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# ASYMPTOTIC SOLUTION FOR A NEW CLASS OF FORBIDDEN $r$-GRAPHS

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We consider the problem of finding $\text{ex}(n; G)$, defined as the maximal number of edges an $r$-graph on $n$ vertices can have that contains no subgraph isomorphic to $G$. We construct certain $r$-graphs $G$ for which we find the coefficient $\tau(G)$ of the asymptotic expansion $\text{ex}(n; G) = (\tau(G)+o(1))\binom{n}{r}$ as $n\to\infty$.

## 1. Basic definitions and notation

Let $V$ be a finite set and $E$ be a family of its $r$-tuples ($r\geqq 2$). The pair $(V, E)$ is said to be an *$r$-graph with vertex set $V$ and edge set $E$*. Let $\mathcal{G}^r$ denote the class of all $r$-graphs. We will write $V(G)$ and $E(G)$, respectively, for the set of vertices and edges of the graph $G$. Set further $v(G)=|V(G)|$ and $e(G)=|E(G)|$.

Let $G, H\in\mathcal{G}^r$. The $r$-graph $G$ is a *subgraph* of the $r$-graph $H$ if $V(G)\subseteqq V(H)$ and $E(G)\subseteqq E(H)$. If, in addition, $E(G)=\{a\in E(H) | a\subseteqq W\}$ and $V(G)=W$, then $G$ is a *subgraph* of $H$ *spanned* by the subset $W$ of the vertices.

Let $G, H\in\mathcal{G}^r$. A map $\varphi\colon V(G)\to V(H)$ is a *homomorphism* from $G$ to $H$ if $\{v_1, \dots, v_r\}\in E(G)$ implies $\{\varphi(v_1), \dots, \varphi(v_r)\}\in E(H)$. The map $\varphi$ is called a *monomorphism* if it is injective. If $\varphi$ is bijective and its inverse is a *homomorphism,* then it is an *isomorphism*. In this case the $r$-graphs $G$ and $H$ are *isomorphic* (or, one of them is an *isomorphic* copy of the other). In what follows we will not make distinction between isomorphic copies of an $r$-graph, provided this would not cause any confusion.

We consider two binary relations on $\mathcal{G}^r$: $G\prec H$ means that there is a homomorphism from $G$ to $H$, and $G\leqq H$ means that there is a monomorphism from $G$ to $H$. It is easy to see that both of these relations are transitive.

The notions "subgraph" and "monomorphism of $r$-graphs" are closely related: $G\leqq H$ if and only if $G$ is isomorphic to a subgraph of the $r$-graph $H$. The notion of "homomorphism of $r$-graphs" is less traditional: $G\prec H$ if and only if identifying certain vertices of $G$ one can obtain an $r$-graph, isomorphic to a subgraph of $H$. In fact, one can identify only those vertices that are incident to distinct edges, for otherwise loops will turn up.

Let $G$ be an $r$-graph with vertices $v_1, \dots, v_n$ and edge set $E$. We denote by $G(m_1, \dots, m_n)$ the $r$-graph obtained from $G$ by repeating every vertex $v_i$ $(i=1, \dots, n)$

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$m_i$ times. More precisely, $G(m_1, \ldots, m_n)$ is the $r$-graph with vertex set $X_1 \cup \ldots \cup X_n$ where $X_1, \ldots, X_n$ are paiwise disjoint sets, $|X_i| = m_i$, and the set of edges is

$$\bigcup_{\{v_{i_1}, \ldots, v_{i_r}\} \in E} A_{i_1, \ldots, i_r},$$

where $A_{i_1, \ldots, i_r} = \{\{x_1, \ldots, x_r\} | x_1 \in X_{i_1}, \ldots, x_r \in X_{i_r}\}$. It is easy to see that $H \prec G$ if and only if there are numbers $m_1, \ldots, m_r$ such that $H \leqq G(m_1, \ldots, m_r)$. It is also clear that $G(m_1, \ldots, m_r) \prec G$ for all $m_1, \ldots, m_r$.

An $r$-graph is *complete* if every $r$-tuple of its vertices is an edge. $K_k^r$ will denote the complete $r$-graph on $k$ vertices. An $r$-graph $G$ is *nonempty* if $E(G)$ is nonempty.

