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SOME GENERALIZATIONS OF THE CRISS-CROSS METHOD FOR THE LINEAR COMPLEMENTARITY PROBLEM OF ORIENTED MATROIDS

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Quadratic programming, symmetry, positive (semi) definiteness and the linear complementary problem were generalized by Morris and Todd to oriented matroids. Todd gave a constructive solution for the quadratic programming problem of oriented matroids. Using Las Vergnas' lexicographic extension and Bland's basic tableau construction Todd generalized Lemke's quadratic programming algorithm for this problem.

Here some generalizations of Terlaky's finite criss-cross method are presented for oriented matroid quadratic programming. These algorithms are based on the smallest subscript rule and on sign patterns, and do not preserve feasibility on any subsets. In fact two variants of the generalized criss-cross method are presented. Finally two special cases (oriented matroid linear programming and the definite case) are discussed.

1. Introduction

Linear and quadratic programming (LP and QP) were examined by hundreds of scientists in the past decades. Several algorithms, especially for QP were developed. The most widely used algorithms for QP are based on the simplex method [2, 3, 8, 9, 10]. Most of the new results of LP, e.g. least index resolution, were applied quickly in the theory of QP as well [7, 28].

Several combinatorial properties of simplex methods were well known from the early time. The most significant step on this field was done by Bland [4, 5] by showing that oriented matroids (OM) can be considered as a combinatorial abstraction of LP. The theory of OM was established by Bland [5], Folkman and Lawrence [6], and Las Vergnas [11]. They proved the generalized duality theorem of OM—LP, but Bland's proof was the first constructive proof. Later Fukuda [12], Jensen [13], Terlaky [27], Todd [31] and Wang [36] gave different constructive proofs for this theorem.

The theory of oriented matroid programming was extended by Todd [30, 31] to OM—QP and OM—LCP. This theory was largely extended by Morris and Todd [21, 22]. By generalizing Lemke's [20] complementary pivoting algorithm Todd proved the duality theory of OM—QP. Up to now, as far as we know this is the only algorithmic proof for this theorem.

Terlaky [26, 27] constructed a finite criss-cross method for LP and OM—LP. Recently Klafszky and Terlaky [16] generalized the criss-cross method for QP in real vector spaces. These results are generalized here for OM—QP. The criss-cross

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method is based on the least subscript rule introduced by Bland [4, 5] and on the sign properties of basic tableaux. In the first two chapters basic properties of OM, symmetry, positive definiteness, QP and LCP in OM are briefly summarized.

In the third chapter the generalized criss-cross method is presented and so a new constructive proof is given for the duality theorem of OM—LCP. Finally in the fourth chapter a modified algorithm and some special cases (OM—LP and positive definite OM) are discussed.

It is assumed that the reader is familiar with the definition and fundamental properties of matroids. Several basic text books exist on this area. All the necessary information can be found for example in Welsh [34].

Oriented Matroids, basic properties. The origins of OM can be found already in Rockafellar's [23] paper. Later Bland, Las Vergnas, Folkman, Lawrence, Fukuda, Morris, Todd and several scientists worked on this field to establish and extend the theory of OM. Without completeness all the necessary results, basic properties and algorithms can be found in [5, 6, 12, 13, 27, 31, 36].

The necessary definitions and basic properties of OMs are summarized as follows.

Let $E=\{e_1, \dots, e_n\}$ be a finite set. An ordered pair $X=(X^+, X^-)$ is called a *signed set* of E, if $X^+\cap X^-=\emptyset$ and $X^+, X^-\subset E$. The *opposite of a signed set* $X=(X^+, X^-)$ is the signed set $-X=(X^-, X^+)$ and the *underlying set* is $\underline{X}=X^+\cup X^-$. A signed set Y is *contained* in the signed set X if $Y^+\subset X^+$ and $Y^-\subset X^-$.

Definition 1.1. Let $\mathcal{O}$ and $\mathcal{O}^*$ be sets of signed sets of E. $M=(E, \mathcal{O})$ and $M^*=(E, \mathcal{O}^*)$ are dual pairs of *oriented matroids* if

(a) $\underline{M}=(E, \underline{\mathcal{O}})$ and $\underline{M}^*=(E, \underline{\mathcal{O}}^*)$ are dual matroids with $\underline{\mathcal{O}}$ and $\underline{\mathcal{O}}^*$ as the sets of circuits and cocircuits.

