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Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree

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We construct a new class of positive definite and compactly supported radial functions which consist of a univariate polynomial within their support. For given smoothness and space dimension it is proved that they are of minimal degree and unique up to a constant factor. Finally, we establish connections between already known functions of this kind.

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1. Introduction

In translation-invariant spaces (cf. [12]) interpolants of the form

$$s_f(x) = \sum_{j=1}^{N} \alpha_j \Phi(x - x_j)$$

are natural tools to solve interpolation problems like $s_f(x_j) = y_j$, $1 \leq j \leq N$, with pairwise distinct points $x_j \in \mathbb{R}^d$. In many cases it is useful to assume $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a radial function $\Phi(x) = \phi(\|x\|)$ with a univariate function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and the Euclidean norm $\|\cdot\|$, and to assume Φ to be positive definite, which means that for all data sets $\{x_1, \ldots, x_N\} \subseteq \mathbb{R}^d$ of pairwise distinct points the interpolation matrix

$$A = (\Phi(x_j - x_k))_{1 \leq j,k \leq N}$$

is positive definite. A continuous function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is said to be positive definite on $\mathbb{R}^d$, $\phi \in \mathbf{PD}_d$, if the induced function $\Phi(x) := \phi(\|x\|)$, $x \in \mathbb{R}^d$, is positive definite.

This method of multivariate interpolation using radial basis functions yields good numerical results (cf. [5]) and was studied extensively in recent years. Some further applications of this theory are the construction of neural networks (cf. [2]) and the construction and modelling of geometric objects (cf. [10]). If Φ is one of the usual basis functions like multiquadrics or thin plate splines still some problems exist, caused by a large number N of centers x_j. Special methods of computation (cf. [4]) and evaluation (cf. [9]) were developed to remedy this defect. These problems could be avoided if the radial basis function Φ is compactly supported. If Φ has compact support the interpolation matrices are sparse and for the evaluation of the interpolants s_f only few terms have to be considered. This leads to efficient algorithms for the computation and the evaluation of the interpolants (cf. [10]). Another advantage of compactly supported basis functions is the principle of locality which is well known from the classical B-splines. The change of one center x_j causes only a local change of s_f.

In this paper we investigate functions $\Phi(x) = \phi(\|x\|)$ of the following form:

$$\phi(r) = \begin{cases} p(r) & 0 \le r \le 1, \\ 0 & r > 1, \end{cases} \tag{1}$$

with a univariate polynomial $p(r) = \sum_{j=1}^N c_j r^j$, $c_N \neq 0$. We call N the degree of Φ or ϕ. Note that due to Bochner a function of this kind is positive definite, if and only if the d-variate Fourier transform

$$\begin{aligned}\hat{\Phi}(x) \equiv \mathcal{F}_d \phi(r) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \Phi(\omega) e^{x^T \omega}\, d\omega \\ &= r^{-(d-2)/2} \int_0^\infty \phi(t) t^{d/2} J_{(d-2)/2}(rt)\, dt\end{aligned}$$

is nonnegative and positive at least on an open subset. Here J_ν denotes the Bessel function of the first kind. Note that the operator $\mathcal{F}_d$ acts in this way on univariate functions. Up to now, only few positive definite functions of the form (1) are known. These are for odd space dimension $d = 2n + 1$:

1. Euclid's Hat $X^{(2n+1)}$ which is generated by d-variate convolution of the characteristic function of the unit ball with itself, and which is in $\mathbf{PD}_{2n+1} \cap C^0$ with a degree $\partial X^{(2n+1)} = 2n + 1$
2. Wu's functions $\phi_{k+n,n} \in \mathbf{PD}_{2n+1} \cap C^{2k}$ with a degree $\partial \phi_{k+n,n} = 2\lfloor d/2 \rfloor + 4k + 1$.

Here, $\lfloor x \rfloor$ stands for the integer n satisfying $n \le x < n + 1$. In the first case it is used that the d-variate convolution of radial functions is again radial. We denote this with

$$(\phi *_d \psi)(\|x\|) = \int_{\mathbb{R}^d} \phi(\|y\|)\psi(\|x - y\|)\, dy.$$

Both of these function types are handled in section 4, and a connection between them is given.

Another known function is the truncated power function

$$\psi_\ell(r) = (1 - r)_+^\ell, \tag{2}$$

which is in $\mathbf{PD}_d$ if $\ell \ge \lfloor d/2 \rfloor + 1$ (cf. [1,3,7]). Starting with this function we construct in the following section a class of new functions $\psi_{\ell,k}$. In section 3 we show that these functions are in $\mathbf{PD}_d \cap C^{2k}$ and of degree $\partial\psi_{\ell,k} = \lfloor d/2 \rfloor + 3k + 1$. The degree turns out to be minimal under the requirement $\phi \in C^{2k}$, and the minimal-degree solution is unique up to a constant factor. Further, it will become obvious why we always talk about even degrees of smoothness.