A pair of vertices of an $r$-graph is called *covered* if it has an edge containing these two vertices, it is called *uncovered* otherwise. An $r$-graph is *covering* if any pair of its vertices is covered. Observe that if $G$ is a covering $r$-graph, then every homomorphism from $G$ to $H$ is a monomorphism.

The *extension* $\tilde{H}$ of an $r$-graph $H$ is the $r$-graph obtained from $H$ by adding $r-2$ new vertices and a new edge for each uncovered pair: the new edge is formed by the new vertices plus the pair. In this way, $(r-2)l$ new vertices and $l$ new edges are added to $H$ if there were $l$ uncovered pairs in it. Clearly, for a covering $r$-graph $G$ one has $\tilde{H} \prec G$ if and only if $H \leqq G$.

A class of $r$-graphs $\mathscr{H} \subseteqq \mathscr{G}^r$ is called *hereditary (strongly hereditary)* if $G \leqq H$ ($G \prec H$, respectively) and $H \in \mathscr{H}$ imply $G \in \mathscr{H}$. Let $\mathscr{A} \subseteqq \mathscr{G}^r$ and set

$$Z(\mathscr{A}) = \{G \in \mathscr{G}^r | \forall H \in \mathscr{A}: H \not\leqq G\}$$

$$\hat{Z}(\mathscr{A}) = \{G \in \mathscr{G}^r | \forall H \in \mathscr{A}: H \not\prec G\}.$$

It is easy to see that $Z(\mathscr{A})$ is hereditary and $\hat{Z}(\mathscr{A})$ is strongly hereditary. On the other hand, all hereditary (strongly hereditary) classes are of this form. Indeed, it follows from the transitivity of the relations $\leqq$ and $\prec$ that for every hereditary (strongly hereditary) class of $r$-graphs $\mathscr{H}$ one has $\mathscr{H} = Z(\mathscr{G}^r \setminus \mathscr{H})$ ($\mathscr{H} = \hat{Z}(\mathscr{G}^r \setminus \mathscr{H})$, respectively). Observe that if every $r$-graph in $\mathscr{A}$ is covering, then $Z(\mathscr{A})$ and $\hat{Z}(\mathscr{A})$ coincide. Moreover, it can be shown that any strongly hereditary class $\mathscr{H}$ of $r$-graphs can be represented in the form $\mathscr{H} = Z(\mathscr{A})$ where $\mathscr{A}$ consists of covering $r$-graphs only.

Below we will consider only those hereditary classes $\mathscr{H}$ that contain, for all $n$, at least one $r$-graph on $n$ vertices. This is equivalent to saying that the set $\mathscr{A}$ in the representation $\mathscr{H} = Z(\mathscr{A})$ (or $\mathscr{H} = \hat{Z}(\mathscr{A})$) consists of nonempty $r$-graphs.

A lot of discrete extremal problems can be reduced to this question: Find $e_n(G) = \max\{e(G) | G \in \mathscr{H}, v(G) = n\}$ where $\mathscr{H}$ is a hereditary class of $r$-graphs. Let $\mathscr{A}$ be a set of $r$-graphs; the *coefficient of saturation* for this class is defined as the limit

$$\tau(\mathscr{A}) = \lim_{n \to \infty} e_n(Z(\mathscr{A})) \binom{n}{r}^{-1}$$

(it is well-known [8] that the expression after the lim sign does not increase as $n \to \infty$).

The case $r=2$ is that of the ordinary graphs. The usual terminology will be used for them: e.g., tree, forest, path, connected component, degree of a vertex, etc.

When $r=2$, the coefficient of saturation is given by a well-known theorem of Erdős and Simonovits [6] (cf. Corollary 2.7 below). When $r \geqq 3$, there is only one complete result: Bollobás [1] found $e_n$ for the class of 3-graphs where no edge is contained in the symmetric difference of two others. In this paper we are going to find the coefficient of saturation $\tau(\{H\})$ for a number of $r$-graphs $H$ $(r \geqq 3)$. We will write $\tau(H)$, $Z(H)$ and $\hat{Z}(H)$ instead of $\tau(\{H\})$, $Z(\{H\})$, and $\hat{Z}(\{H\})$.

