8. *If α is a formula of any logic $\mathscr{M}$ and β is a $\mathscr{K}$-reduct of α, then α and β are frame-equivalent.*

PROOF. Without loss of generality, we may assume that $\mathscr{M}$ is stan) dard. To show that α and β are frame-equivalent, it suffices to show that whenever $\eta \succ \eta'$ then η and η' are frame-equivalent. There are five cases to consider, corresponding to (A)–(E) in the definition of $\succ$; of course (E) is trivial.

(A) If (W, R) is a frame and $w \in W$, let $R^0(w) = \{w\}$, $R^{n+1}(w) = R^n(w) \cup \{v \mid (\exists z)\, (z \in R^n(w)\ \&\ zRv)\}$. Then, for any V,

$$\begin{aligned} V\Big(\bigwedge_{j=0}^{n} \Box^j \delta_*(a_1, \ldots, a_m, q), w\Big) \in D_{\mathscr{M}} & \\ &\Leftrightarrow \big(\forall v \in R^n(w)\big)\ [V(\delta_*(a_1, \ldots, a_m, q), v) \in D_{\mathscr{M}}] \\ &\Leftrightarrow \big(\forall v \in R^n(w)\big)\ [V(q, v) = V(*a_1 \ldots a_m v,)] \\ &\Leftrightarrow V(\beta, w) = V(\beta(*a_1 \ldots a_m/q), w) = V(\eta, w) \end{aligned}$$

Now if w and V are such that $V(\eta, w) = V\big(\beta(\alpha/q), w\big) \notin D_{\mathscr{M}}$, then, since q does not occur in α, we may assume that $V(q, v) = V(\alpha, v)$ for all $v \in W$. Then $V\big(\bigwedge_{j=0}^{n} \Box^j \delta_*(a_1, \ldots, a_m, q), w\big) \in D_{\mathscr{M}}$, $V(\beta, w) = V\big(\beta(\alpha/q), w\big) \notin D_{\mathscr{M}}$, and $V(\eta', w) \notin D_{\mathscr{M}}$. Conversely, if w and V are such that $V(\eta', w) \notin D_{\mathscr{M}}$ then $V(\beta, w) \notin D_{\mathscr{M}}$ but $V\big(\bigwedge_{j=0}^{n} \Box^j \delta_*(a_1, \ldots, a_m, q), w_{\mathscr{M}}\big) \in D_{\mathscr{M}}$. The last fact implies that $V(\eta, w) = V(\beta, w) \notin D_{\mathscr{M}}$. Hence $(W, R) \models \eta \Leftrightarrow (W, R) \models \eta'$.

(B) Since δ_a^α is designation-equivalent to $\tau_a \alpha$, and β has only $\neg$, $\vee$, and $\Box$ as connectives, $\beta(\delta_a^\alpha/q)$ is designation-equivalent to $\beta(\tau_a \alpha/q)$. Hence $\eta' = \beta(\delta_a^\alpha, \alpha_1, \ldots, \alpha_m/q, q_1, \ldots, q_m)$ is designation-equivalent, and hence frame-equivalent, to $\eta = \beta(\tau_a \alpha, \sigma_1, \ldots, \alpha_m/q, q_1, \ldots, q_m)$.

(C) Since $\tau(q)$ is designation-equivalent to q, and the substitution takes place within the scope of no connective other than $\neg$, $\vee$, and $\Box$, η' is designation-equivalent to η.

(D) If (W, R) *non* $\models \eta$, let V be an $\mathscr{M}$-valuation, and $w \in W$, such that $V(\eta, w) \notin D_{\mathscr{M}}$. Let V' be a $\mathscr{K}$-valuation for (W, R) satisfying

$$V'(q_i^r, v) = 1 \quad \Leftrightarrow \quad V(p_i, v) = a_r \quad \Leftrightarrow \quad V(\tau_{a_r} p_i, v) \in D_{\mathscr{M}},$$

so that $V'\big(\bigwedge_{j=0}^{n}\Box^j(\bigwedge_{i=1}^{k}\delta(q_1, \ldots, q_i^m)), w\big) = 1$. Since η is constructed from the formulas $\tau_{a_r}p_i$ using just $\neg$, $\vee$, and $\Box$, $V'(\beta, w) = 0$. So (W, R) *non* $\models \eta'$. Conversely, if V' is a $\mathscr{K}$-valuation for some (W, R), and $w \in W$, and $V'(\eta', w) = 0$, then $V'(\bigwedge_{j=0}^{n}\Box^j(\bigwedge_{i=0}^{k}\delta(q_i, \ldots, q_m^i)), w) = 1$ so there is an $\mathscr{M}$-valuation V satisfying, for all $v \in R^n(w)$, $V(p_i, v) = a_r \Leftrightarrow V'(q_i^r, v) = 1$. Then $V(\eta, w) \notin D_{\mathscr{M}}$, since $V'(\beta, w) = 0$. Thus $(W, R) \models \eta \quad \Leftrightarrow \quad (W, R) \models \eta'$, for all frames (W, R).