2. Construction

We start with some notation. For $\alpha \in \mathbb{R}\backslash\{-1\}$, $\nu \in \mathbb{R}$ and $\ell \in \mathbb{N}$ we define bracket-operators

$$[\alpha]_{-1} := \frac{1}{\alpha+1}, \qquad [\alpha]_0 := 1, \qquad [\alpha]_\ell := \alpha(\alpha-1)\cdots(\alpha-\ell+1), \qquad \alpha \geq \ell - 1$$

and

$$(\nu)_0 := 1, \qquad (\nu)_\ell := \nu(\nu+1)\cdots(\nu+\ell-1).$$

Then we have simple relations such as $[\alpha]_\ell = (\alpha-\ell+1)_\ell$, $[\alpha]_\ell[\alpha-\ell+1]_n = (\alpha-\ell+1)[\alpha]_{\ell+n-1}$ and $(\nu)_\ell(\nu+\ell)_n = (\nu)_{\ell+n}$. After this we introduce the operators

$$I(f)(r) := \int_r^\infty sf(s)\,ds, \tag{3}$$

$$D(f)(r) := -\frac{1}{r}\frac{d}{dr}f(r), \tag{4}$$

for all functions f for which the terms on the right-hand side are well defined. The operators I, D, $\mathcal{F}_d$ and $*_d$ are connected in the following way.

Lemma 2.1

(1) If $\phi \in C^{2k}$ then $D^k I^k \phi = I^k D^k \phi = \phi$.
(2) If $I\phi \in L_1(\mathbb{R}^d)$ then $\mathcal{F}_d(I\phi) = \mathcal{F}_{d+2}(\phi)$.
(3) If $D\phi \in L_1(\mathbb{R}^d)$ then $\mathcal{F}_d(D\phi) = \mathcal{F}_{d-2}(\phi)$.
(4) If $I\phi$, $I\psi \in L_1(\mathbb{R}^d)$ then $I(\phi *_{d+2} \psi) = (I\phi) *_d (I\psi)$.
(5) If $D\phi$, $D\psi \in L_1(\mathbb{R}^d)$ then $D(\phi *_{d-2} \psi) = (D\phi) *_d (D\psi)$.

A proof of these facts can be found in [13].

Note that assertions (2) and (3) give rise to construction methods of compactly supported positive definite functions via integration or differentiation. Actually, Wu used statement (3) in [15] to construct such functions. He started with very smooth, positive definite functions in $\mathbb{R}^1$ and gained less smooth functions that are positive definite in higher dimensional spaces. Here, we take the opposite direction. We start with the function in (2) and construct

$$\psi_{\ell,k} := I^k \psi_\ell, \tag{5}$$

or, more general, $\psi_{\nu,k} := I^k \psi_\nu$ with $\nu > 0$ and $\psi_\nu(r) = (1-r)_+^\nu$. Therefore, we need first of all a lemma about the incomplete Beta function with one natural argument, which can be proved by simple induction.

Lemma 2.2
For $k \in \mathbb{N}$ and $\nu \in \mathbb{R}_{>0}$ we define

$$M_{k,\nu}(r) := \int_r^1 s^k \psi_\nu(s)\, ds.$$

Then we obtain the identity

$$M_{k,\nu}(r) = \sum_{\ell=0}^{k} \frac{[k]_\ell}{(\nu+1)_{\ell+1}} r^{k-\ell} \psi_{\nu+\ell+1}(r).$$

Using this lemma we get the following representation of $\psi_{\nu,k}$, which can again be proved by induction.

Theorem 2.3
We have the representation

$$\psi_{\nu,k}(r) = \sum_{n=0}^{k} \beta_{n,k} r^n \psi_{\nu+2k-n}(r). \tag{6}$$

The coefficients satisfy the recursion

$$\beta_{0,0} = 1,$$

$$\beta_{j,k+1} = \sum_{n=j-1}^{k} \beta_{n,k} \frac{[n+1]_{n-j+1}}{(\nu+2k-n+1)_{n-j+2}}, \qquad 0 \le j \le k+1,$$

if the term for $n = -1$ for $j = 0$ is ignored.

If we now choose $\nu = \ell \in \mathbb{N}$ we get

Lemma 2.4
1. The function $\psi_{\ell,k}$ has support in $[0,1]$, and consists there of a polynomial of degree $\partial\psi_{\ell,k} = \ell + 2k$.
2. It possesses $2k$ continuous derivatives around zero.
3. It possesses $k + \ell - 1$ continuous derivatives around 1.