(b) $X\in\mathcal{O}\Rightarrow -X\in\mathcal{O}$ and $Y\in\mathcal{O}^*\Rightarrow -Y\in\mathcal{O}^*$.

(c) $X_1, X_2\in\mathcal{O}$ and $\underline{X}_1=\underline{X}_2\Rightarrow X_1=\pm X_2$
$Y_1, Y_2\in\mathcal{O}^*$ and $\underline{Y}_1=\underline{Y}_2\Rightarrow Y_1=\pm Y_2$.

(d) $X\in\mathcal{O}$ and $Y\in\mathcal{O}^*$ implies

$$(X^+\cap Y^+)\cup(X^-\cap Y^-)=\emptyset \Leftrightarrow (X^+\cap Y^-)\cup(X^-\cap Y^+)=\emptyset.$$

The elements of $\mathcal{O}$ and $\mathcal{O}^*$ are called *oriented circuits* and *oriented cocircuits* respectively. Assumption (d) is called the *orthogonality condition* and in this case notation $X\perp Y$ is used. It is known that the orthogonality condition is equivalent with the so called *elimination* property. The elimination property is often formulated in one of the following two ways.

(e1) For all $X_1, X_2\in\mathcal{O}$ such that $X_1\neq -X_2$ and $e\in(X_1^+\cap X_2^-)\cup(X_1^-\cap X_2^+)$ there exists $X_3\in\mathcal{O}$ such that $X_3^+\subset(X_1^+\cup X_2^+)\setminus\{e\}$ and $X_3^-\cup(X_1^-\cup X_2^-)\setminus\{e\}$.

(e2) For all $X_1, X_2\in\mathcal{O}$ such that $X_1\neq -X_2$, $e'\in(X_1^+\cap X_2^-)\cup(X_1^-\cap X_2^+)$ and $e''\in(X_1^+\cap X_2^+)\cup(X_1^-\cap X_2^-)$ there exist $X_3\in\mathcal{O}$ such that $X_3^+\subset(X_1^+\cup X_2^+)\setminus\{e'\}$, and $X_3^-\subset(X_1^-\cup X_2^-)\setminus\{e'\}$ and $e''\in X_3$.

The signed set $Z=Z_1\circ Z_2=(Z_1^+\cup(Z_2^+\setminus Z_1^-),\ Z_1^-\cup(Z_2^-\setminus Z_1^+))$ is called the composition of the signed sets Z_1 and Z_2. It is obvious that composition is not a symmetric operation. Denote $\mathcal{K}(\mathcal{O})$ the set of signed sets obtained by repeated composition of oriented circuits of M The elements of $\mathcal{K}(\mathcal{O})$ and $\mathcal{K}(\mathcal{O}^*)$ are called *oriented cycles* and *oriented cocycles* respectively. Bland [5] proved that for all

$K \in \mathcal{K}(\mathcal{O})$ and $L \in \mathcal{K}(\mathcal{O}^*)$ we have $K \perp L$. Furthermore the elimination property is valid for cycles as well.

(e*) For all $K_1, K_2 \in \mathcal{K}(\mathcal{O})$ and $e \in (K_1^+ \cap K_2^-) \cup (K_1^- \cap K_2^+)$ there exist $K_3 \in \mathcal{K}(\mathcal{O})$ such that $e \notin \mathcal{K}_3$, $K_3^+ \subset K_1^+ \cup K_2^+$, $K_3^- \subset K_1^- \cup K_2^-$ and $(K_1 \cup K_2) \setminus [(K_1^+ \cap K_2^-) \cup (K_1^- \cap K_2^+)] \subset K_3$.

The oriented matroid

$$\mathcal{K}_F(\mathcal{O}) = \{[(K^+ \setminus F) \cup (K^- \cap F), (K^- \setminus F) \cup (K^+ \cap F)] | K \in \mathcal{K}(\mathcal{O})\}$$

is obtained from $\mathcal{K}(\mathcal{O})$ by *sign reversing on F.* If G and H are disjoint subsets of E, then $M \setminus G$ is the oriented matroid $M \setminus G = (E \setminus G, \overline{\mathcal{O}})$ where $\overline{\mathcal{O}} = \{X | X \cap G = \emptyset;\ X \in \mathcal{O}\}$ and M/H is the oriented matroid $M/H = \{E \setminus H, \overline{\mathcal{O}}\}$ where $\overline{\mathcal{O}} = \{X \setminus H | X \in \mathcal{O}$ and $X \setminus H$ is a minimal dependent set$\}$. Operation "$\setminus$" is called *deletion* and "/" is called *contraction.* Combining these two operations we have oriented matroid $M \setminus G/H = (E \setminus (G \cup H), \{X \setminus H | X \cap G = \emptyset,\ X \in \mathcal{O}$ and $X \setminus H$ is a minimal dependent set$\})$.

