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On the Algebraization of a Feferman's Predicate

(*The algebraization of theories which express Theor*; X)

Summary. This paper is devoted to the algebraization of an arithmetical predicate introduced by S. Feferman. To this purpose we investigate the equational class of Boolean algebras enriched with an operation ϱ, which translates such predicate, and an operation τ, which translates the usual predicate Theor. We deduce from the identities of this equational class some properties of ϱ and some ties between ϱ and τ; among these properties, let us point out a fixed-point theorem for a sufficiently large class of ϱ-τ polynomials. The last part of this paper concerns the duality theory for ϱ-τ algebras.

Introduction

Recently, R. Magari and other authors have studied how some metatheorems of Peano arithmetic $\mathscr{P}$ (and in general of theories satisfying some derivability conditions) can be expressed in algebraic terms. It has been emphasized that many results depend only on three properties of the predicate Theor, which can be written as identities of the Lindenbaum algebra of $\mathscr{P}$ enriched with a unary operation τ expressing Theor. Among these results, let us point out the two Gödel's Theorems, Löb's condition,[1] the existence and uniqueness (up to provable equivalence) of the fixed-point for formulas built (in a reasonable sense) from variables, Theor, and Boolean connectives, also in the intuitionistic logic. The above-mentioned papers provide us with relatively simple techniques in the part of the proof-theory which is related to the Gödel's theorems.

In this paper, we study, with the same techniques, the algebraization of a predicate which has been introduced by S. Feferman in [3], and is similar, in a sense, to Rosser's predicate. Namely, we define, in the Lindenbaum sentence algebra of $\mathscr{P}$, a unary operation ϱ, which can be regarded as an algebraic translation of Feferman's predicate. Then, in paragraph 2, we study the equational class of Boolean algebras enriched with such an operator ϱ (the algebras of this class will be called ϱ-algebras); that allows us to obtain an algebraic counterpart of Rosser's theorem. In paragraph 3 we consider the relations expressed by identities between the operations ϱ and τ in the Lindenbaum sentence algebra of $\mathscr{P}$; these identities define the equational class of ϱ-τ algebras. In paragraph 4 we discuss both the problem of introducing an operator with the property of ϱ into a diagonalizable algebra, and that of introducing an operation with the properties of τ into a ϱ-algebra. In this way we translate the logic problem

[1] $\vdash_{\mathscr{P}} \dot{T}(\overline{p}) \to \dot{T}(\overline{\dot{T}(\overline{p})})$ for every sentence p of $\mathscr{P}$.

of building a "Theor predicate" starting from a "Rosser predicate" (see the sequel for the definitions) and the inverse problem. Then (n. 5), we prove a fixed-point theorem in every ϱ-τ algebra for a sufficiently large class of ϱ-τ polynomials. Finally, we characterize the dual space of a Boolean algebra enriched with the operations ϱ and τ. With regard to the last problem, we recall that $\tilde{\varrho} \equiv \nu\varrho\nu$ and $\sigma \equiv \nu\tau\nu$[2] are hemimorphisms in the sense of P. R. Halmos, [5], and hence they are associated with Boolean relations defined in the dual space.

For the sake of simplicity, we always refer to Peano arithmetic $\mathscr{P}$, although there exists a large class of theories for which the some results can be obtained: for instance, all these theories in which $\mathscr{P}$ is relatively interpretable.

1. Preliminary notes

We say that a formula $\dot{T}(x)$ of $\mathscr{P}$, with just one free variable x, is a *Theor predicate* if the set $\{p:\ \vdash_{\mathscr{P}}\dot{T}(\overline{p})\}$ is exactly the set of theorems of $\mathscr{P}$, and if, for every two propositions p, q of $\mathscr{P}$, we have: a) $\vdash_{\mathscr{P}}[\dot{T}(\overline{p})\wedge\dot{T}(\overline{p\to q})]\to\dot{T}(\overline{q})$ and b) $\vdash_{\mathscr{P}}\dot{T}(\overline{p})\to\dot{T}(\overline{\dot{T}(p)})$. From a) and b), by diagonalization Lemma, we obtain both Löb's theorem and its formalization: $\vdash_{\mathscr{P}}\dot{T}(\overline{\dot{T}(\overline{p})\to p})\to\dot{T}(\overline{p})$. Moreover, let us note that, as a particular case of Löb's theorem, we have the second theorem of Gödel, that is: *not* $\vdash_{\mathscr{P}} Con_T$, where Con_T is the sentence $x\forall x\neg[\dot{T}(x)\wedge\dot{T}(\neg x)]$.

On the other hand, we shall say that $\dot{R}(x)$ is a *Rosser predicate*, if the set $\{p:\ \vdash_{\mathscr{P}}\dot{R}(\overline{p})\}$ is the set of theorems of $\mathscr{P}$, and $\vdash_{\mathscr{P}}Con_R$. In order to obtain an algebraization of Rosser's theorem, we have to define in the Lindenbaum algebra of $\mathscr{P}$ an operation ϱ associated with a Rosser predicate $\dot{R}(x)$ as follows: $\varrho[p] = \dot{R}(\overline{p})]$, where, for every sentence of $\mathscr{P}$, $[p]$ denotes the equivalence class of p with respect to provable equivalence. Since the above definition is unambiguous, $\varrho[p]$ must depend only on $[p]$and not on p. In other words, a necessary condition is: if $\vdash_{\mathscr{P}}p\leftarrow q$, then $\vdash_{\mathscr{P}}\dot{R}(p)\leftrightarrow\dot{R}(q)$. On the ground of this remark, it is useful to consider the *Feferman's predicate* $\dot{F}(x)$ (see [3][3]), which can be informally defined in the following way: let $\Pi(x)$ be a formula which binumerates "in a natural way" the set A of axioms of $\mathscr{P}$, and let $\Pi^*(x)$ be the formula $\Pi(x)\wedge\forall y(y$

[2] We use terminology and notations of R. Magari [7], [8]. In particular, $+, \cdot, \nu$, denote respectively the operations of join, meet, complementation. With regard to the representation of recursive and recursively enumerable relations in a theory, we sometime refer to S. Feferman, [3]. For the sake of simplicity, we consider as Gödel-numbering of the set of propositions of $\mathscr{P}$ a primitive recursive bijection between this set and the set of natural numbers.

[3] In [3], this predicate is denoted by $Pr_{\Pi^*}^{(M)}(x)$.

$\leqslant x \to Con_{\Pi/y})$ where $Con_{\Pi/y}$ expresses the consistency of the theory having the set $\{p \in A: \overline{p} \leqslant y\}$ as axiom system. Then, by definition, $\dot{F}(x)$ is the formula which numerates "in a natural way" the set of theorems of the theory whose axiom system is the set binumerated by the formula $\Pi^*(x)$[4]. Since $\mathscr{P}$ is reflexive, $\dot{F}(x)$ numerates in $\mathscr{P}$ the set of theorems of $\mathscr{P}$; moreover, we have also $\vdash_{\mathscr{P}} Con_F$ (see S. Feferman [3]). Hence $\dot{F}(x)$ is a Rosser's predicate. Then, one can prove $\vdash_{\mathscr{P}}[\dot{F}(\overline{p}) \wedge \dot{F}(\overline{p \to q})] \to \dot{F}(\overline{q})$ (see S. Feferman, [3]), and hence, for every propositions p, q of $\mathscr{P}$, if $\vdash_{\mathscr{P}} p \leftrightarrow q$, then $\vdash_{\mathscr{P}} \dot{F}(\overline{p}) \leftrightarrow \dot{F}(\overline{q})$.