§2. Let us turn now to the problem of a syntactical reduction of formulas of $\mathscr{M}$ to formulas of $\mathscr{K}$. Until further notice, $\mathscr{M}$ is a fixed standard logic. {But all our results would hold so long as the standard connectives were *definable* in $\mathscr{M}$; this is the case, for example, if $t(\mathscr{M})$ is functionally complete, or if $t(\mathscr{M})$ is any of the finitely-many-valued logics of Łukasiewicz.}

The *$\mathscr{M}$-axioms* are the following (finitely many) formulas of $\mathscr{M}$:

(a) $p \Rightarrow (q \Rightarrow p)$,
(b) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$,
(c) $(\neg p \Rightarrow \neg q) \Rightarrow (q \Rightarrow p)$,
(d) $\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$,
(e) $\delta_*(p_1, \ldots, p_m, *p_1 \ldots p_m)$, for each m-ary connective $*$ of $t(\mathscr{M})$,
(f) $p \Leftrightarrow \tau p$,
(g) $\delta(\tau_{a_1}p, \ldots, \tau_{a_m}p)$,
(h) $\delta_a^{\Box p} \quad \Leftrightarrow \quad \tau_a \Box p$, for each $a \in T_{\mathscr{M}}$.

The *$\mathscr{M}$-rules* are the rules:

Substitution: from α infer $\alpha(\beta/q)$,
Detachment: from α and $\alpha \Rightarrow \beta$, infer β,
Necessitation: from α infer $\Box\alpha$,
Elimination: from $\alpha(\tau_{a_1}p, \ldots, \tau_{a_m}p/q_1, \ldots, q_m)$ infer $\big(\bigwedge_{j=0}^{n}\Box^j\delta(q_1, \ldots, q_m)\big) \Rightarrow \alpha$, provided α is a standard formula of rank zero and $n = \max\{d(q_i, \alpha) \mid 1 \leqslant i \leqslant m\}$, p does not occur in α and no $\tau_a q$ occurs in α.

An *$\mathscr{M}$-system* is a formal system whose language is that of $\mathscr{M}$, whose axioms include the $\mathscr{M}$-axioms, and whose rules are just the $\mathscr{M}$-rules. The weakest $\mathscr{M}$-system, whose axioms are just the $\mathscr{M}$-axioms, is called $K^{\mathscr{M}}$. If Σ is a formal system and α is a formula of the language of Σ, then $\Sigma \vdash \alpha$ means that α is a thesis of Σ, i.e. that α is provable from the axioms of Σ via the rules of Σ. Two formal systems are *equivalent* if they have the same theses.

LEMMA 9. (*Completeness Theorem for $K^{\mathscr{M}}$*) *A formula of $\mathscr{M}$ is a thesis of $K^{\mathscr{M}}$ if and only if it is valid on all frames.*

PROOF. The $\mathscr{M}$-axioms are valid on all frames, and the $\mathscr{M}$-rules preserve validity on each frame, so every thesis of $K^{\mathscr{M}}$ is valid on all frames. {The Elimination rule preserves validity for the same reason that $(W, R) \models \eta \Rightarrow (W, R) \models \eta'$ in case (D) of the proof of Theorem 8.}

By the well-known completeness theorem for the two-valued modal system K (the system in the language of $\mathscr{K}$ having (a)–(d) as axioms and Substitution, Detachment, and Necessitation as rules), every formula of $\mathscr{K}$ which is valid on all frames is a thesis of K. But the axioms and rules of $K^{\mathscr{M}}$ include those of K, so every such formula is a thesis of $K^{\mathscr{M}}$ as well.

Now suppose $\alpha \in Fla_{\mathscr{M}}$ is valid on all frames. Let β be a $\mathscr{K}$-reduct of α. By Theorem 8, β is valid on all frames, and by the above remark β is a thesis of $K^{\mathscr{M}}$. To complete the proof of Lemma 9, it suffices to show that whenever $\eta \succ \eta'$ and η' is a thesis of $K^{\mathscr{M}}$ then so is η.

Write $\vdash$ for $K^{\mathscr{M}} \vdash$. Again there are four non-trivial cases.