Finally Bland's basic tableau construction is presented. Let $\mathcal{B}$ be the set of bases of $\underline{M}$, and let m be the rank of $\underline{M}$. For all $B = \{e_{b_1}, \ldots, e_{b_m}\} \in \mathcal{B}$ and for all $e_{b_i} \in B$ there exists a unique cocircuit $\underline{Y}_{b_i} \in \mathcal{O}^*$ of M^* such that $\underline{Y}_{b_i} \cap B = \{e_{b_i}\}$. The set of circuits $\{Y_{b_1}, \ldots, Y_{b_m}\}$ is called *fundamental set of cocircuits* of the dual base $E \setminus B$ if $e_{b_i} \in Y_{b_i}^+$ for all $e_{b_i} \in B$ and $\underline{Y}_{b_i} \subset (E \setminus B) \cup \{e_{b_i}\}$. The notation $Y_{b_i} = C(\overline{B}, e_{b_i})$ is used as well, where $\overline{B} = E \setminus B$.

Let $T(B)$ be the matrix formulated by the signed incidence vectors of $C(\overline{B}, e_{b_i})$. $T(B)$ is called the *basic tableau* corresponding to the base B. As we did in our previous papers [15, 16, 26, 27], we will refer as the b_i row to the row corresponding the basic element e_{b_i}.

2. Symmetry and positive definiteness

Morris and Todd [21, 22, 30, 31] gave a combinatorial abstraction of complementarity, symmetry and positive definiteness. This generalization made it possible to formulate OM—QP and OM—LCP. These results are briefly summarized here.

The definitions are slightly modified, in order to make them a little more symmetric. All the results of Morris and Todd remain valid, only some of the assumptions are modified. This can be checked easily following Morris' and Todd's proofs.

Definition 2.1. Let $M = (E, \mathcal{O})$ be an OM, where $E = S \cup T$, $S \cap T = \emptyset$, $S = \{s_1, \ldots, s_n\}$ and $T = \{t_1, \ldots, t_n\}$. A subset $F \subset E$ is called *complementary if* $|F \cap \{s_i, t_i\}| \leq 1$ for all $i = 1, \ldots, n$, that is at most one of the complementary elements is in the set.

Definition 2.2. An OM $M = (E, \mathcal{O})$ is called a *square* OM (with respect to S and T) if $E = S \cup T$ as above, $\mathrm{rank}(M) = n$ and there is a complementary base of M.

The particular ordering of the element s_i and t_i is significant but we do not require that S is a base of M as Morris and Todd did.

Definition 2.3. The *switch* of a signed set X is the signed set

$$\text{sw}\, X = (\{s_i | t_i \in X^-\} \cup \{t_i | s_i \in X^+\},\ \{s_i | t_i \in X^+\} \cup \{t_i | s_i \in X^-\})$$

which is obtained by switching the signs on T and exchanging elements s_i and t_i.

The switch of a square OM M is the OM sw M if the circuits of sw M are the switches of the circuits of M.

Definition 2.4. The OM $M_{ST}^t = \text{sw}\, M^*$ is called the *transpose* of the square OM M.

Definition 2.5. Let M be a square OM. M is called *bisymmetric* (with respect to (S, T)) if $M = M_{ST}^t = \text{sw}\, M^*$. If S or T is a base of M, then M is called *symmetric.*

The following theorem describes the structure of basic tableaus of complementary bases.

Theorem 2.6. *Let B be a complementary base of a bisymmetric OM $M = (S \cup T, \mathcal{O})$. Then*

(a) *If* $s_i, s_j \in B$, *then* $s_j \in C(B, t_i)^+ \Leftrightarrow s_i \in C(B, t_j)^+$ *and*
$s_j \in C(B, t_i)^- \Leftrightarrow s_i \in C(B, t_j)^-$.

(b) *If* $s_i, t_j \in B$, *then* $s_i \in C(B, s_j)^+ \Leftrightarrow s_j \in C(B, t_i)^-$ *and*
$s_i \in C(B, s_j)^- \Leftrightarrow t_j \in C(B, t_i)^+$.