Let us note that for $\dot{F}(x)$ *Godel's* second *theorem and Löb's theorem do not hold*. Hence, we can conclude that for some sentences p of $\mathscr{P}$, $\dot{F}(\overline{p}) \to \dot{F}(\overline{\dot{F}(\overline{p})})$ is not a theorem of $\mathscr{P}$.

2. The equational classes of ϱ-algebras and of diagonalized ϱ-algebras.

Taking the remarks of the preceding paragraph into account, we can define in the Lindenbaum sentence algebra of $\mathscr{P}$ a unary operation ϱ as follows: $\varrho[p] = [\dot{F}(\overline{p})]$ (In the sequel we shall omit square brackets). The operation ϱ has the following properties

ϱ 1) $\quad \varrho\, 1 = 1$

ϱ 2) $\quad \varrho\, 0 = 0$

ϱ 3) $\quad \varrho(p \to q) \leqslant \varrho p \to \varrho q$

Let us note that, assuming ϱ 1), ϱ 3) is equivalent to the following identity:

ϱ 4) $\quad \varrho(p \cdot q) = \varrho p \cdot \varrho q$

(See G. Sambin, [13]). Now, we call ϱ algebra every Boolean algebra with a unary operation ϱ which satisfies ϱ 1), ϱ 2), ϱ 3) or, equivalently, ϱ 1), ϱ 2), ϱ 4). Then, the ϱ algebras constitute an equational class.

Let us note that, by diagonalization Lemma, for every polynomial $F(x)$ in which x appears under the scope of ϱ, there exists, in the ϱ-algebra of $\mathscr{P}$, at least a p_f such that $p_f = Fp_F$. Hence, it is suitable to consider also the equational class obtained from the one of ϱ algebras by adding, for every polynomial $f(x)$ with x under the scope of ϱ, a 0-adic operation p_f and the identity $p_f = Fp_f$. We call the algebras of this equational class diagonalized ϱ -algebras. Now, we have the following algebraization of *Rosser's theorem*:

[4] The words "in a natural way" as well as many others informal definitions we give, can be made precise (see for instance S. Feferman, [3]).

THEOREM 1. *In every non-trivial diagonalized* ϱ *-algebra, there is a* p *such that* $0 < p < 1$.

PROOF. Let p be an element of a non-trivial ϱ -algebra such that $p = \nu\varrho p$. Then, from $p = 0$ it follows that $0 = \nu\varrho 0 = \nu 0 = 1$, and from $p = 1$ it follows that $1 = \nu\varrho 1 = \nu 1 = 0$; hence $p \neq 0, p \neq 1$. Q.E.D.

Moreover, it can be of some interest to compare $\mathfrak{F}_0$, the free diagonalizable algebra on the empty set, which can be regarded as the algebra built from the (equivalence class of the) well-known Gödel formula, and the ϱ algebra $\mathscr{R}$, freely generated by an element which represents the undecidable formula suggested by Feferman, namely a p such that $p = \nu\varrho p$[5]. In the former algebra, the elements $0, \tau 0 \ldots\ldots \tau^n 0 \ldots$ and their complements constitute two chains (ordered by $<$) with the order type of ω and $-\omega$ respectively. In the latter algebra, the elements $p, \varrho p \ldots\ldots \varrho^n p \ldots$ and their complements constitute two chains with the order type of $\mathbf{Z}$. Namely, we have: $\varrho^{2n+2}p \leqslant \varrho^{2n}p \leqslant \nu\varrho^{2n-1}p \leqslant \nu\varrho^{2n+1}p$, and hence $\varrho^{2n+1}p \leqslant \varrho^{2n-1}p \leqslant \nu\varrho^{2n}p \leqslant \nu\varrho^{2n+2}p$ for every natural number n different from 0. In fact, from ϱ 2) and ϱ 4) it follows that $\varrho x \leqslant \nu\varrho\nu x$, and hence $\varrho^2 p \leqslant \nu\varrho\nu\varrho p = \nu\varrho p$, i. e. $\varrho^2 p \leqslant \nu\varrho p = p$. Since ϱ is a non-decreasing function, by the last relation, we have $\varrho^{2n+2}p \leqslant \varrho^{2n}p \leqslant p$, and $\varrho^{2n+1}p \leqslant \varrho^{2n-1}p \leqslant \varrho p$, hence the statement easily follows. Let us also note that, for every natural number n different from 0, $\varrho^{2n+2}p \neq \varrho^{2n}p$, $\varrho^{2n+1}p \neq \varrho^{2n-1}p$ and $\varrho^n p \neq 0$. Indeed, in the opposite case, the equality $\varrho^{2n+2}p = \varrho^{2n}p$ (respectively $\varrho^{2n+1}p = \varrho^{2n+1}p$ or $\varrho^n p = 0$) would follow from the identities of ϱ algebras and from the identity $p = \nu\varrho p$; hence, it would hold in every ϱ algebra containing a p such that $p = \nu\varrho p$. But, in paragraph 6, we shall show this is not the case (see example b). Finally, let us note that in $\mathfrak{F}_0$ every polynomial $f(x)$ with x under the scope of τ has a fixed point, whereas in $\mathscr{R}$ there are polynomials $f(x)$ with x under the scope of ϱ which have no fixed point. To prove this, by the above argument, it suffices to exhibit a ϱ algebra $\mathscr{A}$ with a p such that $p = \nu\varrho\, p$ and a polynomial $f(x)$ with x under the scope of ϱ, which has no fixed point in $\mathscr{A}$. To this purpose, let us consider the algebra $\mathscr{A} = \langle \{0, 1, p, \nu p\}, +, \cdot, \nu, \varrho \}$ where $\varrho p = \nu p$, $\varrho\nu p = p$. It can easily be seen that $\nu\varrho^2 x$ admits no fixed point in this algebra.

REMARK. One can also consider the equational class obtained from that of ϱ-algebras by adding, for every polynomial $f(x, y, \ldots y)$ with x under the scope of ϱ, an n-ary operation $x(y \ldots y)$ and the identity $x(y \ldots y) = f(x(y \ldots y)y \ldots y)$. In this way, we should have a stronger algebraic counterpart of diagonalization lemma.

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[5] Strictly speaking, $\mathscr{R}$ is the free algebra on the empty set in the equational class which is obtained from the one of ϱ-algebras by adding a new 0-adic operation p and the identity $p = \nu\varrho p$.

3. The equational class of ϱ-τ algebras

Now, we study the identities which emphasize ties between the operation ϱ and τ defined in the Lindenbaum sentence algebra of Peano arithmetic. To this purpose, it is suitable to consider some relations between the formulas $\dot{T}(x)$ and $\dot{F}(x)$. First, we have the following preliminary lemmas:

LEMMA 1 (Essentially due to S. Feferman). *If p is probably equivalent to an RE-formula, then* $\vdash_{\mathscr{P}} p \to \dot{F}(\overline{p})$.

Proof. Let q be an RE-formula such that $\vdash_{\mathscr{P}} p \leftrightarrow q$, and let $Pr_{[Q]}(x)$ be the formula which numerates "in a natural way" the set of theorems of Robinson arithmetic. It is known that $\vdash_{\mathscr{P}} q \to Pr_{[Q]}(\overline{q})$ (see S. Feferman, [3]). Moreover, if $n = \max\{i$: *i is a Gödel number of an axiom of Robinson arithmetic*$\}$, we have $\vdash_{\mathscr{P}} Con_{\Pi/n}$, and hence $\vdash_{\mathscr{P}} \forall x[\mathscr{P}r_{[Q]}(x) \to \dot{F}(x)]$, and $\vdash_{\mathscr{P}} q \to \dot{F}(\overline{q})$. But, since $\vdash_{\mathscr{P}} q \leftrightarrow p$, we have $\vdash_{\mathscr{P}} \dot{F}(\overline{p}) \leftrightarrow \dot{F}(\overline{q})$ and thus we conclude $\vdash_{\mathscr{P}} p \to \dot{F}(\overline{p})$. Q.E.D.