Proof
The proof can be given either directly by theorem 2.3 or by means of induction, taking into account that a function of the form (1) is in C^{2k} around zero iff the first k odd coefficients of p vanish. □

3. Main results

In this section we shall prove our main assertions by choosing ℓ for $\psi_{\ell,k}$ appropriately. The shortness of the proofs demonstrates the strength of the operators I and D.

Theorem 3.5
For every space dimension $d \in \mathbb{N}$ and every $k \in \mathbb{N}$ there exists a function ϕ of the form (1) with degree $\partial\phi = \lfloor d/2 \rfloor + 3k + 1$ and $\phi \in \mathbf{PD}_d \cap C^{2k}$.

Proof
We choose $\ell = \lfloor (d+2k)/2 \rfloor + 1 = \lfloor d/2 \rfloor + k + 1$. Then we get $\psi_\ell \in \mathbf{PD}_{d+2k} \cap C^0$ by the remarks in the introduction. Using lemma 2.1 leads us to $\psi_{\ell,k} \in \mathbf{PD}_d$. By lemma 2.4 we achieve that $\psi_{\ell,k}$ possesses $2k$ smooth derivatives around zero and $2k + \lfloor d/2 \rfloor$ smooth derivatives around 1. This completes the proof. □

Before we prove the optimality and the uniqueness of these functions, we wish to emphasize some additional features. If we take space dimension $d = 1$ our C^2-function $\psi_{2,1}$ has degree $\partial\psi_{2,1} = 4$ which is just one degree higher than the degree of the classical cubic B-splines.

The d-variate Fourier transform is computable by means of lemma 2.1 as

$$\begin{aligned}\mathcal{F}_d \psi_{\ell,k}(r) &= \mathcal{F}_d I^k \psi_\ell(r) = \mathcal{F}_{d+2k} \psi_\ell(r) \\ &= r^{-d-2k-\ell} \int_0^r (r-t)^\ell t^{(d/2)+k} J_{(d/2)+k-1}(t)\, dt,\end{aligned}$$

which is, except for $\psi_{1,0}$, strictly positive (see Gasper [6]), so that the techniques of Narcowich and Ward [8] and Schaback [11] can be used to obtain bounds for the condition of the interpolation matrix. The existence of zeros of the Fourier transforms was a very unpleasant feature of Wu's functions and the Euclidean Hat.

Figure 1 shows the function $\psi_{3,1}(r) \doteq (1-r)_+^4(1+4r) \in \mathbf{PD}_3 \cap C^3$. Table 1 contains a table of some of the new functions. Here, $\doteq$ means equality up to a constant.

Theorem 3.6
There exists no function ϕ of the form (1) with $\phi \in \mathbf{PD}_d \cap C^{2k}$ and $\partial\phi < \lfloor d/2 \rfloor + 3k + 1$.

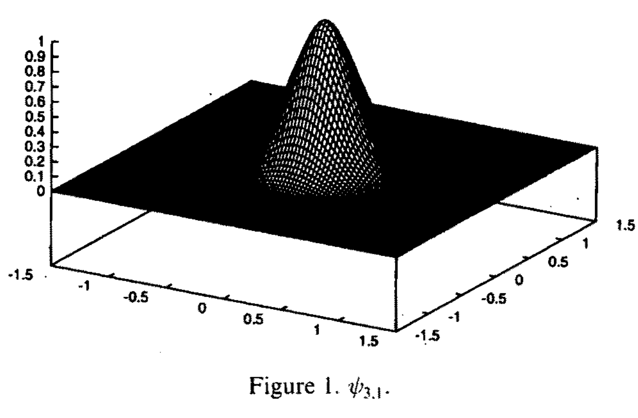

Figure 1. $\psi_{3,1}$.

Table 1
Table of functions.

$d=1$	$\psi_{1,0} = (1-r)_+$	C^0
	$\psi_{2,1} \doteq (1-r)_+^3(3r+1)$	C^2
	$\psi_{3,2} \doteq (1-r)_+^5(8r^2+5r+1)$	C^4
$d=3$	$\psi_{2,0} = (1-r)_+^2$	C^0
	$\psi_{3,1} \doteq (1-r)_+^4(4r+1)$	C^2
	$\psi_{4,2} \doteq (1-r)_+^6(35r^2+18r+3)$	C^4
	$\psi_{5,3} \doteq (1-r)_+^8(32r^3+25r^2+8r+1)$	C^6
$d=5$	$\psi_{3,0} = (1-r)_+^3$	C^0
	$\psi_{4,1} \doteq (1-r)_+^5(5r+1)$	C^2
	$\psi_{5,2} \doteq (1-r)_+^7(16r^2+7r+1)$	C^4