(c) *If* $t_i, t_j \in B$, *then* $t_i \in C(B, s_j)^+ \Leftrightarrow t_j \in C(B, s_i)^+$ *and*
$t_i \in C(B, s_j)^- \Leftrightarrow t_j \in C(B, s_i)^-$. ∎

Corollary. *If S or T is a complementary base of M, then the corresponding basic tableau is symmetric.*

Definition 2.7. A square OM is called *nonsingular* if both of S and T are bases of M.

Theorem 2.8. *Let M be a bisymmetric OM. If there is a complementary base B with $s_i \in B$ and $t_i \notin B$ for some index i, then $M \setminus t_i / s_i$ is a bisymmetric OM.*

It is bovious that the role of s_i and t_i can be changed in Theorem 2.8, and it can be applied for arbitrary subset of S or T and the corresponding complementary subset.

Definition 2.9. A cycle of a square, bisymmetric OM is called *sign reversing* if $\{s_i, t_i\} \not\subset K^+$ and $\{s_i, t_i\} \not\subset K^-$ for all $1 \leqq i \leqq n$. Cycle K is called *strictly sign reversing* if it is sign reversing and $\{s_i, t_i\} \subset \underline{K}$ for an index i, that is either $s_i \in K^+$, $t_i \in K^-$ or $s_i \in K^-$ and $t_i \in K^+$.

Definition 2.10. A bisymmetric OM is called *positive (semi) definite* if it does not contain any sign reversing (strictly sign reversing) cycle.

Properties of square, symmetric and bisymmetric matrices are preserved by this definition.

Lemma 2.11. *Let M be a bisymmetric, positive semidefinite OM. If B is a complementary base of M, then:*

(a) *If* $s_i, s_j \in B$, *then* $s_i \notin \underline{C}(B, t_j) \Rightarrow s_j \notin \underline{C}(B, t_i)$.

(b) *If* $t_i, t_j \in B$, *then* $t_i \notin \underline{C}(B, s_j) \Rightarrow t_j \notin \underline{C}(B, s_i)$.

(c) *If* $s_i \in B$, *then* $s_i \notin C(\bar{B}, t_i)^-$ *and* $t_i \notin C(\bar{B}, s_i)^+$.

(d) *If M is nonsingular and* $s_i \in B$, *then* $s_i \in \underline{C}(B, t_i)$; *and if* $t_i \in B$, *then* $t_i \in \underline{C}(B, s_i)$. ∎

Lemma 2.12. *If M is a positive (semi) definite, bisymmetric OM, then $M \setminus t_i/s_i$ is positive (semi) definite bosymmetric if there is a complementary base B with $s_i \in B$. Conversely if $t_i \in B$ for a complementary base B then $M \setminus s_i/t_i$ is a positive (semi) definite bisymmetric OM.* ▮

Teorem 2.13. *A positive semidefinite OM is positive definite iff it is nonsingular.* ▮

Quadratic programming in OM. Let $\hat{M}=(\hat{E}, \hat{\mathcal{O}})$ be an OM with the following properties:

(a) $\hat{E}=\{e\} \cup S \cup T$, denote $E=S \cup T$.

(b) $M=\hat{M} \setminus e=M(E, \mathcal{O})$ is a bisymmetric positive semidefinite OM.

A subset $\hat{F} \subset \hat{E}$ will be called complementary if $F=\hat{F} \setminus \{e\}$ is complementary in M.

Problem OM—QP. Find a nonnegative complementary oriented circuit $\hat{X} \in \hat{\mathcal{O}}$ of $\hat{M}$ for which $e \in \hat{X}$.

First Todd [31] gave a constructive proof for this problem. Todd generalized Lemke's [20] complementary pivot algorithm, using Las Vergnas' [17] lexicographic extension and he proved the following Main Theorem.

Main Theorem. *Given an OM—QP one and only one of the following alternatives holds:*

(a) *There is a nonnegative complementary oriented circuit which solves problem OM—QP, that is $e \in \hat{X} \in \hat{\mathcal{O}}$.*

(b) *There is a nonnegative cocircuit $e \in \hat{Y} \in \hat{\mathcal{O}}^*$ for which either $S \cap \underline{\hat{Y}}=\emptyset$ or $T \cap \underline{\hat{Y}}=\emptyset$.*

In the third chapter a new constructive proof is presented for the Main Theorem, which is the main result of this paper. Our algorithms are generalizations of Terlaky's [27] finite criss-cross method and the QP criss-cross methods presented in [16].