COROLLARY 1. $\vdash_{\mathscr{P}} \dot{T}(\overline{p}) \to \dot{F}(\dot{T}(\overline{p}))$, *for every $p \in \mathscr{P}$.*

PROOF. Obvious, since $\dot{T}(\overline{p})$ is provably equivalent to an RE — formula.

LEMMA 2. (Essentially due to S. Feferman). *For every $p \in \mathscr{P}$, we have:*

(a) $\vdash_{\mathscr{P}} [Con_{\mathscr{P}} \wedge \dot{T}(\overline{p})] \to \dot{F}(\overline{p})$ and (b) $\vdash_{\mathscr{P}} \dot{F}(\overline{p}) \to \dot{T}(\overline{p})$.

PROOF. (a) From $\vdash_{\mathscr{P}} Con_{\mathscr{P}} \to \forall x\, Con_{\Pi/x}$, it follows that $\vdash_{\mathscr{P}} Con_{\mathscr{P}} \to \forall x[\Pi(x) \to \Pi^*(x)]$, whence we deduce $\vdash_{\mathscr{P}} Con_{\mathscr{P}} \to \forall x[\dot{T}(x) \to \dot{F}(x)]$ from which the first part of the claim easily follows

(b) We have successively $\vdash_{\mathscr{P}} \forall x[\Pi^*(x) \to \Pi(x)]$, $\vdash_{\mathscr{P}} \forall x[\dot{F}(x) \to \dot{T}(x)]$ and $\vdash_{\mathscr{P}} \dot{F}(\overline{p}) \to \dot{T}(\overline{p})$ for every sentence $p \in \mathscr{P}$. Q.E.D.

By the above lemmas, the operations ϱ and τ defined in the Lindenbaum sentence algebra of $\mathscr{P}$ satisfy the following identities:

ϱ-τ 1): $\varrho x \leqslant \tau x$

ϱ-τ 2): $\tau x \cdot \nu\tau 0 \leqslant \varrho x$

ϱ-τ 3): $\tau x \leqslant \varrho\tau x$

Moreover, ϱ satisfies ϱ 1), ϱ 2), ϱ 3) and τ satisfies the identities

1) $\tau 1 = 1$
2) $\tau(x \cdot y) = \tau x \cdot \tau y$
3) $\tau(\tau x \to x) = \tau x$

(See R. Magari, [8])[6].

These remarks lead us to consider the equational class of Boolean algebras enriched with two unary operations, ϱ and τ, satisfying the above identities. We call these algebras ϱ-τ algebras.

THEOREM 2. *In every ϱ-τ algebra the following properties hold*:

a) $\tau x = \varrho x + \tau 0$ [7]
b) If $x \geqslant \tau 0$, then $\varrho x = \tau x$
c) $\varrho(\tau x \to x) = \varrho x$
d) $\varrho(\nu \tau 0) = 0$

PROOF. a). From $\varrho x \leqslant \tau x$ and $\tau 0 \leqslant x$, we have $\tau x \geqslant \varrho x + \tau 0$. Moreover, from ϱ-τ 2) it follows that $\tau x \cdot \nu \tau 0 + \tau 0 \leqslant \varrho x + \nu \tau 0$, and hence $\tau x \leqslant \varrho x + \tau 0$. Then $\tau x = \varrho x + \tau 0$.

b). Suppose $x \geqslant \tau 0$. From ϱ-τ3) we get $\tau 0 \leqslant \varrho \tau 0$, and consequently $\varrho x \geqslant \varrho \tau 0 \geqslant \tau 0$. Recalling a), we conclude that $\tau x = \varrho x + \tau 0 = \varrho x$.

c). Obviously, $\varrho(\tau x \to x) \geqslant \varrho x$; it suffices to prove $\varrho(\tau x \to x) \leqslant \varrho x$, or, equivalently, $\nu \varrho(\tau x \to x) + \varrho x = 1$. First, we have $\nu \varrho(\tau x \to x) + \varrho x \geqslant \nu \tau(\tau x \to x) + \varrho x = \nu \tau x + \varrho x$. Moreover, $\varrho(\tau x \to x) \leqslant \varrho \tau x \to \varrho x = \nu \tau^2 x + \varrho x$. Thus, $\nu \varrho(\tau x \to x) + \varrho x \geqslant \tau^2 x \cdot \nu \varrho x + \varrho x = \tau^2 x$, so $\nu \varrho(\tau x \to x) + \varrho x \geqslant \nu \tau x + \varrho x + \tau^2 x = 1$.

d). The identity d) can be obtained from d) taking $x = 0$. Q.E.D.

REMARK. We note that from d) it follows that $\vdash_{\mathscr{P}} Con_{F} + \neg Con_{T}$ and hence one gets Theorem 5. 11 of S. Feferman's [3][8].

COROLLARY 2. *The ϱ-τ algebra $\mathfrak{R}_0$, freely generated on the empty set, is the Boolean algebra generated by $\tau 0, \tau^2 0 \ldots \tau^n 0 \ldots$, where τ is defined as in $\mathfrak{F}_0$* (see R. Magari [8]), *and ϱ is defined by*: $\varrho x = 0$ *if* $x \leqslant \nu \tau 0$; $\varrho x = \tau x$ *if* $x \geqslant \tau 0$.

PROOF. The elements of $\mathfrak{F}_0$ are the Boolean combinations of $\tau 0$, $\tau^2 0 \ldots \tau^n 0 \ldots$ and, for every $x \in \mathfrak{F}_0$, $x \geqslant \tau 0$, or $x \leqslant \nu \tau 0$ (See R. Magari, [8]). Hence, the claim easily follows from Theorem 2. Q.E.D.

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[6] We learn from a recent communication that $\mathscr{R}$. Soloway has shown that all the identities which hold in the diagonalizable algebra of $\mathscr{P}$ are consequences of τ1), τ 2), τ3).

[7] It can easily, be seen that the identity a) is equivalent to ϱ- 1) and ϱ- 2), (assuming the others identities), and hence these two identities can be replaced by a).

[8] It is also known that $_{\mathscr{P} \cup \{\neg Con_T\}}$ $Con_{T} + \neg Con_{T}$. Hence, there exist two formulas $\dot{T}'(x)$ and $F'(x)$ both of which binumerate in $\mathscr{P} \cup \{\neg Con_T\}$ this theory, and for which we have: $\vdash_{\mathscr{P} \cup \{\neg Con_T\}} \neg Con_{T''}$ and $\vdash_{\mathscr{P} \cup \{\neg Con_T\}} Con_{F'}$

4. The problems of introducing an operation with the properties of ϱ in any diagonalizable algebra and, conversely, an operation with the properties of τ in any ϱ algebra.

In the first part of this paragraph we try to extend any diagonalizable algebra to a ϱ-τ algebra, by defining a polynomial (or, eventually, an algebraic function) $f(x)$, if it is possible with x under the scope of τ, which has the same properties as ϱ. Let us observe that, if this were possible, then we should be able to construct a Rosser predicate starting from a Theor predicate in a very simple way. Unfortunately, we can not hope to satisfy the requirement that x appear in $f(x)$ under the scope of τ. In fact, such a polynomial (respectively: algebraic function) $f(x)$ must have at least two fixed points, 0 and 1, in every diagonalizable algebra: but this is in contradiction with the uniqueness theorem of C. Bernardi [2] and G. Sambin [13]. Nevertheless, if we remove the requirement that x appear in $f(x)$ under the scope of τ, the problem is solvable in every diagonalizable algebra, as is proved in the following theorem.

