STUDIA LOGICA
Vol. XXXII – 1973

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STRUCTURAL COMPLETENESS OF GÖDEL'S AND DUMMETT'S PROPOSITIONAL CALCULI

The purpose of this paper is to show that all Gödel's many valued propositional calculi and Dummett's linear calculus are structurally complete.

Gödel's calculi G_n are given by matrices $\mathfrak{M}_n$ of Gödel. The sequence of matrices $\mathfrak{M}_n$ was introduced by Gödel in [2] and was axiomatized by Thomas in [5]. Hence G_n are called Thomas's calculi LC_n as well.

Dummett's linear calculus LC was studied in [1] by Dummett. These calculi belong to the class of intermediate or superconstructive (superintuitionistic) propositional calculi. A superconstructive propositional calculus is formed from the intuitionistic propositional calculus H by the addition of finite number of extra axioms.

1. Let At be the set of all propositional variables $p_1, p_2, \ldots$, let S be the set of well-formed formulas built by means of variables $p_1, p_2, \ldots$ and connectives: $\sim$ (negation), $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction). The variables $\alpha, \beta, \ldots$ run over the set S.

If R is a nonempty, finite set of n-ary ($n > 1$) rules r^n (resp. r), where $r^n \subseteq S^n$, and $A \subseteq S$, then the couple $\langle R, A\rangle$ is called the system of propositional calculus with axioms A and primitive rules R. The rule r^n is structural, $r^n \in \text{Struct}$, iff for every $e: At \rightarrow S$ and for every $\alpha_1, \ldots, \alpha_n \in S$: if $r^n(\alpha_1, \ldots, \alpha_n)$, then $r^n(h^e(\alpha_1), \ldots, h^e(\alpha_n))$, where h^e is the extension of e to an endomorphism $h^e : S \rightarrow S$. The set $\text{Cn}(R, X)$ is the least set containing X and closed under each of the rules $r \in R$. The rule r is permissible in $\langle R, A\rangle$, $r \in \text{Perm}(R, A)$, iff $\text{Cn}(R \cup \{r\}, A) \subseteq \text{Cn}(R, A)$; the rule r is derivable in $\langle R, A\rangle$, $r \in \text{Der}(R, A)$, iff $\text{Cn}(R \cup \{r\}, A \cup X) \subseteq \text{Cn}(R, A \cup X)$, for every $X \subseteq S$. The symbol r_0 denotes the modus ponens rule and the symbol r_* denotes the substitution rule, the sets R_0 and R_{0*} are defined by the equations $R_0 = \{r_0\}$, $R_{0*} = \{r_0, r_*\}$ respectively. $\text{Sb}(X)$ is the smallest set containing $X \subseteq S$ and closed under the substitution rule.

2. The system $\langle R, A\rangle$ is structurally complete, i.e. $\langle R, A\rangle \in \text{SCpl}$ iff $\text{Struct} \cap \text{Perm}(R, A) \subseteq \text{Der}(R, A)$ (the notion of structural completeness is introduced by W. A. Pogorzelski in [3]).

T. Prucnal [4] proved that the system $\langle R, A\rangle$ is structurally complete iff for every finite set $\pi \subseteq S$ and for every $\beta \in S$:

$$(*) \qquad \forall_{e: At \rightarrow S} \left(h^e(\pi) \subseteq \text{Cn}(R, A) \Rightarrow h^e(\beta) \in \text{Cn}(R, A)\right) \Rightarrow \beta \in \text{Cn}(R, A \cup \pi).$$

Classical propositional calculus $\langle R_0, \mathrm{Sb}(A_2)\rangle$ is structurally complete [3]. This calculus is the second element of sequence of calculi G_n and these calculi begining with the third one are, as mentioned above, intermediate calculi between classical and intuitionistic. The latter one is known as lacking structural completeness (A. Wroński, T. Prucnal). Hence there arises a problem of structural completeness of Gödel's and Dummett's calculi. (W. A. Pogorzelski proved that $\langle R_{0*}, A_{G_3}\rangle \in \mathrm{SCpl}$, where A_{G_3} is set of axioms of Gödel's matrix $\mathfrak{M}_3$). Till now there is already solved the problem of structural completeness of many valued Łukasiewicz's calculi, Lewis's systems S4, S5 and others.