Before doing so, let us prove that cases (a) and (b) cannot hold simultaneously.

Lemma 2.14. *At most one of alternatives* (a) *and* (b) *of the Main Theorem hold.*

Proof. Suppose to the contrary that both hold, then we have an oriented circuit $e \in \hat{X} \in \hat{\mathcal{O}}$ and an oriented cocircuit $e \in \hat{Y} \in \hat{\mathcal{O}}^*$ with both of them nonnegative. By orthogonality $\hat{X} \perp \hat{Y}$, but $e \in \hat{X}^+ \cap \hat{Y}^+$ while $(\hat{X}^+ \cap \hat{Y}^-) \cup (\hat{X}^- \cap \hat{Y}^+)=\emptyset$ which is a contradiction. ▮

Note that condition $\underline{\hat{Y}} \cap S=\emptyset$ or $\underline{\hat{Y}} \cap T=\emptyset$ was not used in this proof, that is cases (a) and (b) exclude each other without this condition. This property will follow automatically from the algorithm presented in the next chapter.

3. A generalized criss-cross method

The generalized criss-cross method is a simple algorithm. We do not need the extension of matroid $\hat{M}$ as it was necessary using a lexicographic extension to guarantee finiteness of the generalized Lemke method (Todd [31]). We do not need to preserve feasibility of any part of the base. Our algorithm is based only on least index selection and on sign properties of circuits (cocircuits).

Algorithm I. (a generalised criss-cross method)

Initialization: An OM $\hat{M}$ is given as in the Main Theorem. Let us order the elements of E as follows: $(s_1, t_1, s_2, t_2, \ldots, s_i, t_i, \ldots, s_n, t_n, e)$. Let an arbitrary, complementary base B and the corresponding basic tableau $T(B)$ be given, where $e \notin B$.
Output: An oriented circuit $\hat{X}$ satisfying alternative (a) or an oriented cocircuit $\hat{Y}$ satisfying alternative (b) of the Main Theorem.
Pivot rule:

(1) Let $r = \min\{i \mid s_i \in \hat{X}_e^- \text{ or } t_i \in \hat{X}_e^-\}$, where $\hat{X}_e = C(B, e)$ the oriented circuit associated to the nonbasic element e. If there is no r, then return $\hat{X} = \hat{X}_e$, alternative (a) holds.

(2a) If there is an r (we may assume that $s_r \in \hat{X}_e^-$) and $t_r \in \underline{\hat{Y}}_r = \underline{C}(\bar{B}, s_r)$ (the oriented cocircuit associated to $s_r \in B$), then *(diagonal pivot)* s_r is replaced by t_r in the base.

(2b) If there is an r and $t_r \notin \underline{\hat{Y}}_r$ then let $p = \min\{i \mid s_i \in \hat{Y}_r^-\}$. If there is no p, then return $\hat{Y} = \hat{Y}_r$, alternative (b) holds.

(2c) If there is an element s_p then *(exchange pivot)* basic elements s_r and t_p are replaced by t_r and s_p.

Remark, that in case (2a) $t_r \in \hat{Y}_r^-$ since $M = \hat{M} \setminus e$ is a positive semidefinite OM (Lemma 2.11) and in case (2b) $\underline{\hat{Y}}_r \cap S = \emptyset$ or $\underline{\hat{Y}}_r \cap T = \emptyset$ (Lemma 2.11) depending on $s_r \in \hat{X}_e^-$ or $t_r \in \hat{X}_e^-$. The last property holds in case (2c) as well.

The above remarks show that Algorithm I. is consistent and the complementarity property of the base is preserved through the algorithm, so the sign patterns of the basic tableaux are described by Theorem 2.6 and Lemma 2.11.

To prove the Main Theorem one has only to show that Algorithm I. is finite.

Theorem 3.1. *The generalized criss-cross method is finite.*

Proof. Since there is only a finite number of different bases one only has to show that cycling cannot occur. Suppose to the contrary that cycling occurs, that is starting from a base B, after a finite number of pivots base B returns. Denote $I^* = \{i \mid s_i \text{ entered the base through the cycle}\}$. Then obviously s_i left the base in some step, and the same holds for t_i since all the bases are complementary.