THEOREM 3. *Let $\mathscr{A}$ be a diagonalizable algebra. Then $\mathscr{A}$ can be extended to a ϱ-τ algebra if a new operation $f(x)$ is defined as follows: $f(x) = \tau x(x + \nu\tau 0)$. Moreover, if $g(x)$ is another operation having the same properties as ϱ, we have $f(x) = g(x)$ for every $x \geqslant \tau 0$ and for every $x \leqslant \nu\tau 0$.*

PROOF. It is easily seen that the identities ϱ 1), ϱ 2), ϱ 4) (and hence ϱ 3)), ϱ-τ 1), ϱ-τ 2), ϱ-τ 3) are satisfied by $f(x)$, whence we get the first part of Theorem 3. Now, let $g(x)$ be another operation which satisfies the above identities. Then, by parts b) and d) of Theorem 2, if $x \geqslant \tau 0$, $f(x) = g(x) = \tau x$, if $x \leqslant \nu\tau 0$ $f(x) = g(x) = 0$ Q.E.D.

COROLLARY 3. *For every x of $\mathfrak{R}_0$, $\varrho x = \tau x(\nu\tau 0 + x)$.*

PROOF. Obvious, since, for every $x \in \mathfrak{R}_0$, either $x \geqslant \tau 0$ or $x \leqslant \nu\tau 0$. Q.E.D.

REMARK 1. Let us observe that, taking $F(x) = \nu f \nu f(x)$, we obtain in $\mathfrak{R}_0$ (and hence also in $\mathfrak{F}_0$) the characteristic function for the filter generated by $\tau 0$. Namely, $F(x) = 1$ if $x \geqslant \tau 0$, $F(x) = 0$ if $x \leqslant \nu\tau 0$.

REMARK 2. In the ϱ-τ algebra of $\mathscr{P}$, the polynomial $f(x)$ is not identical to ϱx. In fact $\nu\varrho(x)$ has at least a fixed-point in the ϱ-τ algebra of $\mathscr{P}$, where $\nu f(x)$ has no fixed-point in a non trivial diagonalizable algebra; indeed, from $p = \nu f p = \nu\tau p + \nu p \cdot \tau 0$, it would follow $\nu p \cdot \tau 0 = 0$, whence $p = \nu\tau p$ and $p = \nu\tau 0$; but this would imply $\nu\tau 0 = \nu\tau 0 + \tau 0 \cdot \tau 0 = 1$, which is a contradiction because, by the second theorem of Gödel, in every non trivial diagonalizable algebra, $\nu\tau 0 \neq 1$.

Now, we investigate the opposite problem of extending any algebra to a ϱ-τ algebra by defining in it an algebraic function $f(x)$, if it is possible with x under the scope of ϱ, with the same properties as those of τ. In this way, we translate the logic problem of constructing a Theor predicate starting from a predicate analogous to Feferman's predicate. Let us start with the following definitions:

DEFINITION 1. Let $f(x)$ be an algebraic function with x under the scope of ϱ in a ϱ algebra $\mathscr{A}$. We say that $f(x)$ *numerates in $\mathscr{A}$ the set* $\{p \in \mathscr{A}: f(p) = 1\}$.

In the ϱ algebra of $\mathscr{P}$, such an algebraic function $f(x)$ is associated with a predicate $F(x)$ with exactly one free variable, and $f(x)$ numerates the set of the equivalence classes of propositions p such that $\vdash_{\mathscr{P}} F(\overline{p})$

DEFINITION 2. Every ϱ algebra $\mathscr{A}$ such that, for every n-tuple $p_1 \ldots \ldots p_n$ of elements of $\mathscr{A}$, we have $\varrho p_1 + \ldots + \varrho p_n = 1$ iff there exists a least an $i \leqslant n$ such that $p_i = 1$ *is called* ω — *consistent* $(+)$. Alsot every ϱ-τ algebra $\mathscr{B}$ such that, for every n-tuple $q_1 \ldots q_n$ of elements of $\mathscr{B}$, we have $\tau q_1 + \ldots + \tau q_n = 1$ iff there exists at least an $i \leqslant n$ such that, $q_i = 1$ *is called* ω — *consistent* $(++)$.

Let us note that, since $\varrho x \leqslant \tau x$, condition $(++)$ implies condition $(+)$ in every ϱ-τ algebra. Moreover, if $\mathscr{P}$ is ω-consistent, then both the ϱ-algebra and the ϱ-τ algebra of $\mathscr{P}$ are ω-consistent.

Because of the connections with the above logical problem, we require, for ω-consistent ϱ-algebras, that the algebraic function $f(x)$ with the properties of τ satisfy also the following condition: $\tau 4)$ $f(x)$ numerates $\{1\}$.

With regard to the ϱ algebra of $\mathscr{P}$, condition τ 4) corresponds to the requirement that the predicate $F(x)$ associated with $f(x)$ numerate the set of theorems of $\mathscr{P}$. Moreover, in the ϱ algebra of $\mathscr{P}$, the problem is solved on taking $fx = \varrho x + \tau 0$. This fact leads to search in every ϱ-algebra $\mathscr{A}$, for some element p such that the algebraic function $\varrho x + p$ satisfies identities τ 1), τ 2), τ 3), ϱ-τ 1), ϱ-τ 2), ϱ-τ 3) and eventually condition τ 4). Let us note that, on taking $p = 1$, $\varrho x + p$ satisfies the above identities, but not condition τ 4); on the other hand, there are ϱ algebras in which the only p such that $\varrho x + p$ verifies the above identities is 1. Clearly, this case is not of interest for our purposes.

The properties of the set of all such elements p are described in the following theorem:

THEOREM 6. *Let $\mathscr{A}$ be a ϱ algebra, and let $P = \{p \in \mathscr{A}: \varrho x + p$ satisfies $\tau 1)$, $\tau 2)$, $\tau 3)$, ϱ-$\tau 1)$, ϱ-$\tau 2)$, ϱ-$\tau 3)\}$. Then, P is closed under the operation $\cdot$. Moreover, if F is the filter generated by P, $\varrho F \subseteq P$. Finally, if $\mathscr{A}$ is ω-consistent and $p \in P$, $p \neq 1$, $\varrho x + p$ satisfies also condition $\tau 4)$.*

PROOF. Let p_1 and p_2 be elements of P. It is easily seen that the algebraic function $g(x) = \varrho x + p_1 p_2$ satisfies $\tau 1)$, $\tau 2)$, ϱ-$\tau 1)$, ϱ-$\tau 2)$. Let us prove that $g(x)$ satisfies ϱ-$\tau 3)$, that is $g(x) \leqslant \varrho g(x)$, or equivalently, $\varrho x + p_1 p_2$ $\varrho(\varrho x + p_1 p_2)$. By our hypothesis, $\varrho x + p_i \leqslant \varrho(\varrho x + p_i)$ $(i = 1, 2)$, and hence

$(\varrho x+p_1)\cdot(\varrho x+p_2) \leqslant \varrho(\varrho x+p_1)\cdot\varrho(\varrho x+p_2)$, from which the claim easily follows. Let us prove τ 3), that is $g(g(x)\to x) = g(x)$. This equality is equivalent to the following conditions: a) $g(x) \leqslant gg(x)$ and b) if $x \geqslant g(x)$, then $x = 1$ (See R. Magari, [9]). Condition a) is an obvious consequence of ϱ-τ3); now, let x be an element of $\mathscr{A}$ such that $x \geqslant \varrho x+p_1p_2$. We have $x \geqslant (\varrho x+p_1)(\varrho x+p_2)$, and hence $x+\nu\varrho x\cdot\nu p_1 \geqslant \varrho x+p_2$. From this, by property c) of theorem 2), we deduce $\varrho x = \varrho(\nu(\varrho x+p_1+x) = \varrho(\nu\varrho x\cdot\nu p_1 + x) \geqslant \varrho(\varrho x+p_2)$, whence $\varrho x+p_2 \geqslant \varrho(\varrho x+p_2)+p_2$, and $\varrho x+p_2 = 1$, since $p_2 \in P$ and condition b) holds for $\varrho x+p_2$. So, $\varrho x \geqslant \nu p_2$, and $x \geqslant \varrho x+p_1p_2 = \varrho x+\nu p_2+p_1p_2 = \varrho x+\nu p_2+p_1 \geqslant \varrho x+p_1$. Therefore, $x = 1$, because $p_1 \in P$.