Notice that the number of edges of the $r$-graph $G(m_1, \ldots, m_n)$ is a homogeneous polynomial of degree $r$ of the variables $m_1, \ldots, m_n$. It will be convenient to consider this polynomial as an ordinary function of $n$ real variables. In this way we assign to each $r$-graph $G(\{v_1, \ldots, v_n\}, E)$ its *density function*

$$F_G(x_1, \ldots, x_n) = \sum_{\{v_{i_1}, \ldots, v_{i_r}\}} r!\, x_{i_1} \ldots x_{i_r}$$

(the convenience of the factor $r!$ here will be clear later). The *density* of the $r$-graph $G$ is defined as

$$\varrho(G) = \max_{\substack{x_1 + \ldots x_n = 1 \\ x_i \geqq 0}} F_G(x_1, \ldots, x_n).$$

This quantity has been considered in [6, 7, 9, 11]. The $r$-graph $G$ is said to be *dense* if all of its proper subgraphs $G' \neq G$ satisfy the inequality $\varrho(G') < \varrho(G)$.

We will need a number of results from [11]:

**Theorem 1.1.** [11] *Let $\mathscr{A}$ and $\mathscr{B}$ be families of nonempty r-graphs satisfying the condition $\forall H \in \mathscr{B}\ \exists G \in \mathscr{A}$ such that $G \prec H$. Then $\tau(\mathscr{A}) \leqq \tau(\mathscr{B})$.* ∎

**Corollary 1.2** [11]. *For any nonempty set $\mathscr{A}$ of r-graphs*

$$\lim_{n \to \infty} e_n(\hat{Z}(\mathscr{A})) \binom{n}{r}^{-1} = \tau(\mathscr{A}). \quad ∎$$

This shows that for a nondegenerate problem (i.e., when $\tau(\mathscr{A}) = 0$) $e_n(\hat{Z}(\mathscr{A})) = e_n(Z(\mathscr{A}))(1 + o(1))$ as $n \to \infty$. The speed of the convergence of $e_n(\hat{Z}(\mathscr{A})) \binom{n}{r}^{-1}$ to $\tau(\mathscr{A})$ is given in

**Theorem 1.3** [11]. *If $\mathscr{A}$ is a nonempty family of r-graphs and $n \geqq r$, then the following inequalities hold:*

$$e_n(\hat{Z}(\mathscr{A}))\, r!\, n^{-r} \leqq \tau(\mathscr{A}) \leqq e_n(\hat{Z}(\mathscr{A})) \binom{n}{r}^{-1}. \quad ∎$$

**Theorem 1.4** [7, 11]. *Every dense r-graph G is covering.* ∎

When $r = 2$ the covering graph has to be complete so Theorem 1.4 implies that the density of a graph equals the density of its largest complete subgraph. Clearly $\varrho(K_k^2) = (k-1)/k$ and we have

**Corollary 1.5** [9]. *For a 2-graph G with $k = \max\{t \mid K_t^2 \leqq G\}$ one has $\varrho(G) = (k-1)/k$.* ∎

Consequently, every strongly hereditary class with $r = 2$ is of the form $\hat{Z}(K_k^2)$ for some $k$, i.e., it is the class of graphs with chromatic number less than $k$ $(k = 2, 3, \ldots)$.

**Theorem 1.6** [11]. *If $\mathscr{A}$ is a family of nonempty r-graphs, then $\tau(\mathscr{A}) = \sup \varrho(G)$ where the supremum is taken over all dense r-graphs $G \in \hat{Z}(\mathscr{A})$.* ∎

The theorem of Erdős and Simonovits follows from Theorem 1.6 and Corollary 1.5:

**Corollary 1.7** [6]. *Let $\mathscr{A}$ be a set of nonempty 2-graphs and set* $k=\max\{t \mid K_t^2 \in \hat{Z}(\mathscr{A})\}$. *Then* $\tau(\mathscr{A})=(k-1)/k$. ▮

For $r \geqq 3$ not all covering $r$-graphs are complete and this is why it is hard to find $\tau$.