Let $q = \max\{i \mid i \in I^*\}$. So by the above remarks and by the ordering of the elements, element t_q is the largest indexed variable which entered and left the base through the cycle. Depending on, whether $t_q(s_q)$ or some other element was the lowest infeasible (negative) element in oriented circuit $\hat{X}_e$ when $t_q(s_q)$ left or entered the base, one have to consider the following cases.

t_q leaves the base (s_q enters)

A: t_q is the lowest indexed infeasible element of $\hat{X}_e$.

B: s_r (for some $r < q$) is the lowest indexed infeasible element of $\hat{X}_e$.

t_q enters the base (s_q leaves)

C: s_q is the lowest indexed infeasible element of $\hat{X}_e$.

D: t_p (for some $p < q$) is the lowest indexed infeasible element of $\hat{X}_e$.

Before considering the different cases note that in cases A and C both diagonal and exchange pivots are possible, but in cases B and D only exchange pivots may occur.

Since pairs (s_i, t_i) did not influence the algorithm if $i > q$ so basic elements of the set $\{s_i, t_i \mid i > q\}$ can be contracted and nonbasic elements of this set

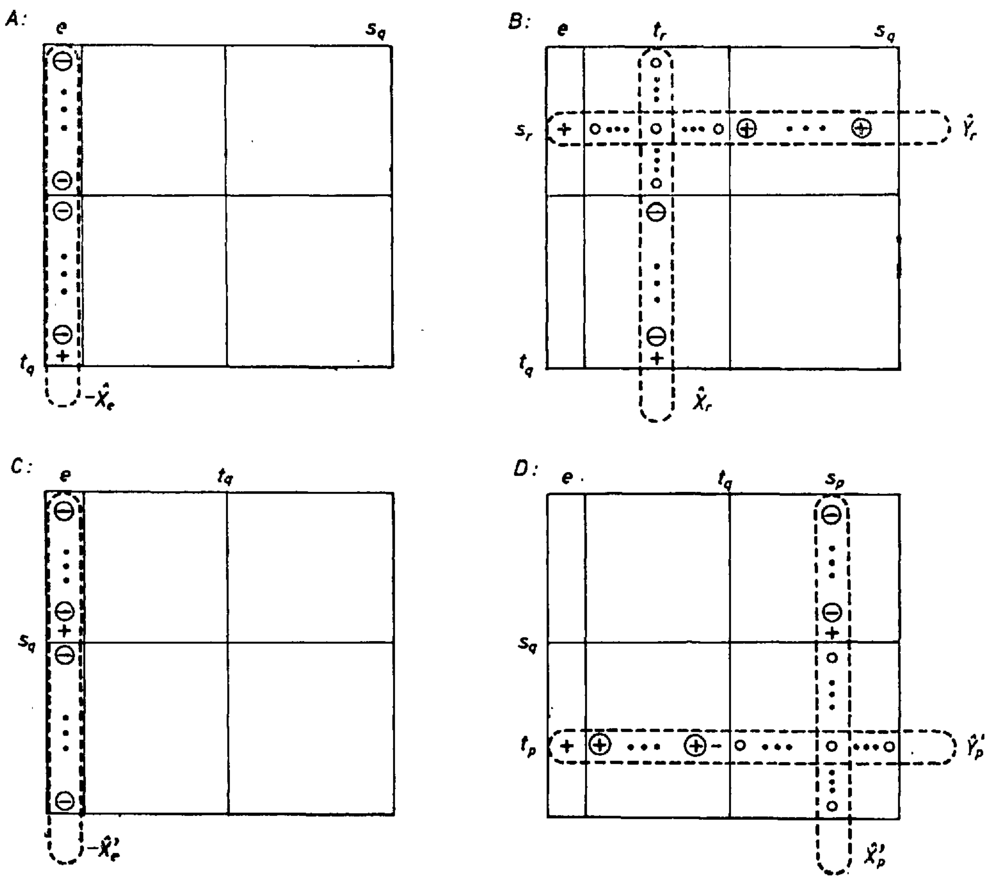

Fig. 1

can be deleted. Basic tableaux in cases A, B, C, D have the following sign properties demonstrated on Figure 1.

To prove the theorem we have to show that cases $A-C$, $A-D$, $B-C$ and $B-D$ cannot hold.