Now, let q be an element of F. Then, there exists a $p \in P$ such that $q \geqslant p$, and hence $\varrho q \geqslant \varrho p$. Since $\varrho x+p$ satisfies ϱ-τ3), we have $\varrho 0+p \leqslant \varrho(\varrho 0+p)$, that is $p \leqslant \varrho p$, whence $\varrho q \geqslant \varrho p \geqslant p$. Let us set $g(x) = \varrho x + \varrho q$. Then, it is easily seen that $g(x)$ satisfies τ1), τ2), ϱ-τ1), ϱ-τ2). Moreover, from the formalization of Löb's theorem, for $\varrho x+p$, it follows that $\varrho(\nu\varrho x\cdot\nu p+x)+p = \varrho x+p$, whence $\varrho(\nu\varrho x\cdot\nu\varrho q+x)+\varrho q \leqslant \varrho(\nu\varrho x\cdot\nu p+x)+p+\varrho q = \varrho x+\varrho q$, that is τ3). So, we have $\varrho x+\varrho q \leqslant \varrho(\varrho x+\varrho q)+\varrho q$ (that is condition a) for $\varrho x+\varrho q$). Moreover, from ϱ-τ3) for $\varrho x+p$, it follows that $\varrho q \leqslant \varrho q+p \leqslant \varrho(\varrho q+p) = \varrho^2 q$ (since $p \leqslant \varrho q$), whence $\varrho q \leqslant \varrho^2 q \leqslant \varrho(\varrho q+x)$. Therefore, $\varrho x+\varrho q \leqslant \varrho(\varrho x+\varrho q)+\varrho q = \varrho(\varrho q+\varrho x)$.

Finally, let us note that if $\mathscr{A}$ is ω-consistent, $p \in P$, $p \neq 1$, then $\varrho x + p \leqslant \varrho x+\varrho p$. Thus, if $\varrho x+p = 1$, then $\varrho x+\varrho p = 1$, and hence, since $\mathscr{A}$ is ω-consistent and $p \neq 1$, $x = 1$. Q.E.D.

5. The fixed point problem in ϱ-τ algebras.

We recall that, in the equational class of diagonalizable algebras, the following fixed-point theorem holds: every polynomial $f(x)$ with x under the scope of τ admits a fixed point in every diagonalizable algebra. Furthermore, this fixed-point is unique. We can formulate the first part of this statement for ϱ-τ algebras as follows: every polynomial $f(x)$with x under the scope of ϱ or of τ admits a fixed-point in every ϱ-τ algebra. Nevertheless this statement can be strongly disproved, in the sense that it fails not only in the equational class of ϱ-τ algebras, but also in every equational class contained in it. Indeed, let us consider the algebra $\mathscr{A} = \langle\{0,1\}, +, \cdot, \nu, \varrho, \tau\rangle$ where obviously $\varrho 0 = 0$, $\varrho 1 = \tau 0 = \tau 1 = 1$. $\mathscr{A}$ is not only a ϱ-τ algebra, but it belongs to every equational class contained in it [9].

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[9] To prove this, let V^* be such an equational class, and $\mathscr{A}^*$ be a non trivial algebra in V^*. Then, denote by F the filter generated by $\tau 0$, and by $\sim$ the relation defined as follows: $x \sim y$ iff $x \leftrightarrow y \in F$. It is easily seen that this relation is a congruence relation. Moreover, since $\tau 0 \neq 0$ by Gödel's second theorem, F is a proper subset of $\mathscr{A}^*$, whence $\mathscr{A}^*/\sim$ is a non trivial algebra of V^* where $\varrho 0 = 0$, $\tau 0 = \varrho 1 = \tau 1 = 1$; we can conclude that $\mathscr{A}$ is a subalgebra of $\mathscr{A}^*/\sim$ and consequently an algebra of V^*.

It is easily seen that in this algebra the polynomial $\nu\varrho x$ has no fixed-point, whence the statement immediately follows.

In this paragraph we shall investigate some classes which admit a fixed-point in every ϱ-τ algebra. First, let us note that a large class of polynomials admits 0 (respectively: 1) as a fixed-point. We denote these classes by 0 and 1, respectively. The following facts are easily proved:

(1) *The identical polynomial and ϱx are both in* 0 *and in* 1.

(2) τx *is in* 1.

(3) *If gx is an arbitrary polynomial and $f(x)$ is in* 0 (*respectively: in* 1) *then $f(x)\cdot g(x)$ is in* 0 (*respectively: $f(x)+g(x)$ is in* 1)

(4) *If $f(x)$ and $g(x)$ are in* 0 (*respectively: in* 1), *then $f(x)+g(x)$ is in* 0 (*respectively: $f(x)\cdot g(x)$ is in* 1).

(5) *If $f(x)$ and $g(x)$ are in* 0 (*respectively: in* 1), *then $f(g(x))$ is in* 0 (*respectively: in* 1).

(6) *For every polynomial $g(x)$, $g(x)$ is in* 0 *iff $\nu g\nu x$ is in* 1.

(7) *If $f(x)$ is in $0\cap 1$, then $\nu f(x)$ has no fixed-point in the algebra $\mathscr{A} = \langle\{0,1\}, +, \cdot, \nu, \varrho, \tau\rangle$.*

(8) *If $f(x)$ is a ϱ polynomial, then $f(x)$ or $\nu f(x)$ is in* 0 *and $f(x)$ or $\nu f(x)$ is in* 1.

Moreover, we have:

THEOREM 7. *The following classes of polynomials have a unique fixed-point in every ϱ-τ algebra:*

a) *The class of ϱ-τ polynomials $f(x)$ with x under the scope of ϱ or of τ, such that $f(x) = g(x)+\tau 0$ for some $g(x)$.*

b) *The class of ϱ-τ polynomials $f(x)$ with x under the scope of ϱ or of τ, such that $f(x) = g(x)\cdot\nu\tau 0$ for some $g(x)$.*

Further, this fixed-point is an element of $\mathfrak{R}_0$.

PROOF. First, let us prove the existence of such a fixed-point. Assume $f(x)$ is in the class a). Let p be a fixed-point of $f(x)$ in the ϱ-τ algebra of Peano arithmetic (which exists by diagonalization lemma). Clearly, $p = fp \geqslant \tau 0$ in this algebra. Now let n be the number of occurrences of ϱ in f and let $\varrho h_1(x)$ be a subpolynomial of $f(x)$ such that ϱ does not occur in $h_1(x)$. By induction on the structure of $h_1(x)$, it is easily seen that either $h_1(p)\geqslant \tau 0$ or $h_1(p)\leqslant \nu\tau 0$; in the case, $\varrho\ h_1p = \tau h_1 p$, in the other one, $\varrho h_1 p = 0$. By replacing $\varrho h_1 x$ with $\tau h_1 x$ (respectively: with 0) in (fx), we obtain a polynomial $f_1(x)$ with $n-1$ occurrences of ϱ, such that $p = f_1 p = fp$. By repeating this procedure n times, we obtain a polynomial $f^*(x)$ with x under the scope of τ and without an occurrence of ϱ, such that $p = f^* p = fp$. By the theorems about existence and uniqueness of a fixed-point in diagonalizable algebras (see C. Bernardi, [1], [2] and G. Sambin, [13]) p is in $\mathfrak{R}_0$ and hence in every ϱ-τ algebra. Moreover, let us note that p and fp are elements of $\mathfrak{R}_0$, and $\mathfrak{R}_0$ is a subalgebra of the ϱ-τ algebra of $\mathscr{P}$. So, since in such algebra $p = fp$, this identity holds also in $\mathfrak{R}_0$, and hence in every ϱ-τ algebra.