Let $\tilde{H}$ be the extension of the $r$-graph $H$. Then a covering $r$-graph belongs to $\hat{Z}(\tilde{H})$ if and only if it belongs to $Z(H)$. Thus Theorems 1.4 and 1.6 imply

**Corollary 1.8.** *If the r-graph $\tilde{H}$ is the extension of the r-graph H, then* $\tau(\tilde{H})=\sup \varrho(G)$ *where the supremum is taken over all dense r-graphs from the class $Z(H)$.* ▮

## 2. The conjecture of Erdős and Sós

Erdős and Sós (see [5]) have the following conjecture:

**Conjecture 2.1.** *If T is a tree or forest on k vertices* $(k \geqq 2)$, *then for all* $G \in Z(T)$

$$e(G) \leqq \frac{1}{2}(k-2)v(G).$$

If a graph $T$ satisfies Conjecture 2.1, then all of its subgraphs $T'$ with $V(T')=V(T)$ satisfy it. It is known (see [9]) that for a path on $k$ vertices Conjecture 2.1 is true. In this section we will prove Conjecture 2.1 for trees having a vertex adjacent to many leaves of the tree. Then we will get an equivalent form of the conjecture in terms of the density function of the graph. These results will be used in Section 3.

**Theorem 2.2.** *If a vertex of a tree T with* $v(T)=k$ *is adjacent to* $l \geqq \frac{1}{2}(k-2)$ *leaves, then T satisfies Conjecture* 2.1.

**Proof** by induction on $l$ (with $k$ fixed). The first step $l=k-1$ of the induction is trivial because $e(G)>\frac{1}{2}(k-2)v(G)$ implies that some vertex of $G$ has degree $k-1$ at least. We use backward induction from $l+1$ to $l$ $(l<k-1)$. Consider a tree $T$ with a vertex $x$ adjacent to $l$ leaves. As $l<k-1$, there is a leaf not adjacent to $x$. Let $y$ be a vertex, different from $x$ and adjacent to the leaf $z$. Delete edge $zy$ and add edge $zx$. The tree obtained this way is $T'$. Consider the graph $G$ critical with respect to condition $e(G)>\frac{1}{2}(k-2)v(G)$. (This means that no subgraph $G' \neq G$ satisfies this condition.) Then all vertices of $G$ have degree at least $\left\lfloor\frac{1}{2}(k-2)\right\rfloor+1=\left\lfloor\frac{1}{2}k\right\rfloor$. We prove that $G$ contains $T$. According to the induction hypothesis $G$ contains $T'$. Let $W$ denote the set of vertices of the subgraph $T'$ except $y$ and the leaves adjacent to $x$. Now $|W|=k-l-2<\left\lfloor\frac{1}{2}k\right\rfloor$ because $l \geqq \frac{1}{2}(k-2)$. But the degree of $y$ is at least $\left[\frac{1}{2}k\right]$ and so there is a vertex in $G$ adjacent to $y$ and not in $W$. This means that $G$ contains $T$ as a subgraph.

We proved at the same time that Conjecture 2.1 is true for all subgraphs $T'$ of a tree $T$ with $v(T')=v(T)$ if $T$ satisfies the conditions of Theorem 2.2. The following theorem enlarges the list of forests for which the conjecture holds. ▮

**Theorem 2.3.** *Assume $T'$ satisfies Conjecture* 2.1. *Then any graph $T$ obtained from $T'$ by adding a few isolated edges satisfies Conjecture* 2.1, *too.*

**Proof.** It is enough to consider the case when $T$ is obtained from $T'$ by adding one isolated edge. Let $G\in Z(T)$. We are going to show that $e(G)\leqq\frac{1}{2}(v(T)-2)v(G)$. We may assume (without loss of generality) that $G$ has no isolated vertex. We pick a vertex $x$ from $G$ with degree at most $2e(G)/v(G)$. We pick another vertex $y$ adjacent to $x$. Let $G'$ denote the graph obtained from $G$ by deleting $x$ and $y$ and all edges incident to one of them. Then $G'\in Z(T')$, $v(G')=v(G)-2$, $e(G')\geqq e(G)-\frac{2e(G)}{v(G)}-$ $-(v(G)-2)$. $T'$ satisfies Conjecture 2.1, consequently