3. By n-th Gödel's matrix $\mathfrak{M}_n, (n \in N)$, (cf. [2]), we mean an algebra $\langle|\mathfrak{M}_n|, \Omega_n\rangle$ with designated value $\{1\}$ i.e. $\mathfrak{M}_n = \langle|\mathfrak{M}_n|, \Omega_n, \{1\}\rangle$, where $|\mathfrak{M}_n| = \{1, 2, \ldots, n\}$, $\Omega_n = \{f_n^{\sim}, f_n^{\rightarrow}, f_n^{\wedge}, f_n^{\vee}\}$ and for every $x, y \in |\mathfrak{M}_n|$

$$f_n^{\sim}(x) = \begin{cases} n, \text{ if } x < n, \\ 1, \text{ if } x = n, \end{cases} \qquad f_n^{\rightarrow}(x, y) = \begin{cases} 1, \text{ if } x \geqslant y, \\ y, \text{ if } x < y, \end{cases}$$

$$f_n^{\wedge}(x, y) = \max(x, y), \qquad f_n^{\vee}(x, y) = \min(x, y).$$

(The set of all positive integers is denoted by N). The set of all formulas valid in matrix $\mathfrak{M}_n$ will be denoted by G_n, i.e. $\alpha \in G_n$ iff $v(\alpha) = 1$, for every valuation $v: S \to |\mathfrak{M}_n|$. In [5] Thomas proved that $G_n = \mathrm{Cn}(R_{0*}, A_{G_n})$, $n \in N$, where $A_{G_n} = H \cup \{T_n\}$ and T_n is defined: $T_1 = p_1$, $T_{n+1} = ((p_n \to p_{n+1}) \to p_1) \to T_n$.

By matrix $\mathfrak{M}_\omega$ we mean an algebra $\langle|\mathfrak{M}_\omega|, \Omega_\omega\rangle$ with designated value $\{1\}$, i.e. $\mathfrak{M}_\omega = \langle|\mathfrak{M}_\omega|, \Omega_\omega, \{1\}\rangle$, where $|\mathfrak{M}_\omega| = \{1, 2, \ldots, n, \ldots \omega\}$, $\Omega_\omega = \{f_\omega^{\sim}, f_\omega^{\rightarrow}, f_\omega^{\wedge}, f_\omega^{\vee}\}$ and for every $x, y \in |\mathfrak{M}_\omega|$

$$f_\omega^{\sim}(x) = \begin{cases} \omega, \text{ if } x < \omega, \\ 1, \text{ if } x = \omega, \end{cases} \qquad f_\omega^{\rightarrow}(x, y) = \begin{cases} 1, \text{ if } x \geqslant y, \\ y, \text{ if } x < y, \end{cases}$$

$$f_\omega^{\wedge}(x, y) = \max(x, y), \qquad f_\omega^{\vee}(x, y) = \min(x, y), \text{(cf. [1])}.$$

The set of all formulas valid in the matrix $\mathfrak{M}_\omega$ will be denoted by LC, i.e. $\alpha \in LC$ iff $v(\alpha) = 1$, for every valuation $v: S \to |\mathfrak{M}_\omega|$. In [1] Dummett showed, that $LC = \mathrm{Cn}(R_*, A_{LC})$ where $A_{LC} = H \cup \{(p_1 \to p_2) \vee (p_2 \to p_1)\}$ and that

$$(**) \qquad LC = \bigcap_{n \in N} G_n.$$

From the definition given above it is easy to see that

$$(***) \qquad H \subset LC \subset \ldots \subset G_{n+1} \subset G_n \subset \ldots \subset G_2 \subset G_1 = S$$

where $X \subset Y$ means that $X \subseteq Y$ and $X \neq Y$.

LEMMA 1. *If $\alpha \in G_{n-1} - G_n$; then there exists a valuation $v: S \to |\mathfrak{M}_n|$ such that $v(\alpha) = 2$, $(n \in N)$.*

Proof. We will show that if $\alpha \in G_{n-1} - G_n$ then $v(\alpha) \leqslant 2$, for every $v: S \to |\mathfrak{M}_n|$ It is easy to see that the mapping $h: |\mathfrak{M}_n| \to |\mathfrak{M}_{n-1}|$ defined:

$$h(1) = h(2) = 1; \quad h(k) = k-1, \text{ for } k = 3, \ldots, n.$$

is a homomorphism of an algebra $\langle|\mathfrak{M}_n|, \Omega_n\rangle$ onto an algebra $\langle|\mathfrak{M}_{n-1}|, \Omega_{n-1}\rangle$.