Assume that $f(x)$ belongs to the class b). Let p the fixed-point of $f(x)$ in the ϱ-τ algebra of $\mathscr{P}$. We have $p = f(p) \leqslant \nu\tau 0$, hence, proceeding as above, we find a polynomial $f^*(x)$ with x under the scope of τ such that $p = f^* p = fp$. Then, an argument analogous to the one above allows us to conclude that p is in every ϱ-τ algebra and $p = fp$ in every ϱ-τ algebra.

Finally, we must prove that the fixed-point is unique. Let us observe that the construction of $f^*(x)$ does not depend on p but only on $f(x)$. Moreover, if p is a fixed-point for $f(x)$, then, by the above arguments, $p = f^* p = fp$, so p is also a fixed-point for $f^*(x)$. Therefore p is unique, by the uniqueness theorems of fixed-point in diagonalizable algebras (see C. Bernardi, [2]; and G. Sambin, [13]). Q.E.D.

Corollary 4. *If $f(x)$ is a polynomial with x under the scope of τ then $f(x)$ admits a unique fixed-point in every ϱ-τ algebra.*

Proof. By theorem 7 it sufficies to show that either for every $x, f(x) \geqslant \tau 0$ (whence $f(x) = f(x) + \tau 0$) or, for every x, $f(x) \leqslant \nu\tau 0$, (that is, $f(x) = f(x) \cdot \nu\tau 0$). Since x appears in $f(x)$ under the scope of τ, there exist a ϱ polynomial $h(x_1 \dots x_n)$ and n polynomials $f_1(x), \dots f_n(x)$ which begin with a τ, such that $f(x) = h\big(f_1(x) \dots f_n(x)\big)$.

Now, let us prove the claim by induction on the structure of h. Obviously, $f_i(x) \geqslant \tau 0$ $(i = 1 \dots n)$. If $h_1(x)$ and $h_2(x)$ are both $\geqslant \tau 0$, then also $h_1(x) \cdot h_2(x) \geqslant \tau 0$; if $h_1(x) \leqslant \nu\tau 0$, or $h_2(x) \leqslant \nu\tau 0$, then $h_1(x) \cdot h_2(x) \leqslant \nu\tau 0$; further, if $k(x) \geqslant \tau 0$ (respectively: $k(x) \leqslant \nu\tau 0$), then $\nu k(x) \leqslant \nu\tau 0$ (respectively $\nu k(x) \geqslant \tau 0$). Finally, if $k(x) \geqslant \tau 0$, $\varrho k(x) = \tau k x \geqslant \tau 0$, and, if $k(x) \leqslant \nu\tau 0$, then $\varrho k(x) = 0 \leqslant \nu\tau 0$. Q.E.D.

6. Duality theory for ϱ-τ algebras

We recall that, if $\mathscr{A}$ is a ϱ-τ algebra, then $\tilde{\varrho} \equiv \nu\varrho\nu$ and $\sigma \equiv \nu\tau\nu$ are hemimorphisms from $\mathscr{A}$ to $\mathscr{A}$, in the sense of P. R. Halmos, [5], that is, $\tilde{\varrho}$ and σ are mappings from $\mathscr{A}$ to $\mathscr{A}$ such that $\tilde{\varrho}0 = 0$, $\tilde{\varrho}(x+y) = \tilde{\varrho}(x) + \tilde{\varrho}(y)$; $\sigma 0 = 0$, $\sigma(x+y) = \sigma(x) + \sigma(y)$ $(x, y \in \mathscr{A}.)$ Hence, they are associated with binary relations, $>_\varrho$ and $>_\tau$, in the dual space $\hat{\mathscr{A}}$[10] of $\mathscr{A}$ (see P. R. Halmos, [5]). In order to simplify our notation, let us consider the inverse relations, $<_\varrho$ and $<_\tau$; they are defined by $x <_\varrho y$ iff $\varrho x \leqslant y$ and $x < y$ iff $\sigma x \subseteq y$[11] for every $x, y \in \hat{\mathscr{A}}$. Moreover, we have, for every clopen subset X of $\hat{\mathscr{A}}$: $\tilde{\varrho} X = \{x \in \hat{\mathscr{A}} : \exists y \in X : y <_\varrho x\}$ and $\sigma X = \{x \in \hat{\mathscr{A}} : \exists y \in X : y <_\tau x\}$. In the sequel, by "relation associated with ϱ" (respectively: with τ), we shall mean the above defined relation $<_\varrho$ (respecti-

[10] The dual space of $\mathscr{A}$ is the pair $\langle \hat{\mathscr{A}}, T \rangle$, where T is a suitable topology on $\hat{\mathscr{A}}$. However, we shall often identify such dual space with the set $\hat{\mathscr{A}}$.

[11] Here, $\hat{\mathscr{A}}$ is the set of all ultrafilters of $\mathscr{A}$; hence x, y denote ultrafilters of $\mathscr{A}$, and $\hat{\varrho} x$, σx are the sets $\{\hat{\varrho} p : p \in x\}$ and $\{\sigma p : p \in x\}$ respectively.

vely: $<_\tau$). Conversely, by "operation associated with $<_\varrho$ (respectively: $<_\varrho$)", we shall mean the operation $\varrho \equiv \nu\tilde{\varrho}\nu$ (respectively: $\tau \equiv \nu\sigma\nu$), where $\tilde{\varrho}$ and σ are defined as above. Finally, we recall that the relation $<_\tau$ is transitive and relatively founded (see R. Magari, [9]). In the following theorem, we characterize the properties of $<_\varrho$ and the ties between $<_\varrho$ and $<_\tau$.

THEOREM 8. *Let $\mathscr{A}$ be a Boolean algebra with two unary operations, ϱ and τ, such that $\tilde{\varrho} \equiv \nu\varrho\nu$ and $\sigma \equiv \nu\tau\nu$ are hemimorphisms. Then $\mathscr{A}$ is a ϱ-τ algebra iff all the following conditions are satisfied:*

(1) *$<_\tau$ is transitive and relatively founded*

(2) *$<_\varrho$ is not founded, and, for every $x \in \hat{\mathscr{A}}$, there is a $y \in \hat{\mathscr{A}}$ such that $y <_\varrho x$*

(3) *$<_\tau \subseteq <_\varrho$, and $<_\varrho - <_\tau \subseteq \tau 0 \times \tau 0$, that is, if $x <_\varrho y$ but $x \not<_\tau y$, then $x, y \in \tau 0$.*

PROOF. Let $\mathscr{A}$ be a ϱ-τ algebra; then (1) has been proved by R. Magari in [9]. Moreover, from $\varrho 0 = 0$ it follows that $\tilde{\varrho} 1 = 1$, that is $\{x : \exists y : y <_\varrho x\} = \hat{\mathscr{A}}$, and hence condition (2). Now, suppose $x <_\tau y$; then $\sigma x \subseteq y$, and hence, for every $X \in x$, $\sigma X \in y$[12]. But, from $\varrho X \leqslant \tau X$ it follows that $\sigma X \leqslant \tilde{\varrho} X$, and we obtain $\tilde{\varrho} X \in y$ since y is a filter. Therefore $\tilde{\varrho} x \subseteq y$, and we can conclude that $x <_\varrho y$. In order to prove the second part of (3) it is useful to establish the following result:

LEMMA 3. *For every $x, y, z \in \hat{\mathscr{A}}$, if $x <_\tau y$, $y <_\varrho z$, then $x <_\tau z$. In particular, if $x \in \tau 0$, $y \in \nu\tau 0$, then $y \not<_\varrho x$.*

PROOF OF LEMMA 3. If $x <_\tau y$, $y <_\varrho z$, then, by the definition of $<_\varrho$ and $<_\tau$, we have $\sigma x \subseteq y$ and $\tilde{\varrho} y \subseteq z$. Then, it suffices to show $\sigma x \subseteq z$, that is, for every $X \in x$, $\sigma X \in z$. Now let X be an element of x; since $\sigma x \subseteq y$, $\sigma X \in y$, and since $\tilde{\varrho} y \subseteq z$, $\tilde{\varrho}\sigma X \in z$. But $\tau X \leqslant \sigma\tau X$ implies $\tilde{\varrho}\sigma X \leqslant \sigma X$, and then $\sigma X \in z$; because z is a filter. Moreover, if $x \in \tau 0 = \nu\sigma 1$, and $y \in \nu\tau 0 = \sigma 1$, then, for every $z \in \hat{\mathscr{A}}$, $z \not<_\tau x$, whereas there exists a z such that $z <_\tau y$. Then, $y <_\varrho x$ would imply, by the first part of Lemma 3), $z <_\tau x$, a contradiction. This completes the proof of Lemma 3).

Let us return to the proof of theorem 8. Suppose $x <_\varrho y$, $x \not<_\tau y$. Then, there is an $X \in x$ such that $\tilde{\varrho} X \in y$, but $\sigma X \notin y$. But from $\tau X = \varrho X + \tau 0$ it follows that $\sigma X = \tilde{\varrho} X \cdot \sigma 1$, and hence $\sigma 1 \notin y$ (indeed, if this is not the case, $\sigma X = \tilde{\varrho} X \cdot \sigma 1 \in y$, since y is a filter. So, $\nu\sigma 1 = \tau 0 \in y$, because y is an ultrafilter, and $y \in \tau 0$; since $x <_\varrho y$, we also have $x \in \tau 0$, by the second part of Lemma 3).

Conversely, let us assume that the relations $<_\varrho$ and $<_\tau$, associated with ϱ and τ respectively, satisfy (1), (2), (3). It is known that τ satisfies

[12] In the sequel we identify every element $p \in \mathscr{A}$ with the set X of all $x \in \hat{\mathscr{A}}$ such that $p \in x$. In this sense, we shall say indifferently $x \in X$ or $X \in x$.

$\tau 1$), $\tau 2$), $\tau 3$) (see R. Magari, [9]). Moreover, $\varrho 1$) and $\varrho 3$) directly follow from the fact that $\nu\varrho\nu$ is a hemimorphism. We have also $\tilde{\varrho}1 = \{x: \exists y \in \hat{\mathscr{A}}, y <_\varrho x\} = \hat{\mathscr{A}}$ (by (2)), that is $\tilde{\varrho}1 = 1$, and hence $\varrho 0 = 0$. Furthermore, $\tilde{\varrho}X = \{x: \exists y \in X: y <_\varrho x\} \supseteq \{x: \exists y \in X: y <_\tau x\}$, since $<_\varrho \subseteq <_\tau$, and hence $\tilde{\varrho}X \geqslant \sigma X$, which implies $\varrho X \leqslant \tau X$. Now, let us prove ϱ-$\tau 2$), that is $\sigma X + \nu\sigma 1 \geqslant \tilde{\varrho}X$; this is equivalent to $\tilde{\varrho}X \cdot \nu\sigma X \leqslant \nu\sigma 1$. Assume $x \in \tilde{\varrho}X$, $x \notin \sigma X$; then, there is a $y \in X$ such that $y <_\varrho x$ and $y \nless_\tau x$. But, by (3), it follows $x, y \in \tau 0 = \nu\sigma 1$. Thus, we have $\tilde{\varrho}X \cdot \nu\sigma X \leqslant \nu\sigma 1$.

Finally, let us prove ϱ-$\tau 3$), that is $\tilde{\varrho}\sigma X \leqslant \sigma X$. If $x \in \tilde{\varrho}\sigma X = \{u: \exists y \in \sigma X: y <_\varrho u\} = \{u: \exists y \exists z: z \in X, z <_\tau y <_\tau y$ and $y <_\varrho u\}$, then there exists a $z \in X$ and a y such that $z <_\tau y$ and $y <_\varrho x$. Now, we have also $y <_\tau x$, because otherwise, by (3), $x, y \in \tau 0$, which is in contradiction with $z <_\tau y$. Since $<_\tau$ is transitive, $z <_\tau x$, and $x \in \sigma X$. Q.E.D.

REMARK. Generally, relations $<_\varrho$ and $<_\tau$ which induce the operations ϱ and τ are not unique. The conditions (1) and (3) of theorem 8 are necessary only in the case when the relations $<_\varrho$ and $<_\tau$ are defined in the standard way[13]. As known there are relations $<_\varrho$ and $<_\tau$ defined in the dual space $\hat{\mathscr{A}}$ of $\mathscr{A}$, which induce the operations ϱ and τ, but do not satisfy (1) and (3). In [9], R. Magari presents a counterexample to (1).

To give a counterexample to (3), let us consider the set $\omega + 1$ in which a topology is defined by taking as clopen sets all finite subsets of ω and their complements in $\omega + 1$. Define $x <_\tau^* y$ iff $x < y$ or $x = y = \{\omega\}$ and $x <_\varrho^* y$ iff $x < y$ or $x = y = 0$, $(x, y \in \omega + 1)$. It is easily seen that the dual algebra of this space is $\mathfrak{R}_0$; in fact, the dual space of $\mathfrak{R}_0$ is the topological space defined above, $<_\tau = <_\tau^*$, and $<_\varrho^* \subseteq <_\varrho$ and $<_\varrho - <_\varrho^* = \langle\{\omega\}, \{\omega\}\rangle$. Thus, for every clopen set X, we have $\varrho X \leqslant \varrho^* X$ (where ϱ^* and ϱ are the operations associated with $<_\varrho^*$ and $<_\varrho$ respectively), and $\varrho^* X - \varrho X \subseteq \subseteq \{\{\omega\}\}$. Since $\{\{\omega\}\}$ is not clopen, whereas $\varrho^* X - \varrho X$ is, we conclude $\varrho^* X = \varrho X$; moreover, $\tau X = \tau^* X$. Hence, the dual algebra is the ϱ-τ algebra $\mathfrak{R}_0$, but $<_\varrho^*$ and $<_\tau^*$ do not satisfy (2), because $\langle\{\omega\}, \{\omega\}\rangle \in <_\tau^* - <_\varrho^*$. If, however, in the dual space of $\mathscr{A}$, every point is a clopen set, then every pair of relations $<_\varrho, <_\tau$ which induce the operations ϱ and τ, satisfy (1), (2), (3). The part of the statement concerning (1) has been proved by R. Magari, [9]; (2) can be obtained exactly as in Theorem 8. Now, let us prove (3); suppose $x <_\tau y$. Then $y \in \sigma\{x\}$, but $\sigma\{x\} \leqslant \tilde{\varrho}\{x\}$, whence $y \in \tilde{\varrho}\{x\} = \{u: \exists z \in \{x\}: z <_\varrho u\} = \{u: x <_\varrho u\}$. So we have $x <_\varrho y$ and $<_\tau \subseteq <_\varrho$. Now, let us assume $x <_\varrho y$ and $x \nless_\tau y$. From this it follows that $y \in \tilde{\varrho}\{x\}$, $y \notin \sigma\{x\}$. But from $\sigma X = \tilde{\varrho}X \cdot \sigma 1$ we obtain $\sigma\{x\} = \tilde{\varrho}\{x\} \cdot \sigma 1$, whence $y \notin \sigma 1$, and, consequently, $y \in \tau 0$. Now, suppose, contrary to our claim, $x \in \nu\tau 0 = \sigma 1$. Then, there exists a z such that $z <_\tau x$,