$$e(G')\leqq\frac{1}{2}(v(T')-2)v(G')=\frac{1}{2}(v(T)-4)(v(G)-2),$$

which implies that $e(G)-\frac{2e(G)}{v(G)}-(v(G)-2)\leqq\frac{1}{2}(v(T)-4)(v(G)-2)$, and

$$\frac{v(G)-2}{v(G)}e(G)\leqq\frac{1}{2}(v(T)-2)(v(G)-2),\quad e(G)\leqq\frac{1}{2}(v(T)-2)v(G),$$

which proves the claim. ▮

For a graph $G$ on $n$ vertices we define the quantity

$$\varrho^*(G)=\max_{\substack{x_i\geqq 0\\ \max\{x_1,\dots,x_n\}=1}}\frac{F_G(x_1,\dots,x_n)}{x_1+\dots+x_n}. \tag{1}$$

**Theorem 2.4.** *For every graph $G$*

$$p^*(G)=\max_{G'}\frac{2e(G')}{v(G')},$$

*where the maximum is taken over all subgraphs $G'$ of $G$.*

**Proof.** Let $V(G)=\{v_1,\dots,v_n\}$ and denote by $G_1$ the subgraph of $G$ for which the ratio $2e(G_1)/v(G_1)$ attains its maximal value. Set $x_i=1$ if $v_i\in V(G_1)$ and $x_i=0$ otherwise. Then

$$\varrho^*(G)\geqq\frac{F_G(x_1,\dots,x_n)}{x_1+\dots+x_n}\geqq\frac{2e(G')}{v(G')}.$$

Suppose, on the other hand, that $2e(G')/v(G')\leqq\gamma$ for all subgraphs $G'$. We will prove that $\varrho^*(G)\leqq\gamma$. Without loss of generality we may assume that $G$ is critical with respect to $\varrho^*$, i.e., $\varrho^*(G')<\varrho^*(G)$ for every proper subgraph $G'$ of $G$. Let

$\bar{y}=(y_1, \ldots, y_n)$ be the vector giving the maximum in (1). If this vector is not unique, then we choose the one with the largest number of components $y_i=1$. To have simpler notation we assume that $0<y_1\leqq y_2\leqq \ldots \leqq y_t<y_{t+1}=\ldots=y_n=1$, (if $y_i=0$, then vertex $v_i$ can be deleted from $G$ without changing $\varrho^*$). If $t=0$, then $\varrho^*(G)=$ $=F_G(1, \ldots, 1)/n=2e(G)/v(G)$. So we assume $t\geqq 1$. Then for $i=1, \ldots, t$:

$$0=\left(\frac{\partial}{\partial x_i}\left(\frac{F_G(x_1, \ldots, x_n)}{x_1+\ldots+x_n}\right)\right)(\bar{y})=\frac{\left(\frac{\partial}{\partial x_i}F_G\right)(\bar{y})-\frac{F_G(\bar{y})}{y_1+\ldots+y_n}}{y_1+\ldots+y_n},$$

i.e.,

$$\left(\frac{\partial}{\partial x_1}F_G\right)(\bar{y})=\ldots=\left(\frac{\partial}{\partial x_t}F_G\right)(\bar{y})=\varrho^*(G).$$

Assume now that there is a pair of nonadjacent vertices $v_i$, $v_j$ with $1\leqq i<j\leqq t$. Then increasing $y_i$ and decreasing $y_j$ with the same amount does not change $F_G(y_1, \ldots, y_n)$ and $y_1+\ldots+y_n$. If $y_i+y_j\leqq 1$, then we may set $y_i'=y_i+y_j$, $y_j'=0$ and this contradicts the fact that $G$ is critical. Now if $y_i+y_j>1$, then we may set $y_i'=1$ and $y_i'=y_i+y_j-1$ and this contradicts the minimality of $t$. Consequently, $1\leqq i<j\leqq t$ implies $\{v_i, v_j\}\in E(G)$. This shows that the functions