Proof
Suppose there exists such a function ϕ. On account of the C^{2k}-smoothness of ϕ we can write $p(r) = q(r^2) + r^{2k+1}h(r)$ with polynomials h, q, $\partial q = k$. But then $D^k p$ is again a polynomial because D operates on $q(r^2)$ as d/dr on q. So $\psi := D^k\phi$ is a function of the form (1) with $\psi \in \mathbf{PD}_{d+2k} \cap C^0$ and $\partial\psi < \lfloor (d+2k)/2 \rfloor + 1$. This is a contradiction to the following lemma. □

Lemma 3.7
If ϕ is a continuous, on $\mathbb{R}^d$ positive definite function of the form (1) then ϕ has degree greater than or equal to $\lfloor d/2 \rfloor + 1$.

Proof
A proof of this was given by Chanysheva [3] and uses the existence of $\lfloor d/2 \rfloor$ smooth derivatives, which again follows in odd dimensional spaces easily from the integral representation for positive definite radial functions. In even dimensional spaces the special form of ϕ has to be considered. □

Note that there is no difference between the optimal function for $d = 2n$ and $d = 2n+1$. This optimal function is in the continuous case unique if it is normalized by $\phi(0) = 1$, namely $\phi(r) = (1-r)^{\lfloor d/2 \rfloor + 1}$. This generalizes to

Theorem 3.8
Up to a constant factor there exists exactly one positive definite function $\Phi(x) = \phi(\|x\|)$ on $\mathbb{R}^d$ with ϕ of the form (1), i.e.

$$\phi(r) = \begin{cases} p(r) & 0 \le r \le 1, \\ 0 & r > 1, \end{cases}$$

consisting of a univariate polynomial p of degree $\partial p = \partial\phi = \lfloor d/2 \rfloor + 3k + 1$ and satisfying $\phi \in C^{2k}$.

Proof
Assume there exist two such functions ϕ, ψ with $\phi \neq c\psi$, $c \neq 0$. Then we must also have $\tilde{\phi} := D^k\phi \neq cD^k\psi =: \tilde{\psi}$ because of $\phi = I^k\tilde{\phi}$ and $\psi = I^k\tilde{\psi}$ by lemma 2.1. But this contradicts the just mentioned uniqueness in the continuous case. □

4. Connection between Wu's functions and Euclid's Hat

In the introduction we mentioned that in the odd dimensional case two additional instances of positive definite functions of the form (1) exist. We will now show that these functions are connected by the same trick we used to construct our optimal functions. Let us suppose in this section that $d = 2\ell + 1$ is odd. As both function types are constructed via convolution, let us suppose for simplicity now that their support is $\{x : 0 \leq \|x\| \leq 2\}$.

The Euclidean Hat $X^{(2\ell+1)}$ is constructed by $2\ell + 1$-variate convolution of the characteristic function χ of the unit ball with itself. Explicit formulas are given in [14] and the proof is sketched in [10]. These functions are of the form (1) and of degree $\partial X^{(2\ell+1)} = 2\ell + 1$. Wu constructed his functions by starting with $f_\ell(r) = (1 - r^2)_+^\ell$, $\phi_\ell = f_\ell * f_\ell$ with univariate convolution and ended with $\phi_{\ell,k} = D^k\phi_\ell$, $0 \leq k \leq \ell$. This led him to functions $\phi_{\ell,k} \in \mathbf{PD}_{2k+1} \cap C^{2\ell-2k}$.

If we mark equality up to a constant again with $\doteq$ we get

Theorem 4.9
Wu's functions $\phi_{\ell,k}$ and Euclid's Hat $X^{(2\ell+1)}$ are connected by

$$\phi_{\ell,k} \doteq I^{\ell-k} X^{(2\ell+1)}.$$

Proof
We use lemma 2.1 to get

$$\begin{aligned}
\phi_{\ell,k} &= D^k(f_\ell *_1 f_\ell) \\
&= (D^k f_\ell) *_{2k+1} (D^k f_\ell) \\
&= (I^{\ell-k} D^\ell f_\ell) *_{2k+1} (I^{\ell-k} D^\ell f_\ell) \\
&= I^{\ell-k}((D^\ell f_\ell) *_{2\ell+1} (D^\ell f_\ell)) \\
&\doteq I^{\ell-k} X^{(2\ell+1)},
\end{aligned}$$

if we take into account that $D^\ell f_\ell(r) = 2^\ell \ell! \chi(r)$. □

Note that this leads to explicit formulas for Wu's functions, but it is obvious that these formulas cannot be as simple as the formulas given in theorem 2.3.

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