(A) If $\vdash \eta'$ then $\vdash \eta'(\alpha/q)$ by Substitution. Now $\vdash \delta_*(p_1, \ldots, p_m, *p_1 \ldots p_m)$, so by Necessitation $\vdash \Box^j \delta_*(p_1, \ldots, p_m, *p_1 \ldots p_m)$ for each j. Since every tautology of $t(\mathscr{K})$ is a thesis of $K^{\mathscr{M}}$, it follows that $\vdash \bigwedge_{j=0}^{n} \Box^j \delta_*(p_1, \ldots, p_m, *p_1 \ldots p_m)$. By Substitution $\vdash \bigwedge_{j=0}^{n} \Box^j \delta_*(\alpha_1, \ldots, \alpha_m, *\alpha_1 \ldots \alpha_m)$. But $\eta'(\alpha/q) = \bigwedge_{j=0}^{n} \Box^j \delta_*(\alpha_1, \ldots, \alpha_m, *\alpha_1 \ldots \alpha_m) \Rightarrow \eta$, so by Detachment $\vdash \eta$.

(B) For $n = d(q, \beta)$, $K \vdash \left[\left(\bigwedge_{j=0}^{n} \Box^j (p_1 \Leftrightarrow p_2)\right) \Rightarrow \left(\beta(p_1/q) \Leftrightarrow \beta(p_2/q)\right)\right]$. Since $\vdash \bigwedge_{j=0}^{n} \Box^j (\delta_\alpha^{\Box p} \Leftrightarrow \tau_\alpha \Box p)$, we infer $\vdash \left(\beta(\delta_\alpha^{\Box p}/q) \Leftrightarrow \beta(\tau_\alpha \Box p/q)\right)$. By Substitution, $\vdash \eta' \Leftrightarrow \eta$.

(C) This is similar to (B), using the axiom $p \Leftrightarrow \tau(p)$ of $K^{\mathscr{M}}$ in place of the axiom $\delta_\alpha^{\Box p} \Leftrightarrow \tau_\alpha \Box p$. Again, $\vdash \eta' \Leftrightarrow \eta$.

(D) Since $\delta(\tau_{\alpha_1} p, \ldots, \tau_{\alpha_m} p)$ is an $\mathscr{M}$-axiom, we have $\vdash \bigwedge_{j=0}^{n} \Box^j \left(\bigwedge_{i=1}^{k} \delta(\tau_{\alpha_1} p_i, \ldots, \tau_{\alpha_m} p_i)\right)$. If $\vdash \eta'$ then, by Substitution and Detachment, $\vdash \eta$.

Notice that all the axioms and rules of $K^{\mathscr{M}}$ were used in the above proof, except for the Elimination rule. Nor is that rule mentioned in any of [1, 2, 3]. Why we want the rule is explained by the next theorem; why it is often unnecessary is made clear in §3. The proof of Lemma 9 establishes, as we shall have occasion to recall in §3, that if a $\mathscr{K}$-reduct of α is provable in any $\mathscr{M}$-system, without using the Elimination rule, then so is α.

THEOREM 10. *Let Σ be an $\mathscr{M}$-system, $\alpha \in Fla_{\mathscr{M}}$, and β a $\mathscr{K}$-reduct of α; then $\Sigma \vdash \beta$ if and only if $\Sigma \vdash \alpha$.*

PROOF. The implication from left to right is proved just as in Lemma 9. To prove the converse, it will suffice to show that whenever $\eta \succ \eta'$ and

$\Sigma \vdash \eta$ then $\Sigma \vdash \eta'$. The cases (B) and (C) follow from what was established in the corresponding cases of the proof of Lemma 9, viz. that $K_{\mathscr{M}} \vdash \eta \Leftrightarrow \eta'$, and of course (E) is trivial. For (A), note that the proof of the same case in Theorem 8 shows that $\eta \Rightarrow \eta'$ is valid on all frames. By Lemma 9, $K^{\mathscr{M}} \vdash \eta \Rightarrow \eta'$, whence $\Sigma \vdash \eta \Rightarrow \eta'$. For (D): finitely many applications of the Elimination rule transform η into a formula tautologically equivalent (in $t(\mathscr{K})$) to η'.

COROLLARY 11. *If Σ and Σ' are $\mathscr{M}$-systems and $\Sigma \vdash \beta \Leftrightarrow \Sigma' \vdash \beta$ for all $\beta \in Fla_{\mathscr{K}}$, then Σ and Σ' are equivalent.*

COROLLARY 12. *Each $\mathscr{M}$-system Σ is equivalent to the $\mathscr{M}$-system whose axioms are the $\mathscr{M}$-axioms together with the $\mathscr{K}$-reducts of the axioms of Σ.*

§3. A *$\mathscr{K}$-system* is a formal system whose language is that of $\mathscr{K}$, whose axioms include the $\mathscr{M}$-axioms (a) – (d), and whose rules are Substitution, Detachment, and Necessitation. An $\mathscr{M}$-system Σ is *analogous to* a $\mathscr{K}$-system S if, for every $\beta \in Fla_{\mathscr{K}}$, $S \vdash \beta \Leftrightarrow \Sigma \vdash \beta$. By Corollary 11, any two $\mathscr{M}$-systems analogous to the same $\mathscr{K}$-system are equivalent.