$$\frac{1}{2}\left(\frac{\partial}{\partial x_i}F_G\right)-x_j \quad \text{and} \quad \frac{1}{2}\left(\frac{\partial}{\partial x_j}F_G\right)-x_i$$

differ only in additive terms of the type $x_k$ where $k\geqq t+1$. Then

$$y_i-y_j=\left(\frac{1}{2}\left(\frac{\partial}{\partial x_i}F_G\right)(\bar{y})-x_j\right)-\left(\frac{1}{2}\left(\frac{\partial}{\partial x_j}F_G\right)(\bar{y})-x_i\right)$$

is an integer and so $y_i=y_j$. Consequently, $y_1=\ldots=y_t$. Let $a$ be the number of edges of the form $\{y_i, y_k\}$ with $1\leqq i\leqq t$, $t+1\leqq k\leqq n$, and let $b$ be the number of edges of the form $\{y_k, y_m\}$ with $t+1\leqq k$, $m\leqq n$. Let $G'$ denote the subgraph spanned by the vertices $v_{t+1}, \ldots, v_n$. Then $2b=2e(G')\leqq\gamma v(G')=\gamma(n-t)$. Set

$$f(z)=\frac{F_G(x_1, \ldots, x_n)}{x_1+\ldots+x_n},$$

where $x_1=\ldots=x_t=z$, $x_{t+1}=\ldots=x_n=1$. Then $f(y_1)=\varrho^*(G)$,

$$f(z)=\frac{t(t-1)z^2+2az+2b}{tz+(n-t)},$$

$$f'(z)=\frac{t^2(t-1)z^2+2(n-t)t(t-1)z+2(n-t)a-2bt}{(tz+(n-t))^2}.$$

The function $f$ takes its maximal value at $z=y_1$ so $f'(y_1)=0$. Hence $2(n-t)t(t-1)y_1+$ $+2a(n-t)-2tb\leqq 0$ and $2t(t-1)y_1+2a\leqq t(2b/(n-t))\leqq t\gamma$. So indeed

$$\varrho^*(G)=\frac{1}{t}\sum_{i=1}^{t}\left(\frac{\partial}{\partial x_i}F_G\right)(\bar{y})=\frac{1}{t}(2t(t-1)y_1+2a)\leqq\frac{1}{t}t\gamma=\gamma. \quad \blacksquare$$

By Theorem 2.4, Conjecture 2.1 is equivalent to this:

$$\forall G\in Z(T)\colon\ p^*(G)\leqq k-2.$$

## 3. Construction of $r$-graphs $G$ with given $\lim \left( \mathrm{ex}(n, G) \Big/ \binom{n}{r} \right)$

For every tree of forest satisfying the Erdős—Sós conjecture we are going to construct a series of $r$-graphs $G_r$ $(r=2, 3, \ldots)$ and find the exact value of $\tau(G_r)$. The basic result is contained in Theorem 3.2.

Set

(2) $\gamma_G(\varepsilon) = \max \{F_G(x_1, \ldots, x_n) | x_1 + \ldots + x_n = 1,\ x_i \geqq 0,\ \max \{x_1, \ldots, x_n\} = \varepsilon\}.$

so that

$$\varrho(G) = \max_{0<\varepsilon\leqq 1} \gamma_G(\varepsilon), \tag{3}$$

$$\varrho^*(G) = \max_{0<\varepsilon\leqq 1} \frac{1}{\varepsilon} \gamma_G(\varepsilon). \tag{4}$$

The *enlargement* of the $(r-1)$-graph $F$ is the $r$-graph $F'$ which is obtained from $F$ by adding a new vertex to $V(F)$ and by adding this new vertex to every edge of $F$. Consider now an $r$-graph $G$ and one of its vertices, $v$. Delete $v$ from $G$ together with all edges not containing $v$ and delete $v$ from all edges containing it. The $(r-1)$-graph obtained this way is the *star* $(r-1)$-*graph* of $G$ at vertex $v$.