Now suppose that for $\alpha \in G_{n-1} - G_n$ there exists a valuation $\bar{v}: S \to |\mathfrak{M}_n|$ such that $\bar{v}(\alpha) = x_0 > 2$. Then for valuation $\bar{\bar{v}}(\alpha) = h(\bar{v}(\alpha))$, $\bar{\bar{v}}: S \to |\mathfrak{M}_{n-1}|$ we have $\bar{\bar{v}}(\alpha) = h(x_0) > 1$, which is impossible.

Let us define such a sequence of formulas:

$$A_1 = \sim(p_1 \to p_1), \quad V = p_1 \to p_1,$$

$$A_2 = p_2 \vee (p_2 \to p_1)$$

$$A_k = p_k \vee (p_k \to A_{k-1}), \quad \text{for } k > 2.$$

We write $\alpha \equiv \beta$ instead of $(\alpha \to \beta) \wedge (\beta \to \alpha)$, and $\alpha \in H$ for α derivable in intuitionistic propositional calculus.

LEMMA 2. *The following formulas are derivable in the intuitionistic propositional calculus*:

(i) $\sim A_i \equiv A_1$, *for* $i > 1$; $\quad \sim V \equiv A_1$; $\quad \sim A_1 \equiv V$,

(ii) a) $(A_{i+j} \to A_i) \equiv A_i$, $\quad (V \to A_i) \equiv A_i$,

b) $(A_i \to A_{i+j}) \equiv V$, $\quad (A_i \to V) \equiv V$,

(iii) $(A_{i+j} \vee A_i) \equiv A_{i+j}$, $\quad (A_i \vee V) \equiv V$,

(iv) $(A_{i+j} \wedge A_i) \equiv A_i$, $\quad (A_i \wedge V) \equiv A_i$.

Proof. (i). Since A_i, $i > 1$, are classical tautologies it follows that $\sim\sim A_i \in H$. From this, using the formula $\sim\sim\alpha \to (\sim\alpha \to \beta) \in H$, we have (i).

(ii) follows by induction on j using the formulas respectively

a) $((\alpha \vee (\alpha \to \beta)) \to \beta) \to \beta \in H$, $((\alpha \to \beta) \to \beta) \to (((\gamma \vee (\gamma \to \alpha)) \to \beta) \to \beta) \in H$

b) $\alpha \to (\beta \vee (\beta \to \alpha)) \in H$, $(\alpha \to \beta) \to (\alpha \to (\gamma \vee (\gamma \to \beta))) \in H$.

(iii) and (iv) follows immediately from (ii) by formulas

$(\alpha \to \beta) \to (\alpha \vee \beta \equiv \beta) \in H$

$(\alpha \to \beta) \to (\alpha \wedge \beta \equiv \alpha) \in H$, respectively.

Let $\{F_n\}$ be a sequence of sets defined by

$$F_1 = \{V\},\ F_2 = \{V, A_1\}, \ldots, F_k = \{V, A_{k-1}, A_{k-2}, \ldots, A_1\}.$$

Consider a matrix $\mathcal{F}_n = \langle F_n, \mho_n, \{V\}\rangle$, where $\mho_n = \{\sim_n, \to_n, \wedge_n, \vee_n\}$ is a set of functions on F_n defined by

$$\begin{aligned} \sim_n B_1 = B_2 \quad &\text{iff} \quad \sim B_1 \equiv B_2 \in H, \\ B_1 \to_n B_2 = B_3 \quad &\text{iff} \quad (B_1 \to B_2) \equiv B_3 \in H, \\ B_1 \wedge_n B_2 = B_3 \quad &\text{iff} \quad (B_1 \wedge B_2) \equiv B_3 \in H, \\ B_1 \vee_n B_2 = B_3 \quad &\text{iff} \quad (B_1 \vee B_2) \equiv B_3 \in H, \\ &B_1, B_2, B_3 \in F_n. \end{aligned}$$

LEMMA 3. *A matrix* $\mathcal{F}_n$ *is isomorphic to a matrix* $\mathfrak{M}_n$, *for every* $n \in N$.