Every $\mathscr{M}$-system is analogous to some $\mathscr{K}$-system, namely the $\mathscr{K}$-system whose axioms are the $\mathscr{K}$-reducts of the theses of the given system. It is *not* obvious that every $\mathscr{M}$-system is analogous to the $\mathscr{K}$-system whose axioms are the $\mathscr{K}$-reducts of the *axioms* of the given system. Consequently, it is not obvious that for every $\mathscr{K}$-system there is an analogous $\mathscr{M}$-system.

If Σ is analogous to S then, using Theorems 8 and 10,

(a) *Σ and S have the same frames,*

(b) *$\Sigma \vdash \alpha \Leftrightarrow S \vdash \beta$, where β is a $\mathscr{K}$-reduct of α,*

(c) *Σ is complete with respect to a given class of frames if and only if S is complete with respect to that class,*

(d) *Σ is decidable if and only if S is decidable,*

(e) *Σ has the finite model property if and only if S has the finite model property,*

Given a $\mathscr{K}$-system S, let $S^{\mathscr{M}}$ be the $\mathscr{M}$-system whose axioms are the axioms of S together with the $\mathscr{M}$-axioms. Then the theses of $S^{\mathscr{M}}$ include those of S. In fact, they include those of any $\mathscr{M}$-system Σ analogous to S; for if $\Sigma \vdash \alpha$ and β is a $\mathscr{K}$-reduct of α then $\Sigma \vdash \beta$ (Theorem 10), $S \vdash \beta$ (definition of "analogous"), $S^{\mathscr{M}} \vdash \beta$, and $S^{\mathscr{M}} \vdash \alpha$ (Theorem 10). Conversely, if Σ is analogous to S then every axiom of $S^{\mathscr{M}}$ is a thesis of Σ. Consequently, if any $\mathscr{M}$-system is analogous to S, then $S^{\mathscr{M}}$ is.

If S is complete, i.e. every non-thesis of S is non-valid on some frame for S, then $S^{\mathscr{M}}$ is analogous to S. For suppose $\beta \in Fla_{\mathscr{K}}$ and S *non* $\vdash$ β. Then there is a frame (W, R) for S on which β is not valid. But $S^{\mathscr{M}}$ and S certainly have the same frames, so (W, R) is a frame for $S^{\mathscr{M}}$ on which β is not valid, and $S^{\mathscr{M}}$ *non* $\vdash$ β.

I conjecture that $S^{\mathcal{M}}$ is always analogous to S.

If $S^{\mathcal{M}}$ is analogous to S, then the Elimination rule is redundant in $S^{\mathcal{M}}$. For suppose $S^{\mathcal{M}} \vdash \alpha$. Then $S^{\mathcal{M}} \vdash \beta$, where β is a $\mathcal{K}$-reduct of α, and $S \vdash \beta$. But any proof in S is a proof in $S^{\mathcal{M}}$ without use of Elimination, so by the remarks following the proof of Lemma 9, α is provable in $S^{\mathcal{M}}$ without use of Elimination. {Of course it does not follow that Elimination is redundant in an arbitrary $\mathcal{M}$-system Σ; only that it is redundant in an $\mathcal{M}$-system equivalent to Σ, a trivial result. In fact Elimination is not always redundant — add to $K^{\mathcal{M}}$ an axiom which is a formula, but not a tautology, of $t(\mathcal{M})$; the resulting system is inconsistent, but without Elimination might well be consistent. No doubt non-trivial examples exist, as well.}

I have not attempted to catalogue the occasions when I have borrowed ideas from Morgan's paper [1]. But I want it to be understood that it is from Morgan that I got the idea that an analysis might be possible of the relationship between two-valued modal logic and modal logic based upon a more-or-less arbitrary many-valued logic. I wish also to call to the reader's attention that Morgan considers "global operators" $\Psi_{\mathcal{M}}$ which do not satisfy my condition $\Psi_{\mathcal{M}}(S) \in D_{\mathcal{M}} \Leftrightarrow S \subseteq D_{\mathcal{M}}$. Roughly speaking, Morgan's results (and mine, presumably) apply whenever an operator satisfying the condition is *definable* in terms of the given $\Psi_{\mathcal{M}}$ (just as it would suffice that the standard connectives be *definable*). I thought it best to forego this generality to simplify the exposition. For philosophical applications, as Morgan points out, the extra generality is important. Moreover, some of the results of Segerberg [3] are special cases of the generalized version of this work, but not the ungeneralized.

References

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Allatum est die 29 Decembris 1976