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[13] That is, they are defined by $x <_\varrho y$ iff $\tilde{\varrho}x \subseteq y$, and $x <_\tau y$ iff $\sigma x \subseteq y$.

whence $x \in \sigma\{z\}$ and $y \in \tilde{\varrho}\sigma\{z\}$. But $\tilde{\varrho}\sigma\{z\} \subseteq \sigma\{z\}$; it would follow that $y \in \sigma\{z\}$, $z <_\tau y$ and $y \in \nu\tau 0$, which is a contradiction.

We conclude this paragraph by giving two examples of ϱ-τ algebras, defined by their dual spaces:

a) Let $\mathbf{Z}$ be the set of integers, and $\mathbf{Z}^*$ be the topological space $\langle \mathbf{Z}, \mathscr{P}(\mathbf{Z})\rangle$. Let ω be the set of natural numbers, $-\omega$ the set of negative integers, $<$ the usual ordering on $\mathbf{Z}$.

Let us define two relations $<_\tau$ and $<_\varrho$ as follows:

$$x <_\tau y \text{ iff } x \in -\omega \text{ and } y \in \omega \text{ or } x, y \in \omega \text{ and } y < y$$

$$x <_\varrho y \text{ iff } x <_\tau y \text{ or } x, y \in -\omega, y = x+1$$

Since $<_\varrho$ and $<_\tau$ satisfy conditions (1), (2), (3) of theorem 8, the dual algebra is a ϱ-τ algebra. Moreover, we have

$$\tau X = \begin{cases} -\omega \text{ if } X \not\supseteq -\omega \\ x: x \leqslant \mu y: y \notin X\} \text{ if } X \supseteq -\omega \end{cases}$$

$$\varrho X = \begin{cases} \tau X \text{ if } X \supseteq -\omega \\ -\omega \cap \{x+1: x \in X\} \text{ if } X \not\supseteq -\omega \end{cases}$$

It is easily seen that the sets $P = \omega \cup \{-2n: n \in \omega\}$ and $P' = \omega \cup \{--2n-1: n \in \omega\}$ are such that $P = \nu\varrho P, P' = \nu\varrho P'$. Moreover, the sets $P_n = \omega \cup \bigcup_{m\in\omega} \{x: -(2m+2)n \leqslant x \leqslant -(2m+1)n\}$ are such that $\nu\varrho^n P_n = P_n$.

b) Let $\mathbf{Z}^*$ be the topological space of example a). Define $<_\tau$ as in a) and $<_\varrho$ as follows:

$$x <_\varrho y \text{ iff } x <_\tau y \text{ or } x, y \in -\omega \text{ and } y = x+1 \text{ or } x = y = -2^n \text{ for some } n > 0.$$

Since $<_\varrho$ and $<_\tau$ satisfy conditions (1), (2), (3) of theorem 8, the dual algebra is a ϱ-τ algebra. Moreover, τ is defined as in a) and

$$\varrho X = \begin{cases} \tau X \text{ if } X \geqslant -\omega \\ [-\omega \cap \{x+1: x \in X\}] - \{-2^n: -2^n \notin X, n > 0\} \text{ if } X \not\supseteq -\omega \end{cases}$$

If $P_n = \{x: x < -2^n\}$ for some $n > 0$, we have $\varrho P_n = P_n$. Hence, the polynomial ϱX has infinitely many fixed-points in the algebra. Moreover, if $P = \omega \cup \{-2n: n \in \omega\}$, we have $P = \nu\varrho P$, and $\varrho^{2n+2}P < \varrho^{2n}P$, $\varrho^{2n+1}P < \varrho^{2n-1}P$, $\bigcap_{n\in\omega-0} \varrho^{2n}P = \bigcap_{n\in\omega-0} \varrho^{2n-1}P = \emptyset$, $\varrho^n P \neq \emptyset$. Thus, the elements $\varrho^n P$ and their complements constitute two chains ordered by $<$, both isomorphic to $\mathbf{Z}$ (see paragraph 2)).

Open Problems

1. In the ϱ-τ algebra $\mathscr{P}^*$ of $\mathscr{P}$, is there a unique p such that $p = \nu\varrho p$? In other words, is it true that if $\vdash_{\mathscr{P}} p \leftrightarrow \vdash \dot{F}(\bar{p})$ $\vdash_{\mathscr{P}} q \leftrightarrow \neg \dot{F}(\bar{q})$ then $\vdash_{\mathscr{P}} p \leftrightarrow q$? Let us note that in the algebra a) of paragraph 6) there are at least two elements P and P' such that $P = \nu\varrho P$ and $P' = \nu\varrho P'$; hence an eventual uniqueness theorem cannot be obtained from the identities of ϱ-τ algebras.

2. Is it true that in the ϱ-τ algebra $\mathscr{P}^*$ of $\mathscr{P}$, if $\varrho p = p$, then either $p = 0$ or $p = 1$? In other words, is it true that if $\vdash_{\mathscr{P}} p \leftrightarrow \dot{F}(\overline{p})$ then either $\vdash_{\mathscr{P}} p$ or $\vdash_{\mathscr{P}} \neg p$? Also this property does not follow from the identities of ϱ-τ algebras since in the algebra b) of paragraph 5 the polynomial ϱx admits infinitely many fixed-points. Moreover, an analogous of Löb's theorem, that is: if $\varrho p \leqslant p$, then either $p = 0$ or $p = 1$, does not hold in the ϱ-τ algebra of $\mathscr{P}$. Indeed, $\varrho\nu\tau 0 = 0 \leqslant \nu\tau 0$, but $0 < \nu\tau 0 < 1$.

3. Find new identities (or prove that it is not possible) which hold in the ϱ-τ algebra $\mathscr{P}^*$ of $\mathscr{P}$, but do not follow from the identities of ϱ-τ algebras.

4. Find minimal conditions for a ϱ-τ algebra to have at least a fixed-point for every polynomial $f(x)$ with x under the scope of ϱ or of τ. As it has been shown, such conditions can not be completely expressed by identities.

5. Is the set of identities of ϱ-τ algebras decidable?

6. Study the properties of the set P of elements of $\mathscr{P}^*$ such that, assuming $\tau_p(x) = \varrho x + p$, ϱ and τ_p satisfy the identities of ϱ-τ algebras. By the results of paragraph 4, if $p \geqslant \tau 0$, then $\varrho p \in P$. Hence $\tau 0, \tau^2 0 \ldots \tau^n 0 \ldots$ are in P. One may expect that in P there are elements $p < \tau 0$ and that P does not have a least (with respect to $<$) element. Since P is closed with respect to $\cdot$, if this is the case there would be a decreasing chain $\ldots p_{n+1} < p_n < \ldots < \tau 0$ of elements of P, and hence a decreasing chain $\ldots \tau_{p_{n+1}}(x) < \tau_{p_n}(x) < \ldots < \tau(x)$ of operations which are associated with arithmetical predicate of theor. type.

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