Proof. For arbitrary $n \in N$ let

$$\mathcal{A}_i = \begin{cases} A_i, & \text{if } i = 1, \ldots, n-1 \\ V, & \text{if } i = n. \end{cases}$$

The mapping $\lambda : |\mathfrak{M}_n| \to F_n$ such that $\lambda(k) = \mathcal{A}_{n-k+1}$ is an isomorphism of $\mathfrak{M}_n$ onto $\mathcal{F}_n$. This is easily seen, since by Lemma 2 we have

(i) $\sim_n \mathcal{A}_{n-k+1} = \mathcal{A}_{n-f_n^{\sim}(k)+1}$,

(ii) $\mathcal{A}_{n-k+1} \to_n \mathcal{A}_{n-l+1} = \mathcal{A}_{n-f_n^{\to}(k,l)+1}$,

(iii) $\mathcal{A}_{n-k+1} \vee_n \mathcal{A}_{n-l+1} = \mathcal{A}_{n-f_n^{\vee}(k,l)+1}$,

(iv) $\mathcal{A}_{n-k+1} \wedge_n \mathcal{A}_{n-l+1} = \mathcal{A}_{n-f_n^{\wedge}(k,l)+1}$,

for $k, l = 1, \ldots, n$.

THEOREM. (a) $\langle R_0, \mathrm{Sb}(A_{G_n})\rangle \in \mathrm{SCpl}$, $(n \in N)$;

(b) $\langle R_0, \mathrm{Sb}(A_{LC})\rangle \in \mathrm{SCpl}$.

Proof. Both cases will be proved by using condition $(*)$. (a). Let $\pi = \{\alpha_1, \ldots, \alpha_j\}$ and assume that $\beta \notin \mathrm{Cn}(R_0, \mathrm{Sb}(A_{G_n}) \cup \pi)$ for some $n \in N$. Then $\Theta = \alpha_1 \to (\alpha_2 \to \ldots \ldots \to (\alpha_j \to \beta) \ldots) \notin \mathrm{Cn}(R_0, \mathrm{Sb}(A_{G_n})) = G_n$. By $(***)$ there is a $k \leqslant n$ such that $\Theta \in G_{k-1} - G_k$. From these by Lemma 1 there exists a valuation $v : S \to |\mathfrak{M}_k|$ such that $v(\Theta) = 2$, i.e.

$$v(\alpha_i) = 1, \quad v(\beta) = 2, \quad i = 1, \ldots, j.$$

Now define a substitution $e_k : At \to F_k$

$$e_k(p_i) = \lambda(v(p_i))$$

where λ is an isomorphism of $\mathfrak{M}_k$ onto $\mathcal{F}_k$ defined as in the proof of Lemma 3. It is easy to show by induction over the length of the formula α that

$$h^{e_k}(\alpha) \equiv \lambda(v(\alpha)) \in H.$$

In particular, by definition of λ, we have $h^{e_k}(\alpha_i) \equiv V \in H$, i.e. $h^{e_k}(\alpha_i) \in H$, and $h^{e_k}(\beta) \equiv A_{k-1} \in H$. It means that $h^{e_k}(\alpha_i) \in G_m$ for $m \geqslant 1$ and $h^{e_k}(\beta) \notin G_m$ for $m \geqslant k$, since $A_s \notin G_t$ for $s < t$.

Hence $h^{e_k}(\pi) \subseteq G_n$ and $h^{e_k}(\beta) \notin G_n$ which proves condition $(*)$ in case (a).

(b). Proof of (b) is quite analogous to that of (a). If $\Theta = \alpha_1 \to (\alpha_2 \to \ldots \to (\alpha_j \to \beta) \ldots) \notin LC = \bigcap_{n \in N} G_n$ then by $(***)$ $\Theta \in G_{k-1} - G_k$ for some $k \in N$. Now applying the same substitution e_k as in case (a) we have $h^{e_k}(\alpha_i) \in G_m$ for each $m \in N$ and $i \leqslant j$. Hence $h^{e_k}(\pi) \subseteq \bigcap_{n \in N} G_n = LC$ and $h^{e_k}(\beta) \notin LC$.

This completes the proof of the Theorem.

It is known that if $\langle R_0, \mathrm{Sb}(A)\rangle \in \mathrm{SCpl}$ then $\langle R_{0*}, A\rangle \in \mathrm{SCpl}$ (converse is not true). Hence, by the above theorem $\langle R_{0*}, A_{G_n}\rangle \in \mathrm{SCpl}$ and $\langle R_{0*}, A_{LC}\rangle \in \mathrm{SCpl}$.

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Allatum est die 22 Junii 1972