**Theorem 3.1.** *For every r-graph $G$ $(r \geqq 3)$ there exists a star $(r-1)$-graph, $G'$, of $G$ with*

$$\gamma_G(\varepsilon) \leqq (1-\varepsilon)^{r-1} \max_{\delta \leqq (r-\varepsilon)} \gamma_{G'}(\delta). \tag{5}$$

**Proof.** Let $\bar{y} = (y_1, \ldots, y_n)$ be the point where (2) attaines its maximal value. Set $\lambda_i = \left(\frac{\partial}{\partial x_i} F_G\right)(\bar{y})$. Observe that $F_G = \frac{1}{r} \sum_{i=1}^{n} x_i \left(\frac{\partial}{\partial x_i} F_G\right)$, consequently, $\gamma_G(\varepsilon) = F_G(\bar{y}) = \frac{1}{2} \sum y_i \lambda_i$. If $y_i < \varepsilon$, then $\lambda_i \leqq \lambda_j$ for all $j = 1, \ldots, n$. Thus

$$\frac{1}{r} \max \{\lambda_i | i \in \{1, \ldots, n\}, y_i = \varepsilon\} \geqq \frac{1}{r} \sum y_i \lambda_i = \gamma_G(\varepsilon).$$

Assume (for the sake of simpler notation) that the maximum in the left hand side of the above inequality is attained for $i = n$. Let $G'$ be the star $(r-1)$-graph of $G$ at the corresponding vertex $v_n$. Set $z_i = \frac{1}{1-\varepsilon} y_i$; then $z_i \leqq \frac{\varepsilon}{1-\varepsilon}$ for $i = 1, \ldots, n-1$, $z_1 + \ldots + z_{n-1} = 1$, and

$$\gamma_G(\varepsilon) \leqq \frac{1}{r} \left(\frac{\partial}{\partial x_n} F_G\right)(y_1, \ldots, y_n) = F_G(y_1, \ldots, y_{n-1}) =$$

$$= (1-\varepsilon)^{r-1} F_{G'}(z_1, \ldots, z_{n-1}) \leqq (1-\varepsilon)^{r-1} \max_{\delta \leqq (r-\varepsilon)} \gamma_{G'}(\delta). \quad \blacksquare$$

Set now

$$f_r(x) = (x+r-3)^{-r} \prod_{i=1}^{r-1} (x+i-2).$$

Let us denote by $M_r$ the last (i.e., the rightmost) maximum of the function $f_r$ on the interval $[2, \infty)$. Now $M_r \geqq M_{r-1}$ because

$$f_r(x) = f_{r-1}(x)\left(\frac{x+r-4}{x+r-3}\right)^{r-1}$$

and the function $\frac{x+r-4}{x+r-3}$ is monoton increasing. In particular, $M_2=M_3=2$, $M_4=2+\sqrt{3}$. We define $M_1=2$.

**Theorem 3.2.** *Let $T$ be a graph satisfying Conjecture 2.1. Assume it has $k \geqq M_r$ vertices. Then $\tau(\tilde{T})=(k-2)f_r(k)$ where $\tilde{T}$ is the extension of the $(r-2)$-fold enlargement of $T$.*

For the proof of this theorem we need the following lemma.

**Lemma 3.3.** *Assume $T$ satisfies Conjecture 2.1 and has $k \geqq M_{r-1}$ vertices. If the $r$-graph $G$ contains no $(r-2)$-fold enlargement of $T$ as a subgraph, then $\gamma_G(\varepsilon) \leqq$ $\leqq (k-2)f_r(x)$ where $x=\max\{(1/\varepsilon)-r+3, k\}$.*

**The proof of the lemma** is by induction on $r$. Let $r=2$, i.e., $f_2(x)=\frac{1}{x-1}$ and $G \in Z(T)$. One gets from (3) and from Corollary 1.5 that $\gamma_G(\varepsilon) \leqq \frac{k-2}{k-1}$. Then (4), Theorem 2.4 and the fact that $T$ satisfies Conjecture 2.1 imply that $\gamma_G(\varepsilon) \leqq (k-2)\varepsilon$. The first step of the induction is proved. Now we prove the induction step from $r-1$ to $r$ $(r \geqq 3)$. According to Theorem 3.1, $G$ has a star $(r-1)$-graph $G'$ satisfying inequality (5). Now $k \geqq M_{r-2}$ because $M_{r-1} \geqq M_{r-2}$. Thus $G'$ satisfies the induction hypothesis for $r-1$. Then $\gamma_G(\delta) \leqq (k-2)f_{r-1}(y)$ where $y=\max\{(1/\delta)-r+4, k\}$. Set $x=\max\{(1/\varepsilon)-r+3, k\}$. If $\delta \leqq \varepsilon/(1-\varepsilon)$, then $1/\delta \geqq (1/\varepsilon)-1$ and $y \geqq x$. The function $f_{r-1}(z)$ decreases when $z \geqq M_{r-1}$ hence $f_{r-1}(y) \leqq f_{r-1}(x)$. So we get