Case A—C: Using elimination property (e) presented in the first section (with $K_1=\hat{X}'_e$, $K_2=-\hat{X}_e$, $e=e$) we have a cycle K with sign properties: $e\notin K$, $t_q\in K^+$, $s_q\in K^-$, $s_i(t_i)\in K^+\Leftrightarrow t_i(s_i)\notin K^+$ and $t_i(s_i)\in K^-\Leftrightarrow s_i(t_i)\notin K^-$. Considering sign properties of $\hat{X}_e$ and $\hat{X}'_e$ these properties of cycle K can be verified easily. Since $e\notin K$, K is a cycle of $M=\hat{M}\setminus e$ and it is strictly sign reversing (Definition 2.9) implying (Definition 2.10) that M is not positive semidefinite contradicting our assumptions.

Case A—D: Reversing signs of elements t_q and s_q, $\hat{X}=\hat{X}_e$ satisfies case (a) and $\hat{Y}=\hat{Y}'_p$ satisfies case (b) of the Main Theorem, which is impossible by Lemma 2.14.

Case B—C: The same way as above $\hat{X}=\hat{X}'_e$ and $\hat{Y}=\hat{Y}_r$ provides a contradiction.

Case B—D: In this case circuit $\hat{X}'_p$ is orthogonal to cocircuit $\hat{Y}_r$, for considering sign properties we have $s_q\in(\hat{X}'^+_p\cap\hat{Y}^-_r)\cup(\hat{X}'^-_p\cap Y^+_r)$ but $(\hat{X}'^+_p\cap\hat{Y}^+_r)\cup(\hat{X}'^-_p\cap\hat{Y}^-_r)=\emptyset$ since $\hat{Y}^-_r=\{s_q\}$ and $\hat{X}'^+_p=\{s_q\}$ which is a contradiction.

All of the possible four cases led to a contradiction, so Algorithm I. is finite. ▮

Notice that in case $B-D$ orthogonality of $\hat{X}_r$ and $\hat{Y}'_p$ could have been used the same way as we used the orthogonality of $\hat{X}'_p$ and $\hat{Y}_r$.

4. Modification

Diagonal pivots were preferred in Algorithm I. as the y assured in step (2a). This was necessary to guarantee the desired oriented cocircuit Y in case (b) of the Main Theorem if case (a) does not hold.

A modification is possible where diagonal pivots are even more prefered than in the original algorithm. It is easy to see that this new Algorithm M is finite again. The proof is the same as in case of Algorithm I.

Algorithm M

Initialization: As in Algorithm I.
Output: Either case (a) or case (b) of the Main Theorem.
Pivot rule.
Steps (1), (2), (2a), (2b) are the same as in Algorithm I.
(2c) If $p<r$ and $s_p \in \underline{\hat{Y}}_p$, then *(diagonal pivot)* t_p leaves and s_p enters the base.
(2d) If $p>r$ or $s_p \notin \underline{\hat{Y}}_p$, then *(exchange pivot)* s_r and t_p leaves, and t_r and s_p enters the base.

4.2. Special cases

Positive definite OM

If the OM $M=\hat{M} \diagdown e$ is positive definite, then Algorithm I. and Algorithm M give a very simple algorithm.

As case (d) of Lemma 2.11 shows, diagonal elements never vanish in the case of a positive definite OM, that is $t_r \in \underline{C}(B, s_r)$ and $s_r \in \underline{C}(B, t_r)$ for all fundamental circuits. So in the above mentioned two algorithms (since diagonal pivots are prefered) only diagonal pivots are performed. Steps (2b), (2c) and (2d) of the algorithms are never used and so it is shown that in case of positive definite OMs the OM—QP always has a solution.

The proof of finiteness also significantly simplifies since only cases A and C, that is case $A-C$ may occur.

Linear programming in OM

It is well known, that LP can be formulated as an LCP (i.e. $Ax \leqq b$, $x \geqq 0$, $yA \geqq c$, $y \geqq 0$, $(yA-c)x=0$, $y(Ax-b)=0$). This idea was generalized for OM by Todd [31], when an OM—LP was transformed into an OM—LCP (OM—QP). In this case we have a specially structured positive semidefinite bisymmetric OM, where symmetric diagonal blocks of any complementary basic tableau are identically zero matrices.

In this case in both of the two algorithms only exchange pivots are performed and Terlaky's [27] criss-cross method is obtained as a special case (the same way as in real spaces [16]). The same pivots are performed, only basic tableau is doubled by construction. The proof of finiteness does not specialize, all the four possible cases occur.

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