$$\gamma_G(\varepsilon) \leqq (1-\varepsilon)^{r-1} \max_{\delta \leqq (r-\varepsilon)} \gamma_{G'}(\delta) \leqq \left(1-\frac{1}{x+r-3}\right)^{r-1} \max_{\delta \leqq (r-\varepsilon)} (k-2)f_{r-1}(y) \leqq$$

$$\leqq \left(1-\frac{1}{x+r-3}\right)^{r-1} f_{r-1}(x) = (k-2)f_r(x). \quad \blacksquare$$

In Section 2 we proved that the statement of Conjecture 2.1 holds for a number of graphs $T$.

The condition $k \geqq M_r$ is automatically fulfilled when $r=3$. Then we have

**Corollary 3.4.** *Assume the graph $T$ satisfies Conjecture 2.1. If, moreover, the 3-graph $\tilde{T}$ is an enlargement of an extension of $T$, then $\tau(\tilde{T})=(k-1)(k-2)/k^2$ where $k=v(T)$.* $\blacksquare$

Now we will show that the last statement holds not only for trees and forests.

**Theorem 3.5.** *Let $T$ be a graph with 4 vertices and 3 edges and such that one of its connected components is a triangle and the other one is an isolated vertex. Let $\tilde{T}$ be a 3-graph which is the extension of an enlargement of $T$. Then $\tau(\tilde{T})=3/8$.*

**Proof.** Let $T'$ denote the enlargement of $T$. Let $T''$ denote the enlargement of the graph $K_3^2$. If $v(G)\geqq 5$, then $G\in Z(T'')=\hat{Z}(T'')$ and by Theorem 1.6, $\varrho(G)\leqq\leqq\tau(T'')\leqq 1/3$ (the last inequality is proved in [10], see [3] as well). If $v(G)\leqq 4$, then $G\leqq K_4^3$ and consequently $\varrho(G)\leqq\varrho(K_4^3)=3/8$. Thus $\varrho(G)\leqq 3/8$ for every $G\in Z(T')$. By Corollary 1.8 we have $\tau(\tilde{T})\leqq 3/8$. But $K_4^3\in\hat{Z}(\tilde{T})$ and Theorem 1.6 implies $\tau(\tilde{T})\geqq\varrho(K_4^3)=3/8$. ∎

It follows from the results of [10] that for $r=4$ the condition $k\geqq M_r=2+\sqrt{3}$ in Theorem 3.2 is not significant.

We mention that our results yield not only the coefficients of saturation $\tau(H)$ but the exact values of $e_n(\hat{Z}(H))$ as well. Indeed, if $K_m^r\in\hat{Z}(\mathscr{A})$ and $\varrho(K_r^m)==\tau(\mathscr{A})$, then by Theorem 1.3, $e_{mt}(Z(\mathscr{A}))\leqq\varrho(K_m^r)(mt)^r/r!$. On the other hand $K_m^r(t, \ldots, t)\in\hat{Z}(\mathscr{A})$. Thus $e_{mt}(\hat{Z}(\mathscr{A}))\geqq e(K_m^r(t, \ldots, t))=\varrho(K_m^r)(mt)^r/r!$. This shows that if $H$ is the extension of an $(r-2)$-fold enlargement of a graph $T$ on $k$ vertices, then under the conditions of Theorem 3.2, 3.5 and Corollary 3.4 we get that for all $n$ which is divisible by $(k+r-3)$

$$e_n(\hat{Z}(\mathscr{A}))=\binom{k+r-3}{r}\left(\frac{n}{k+r-3}\right)^r.$$

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