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TILINGS OF POLYGONS WITH SIMILAR TRIANGLES

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We prove that if a polygon P is decomposed into finitely many similar triangles then the tangents of the angles of these triangles are algebraic over the field generated by the coordinates of the vertices of P. If P is a rectangle then, apart from four "sporadic" cases, the triangles of the decomposition must be right triangles. Three of these "sporadic" triangles tile the square. In any other decomposition of the square into similar triangles, the decomposition consists of right triangles with an acute angle α such that $\tan\alpha$ is a totally positive algebraic number. Most of the proofs are based on the following general theorem: if a convex polygon P is decomposed into finitely many triangles (not necessarily similar) then the coordinate system can be chosen in such a way that the coordinates of the vertices of P belong to the field generated by the cotangents of the angles of the triangles in the decomposition.

Introduction

We shall say that a triangle Δ tiles the polygon P, if P can be decomposed into finitely many non-overlapping triangles similar to Δ. The following problem, which was the starting point of our investigations, was posed by L. Pósa [4]: does the triangle with angles 30°, 60°, 90° tile the square? As we shall see, the answer is no, and thus the question arises, exactly which triangles tile the square? This problem proves to be surprisingly difficult, and we only give a partial answer.

In Section 5 we shall prove that each of the triangles with angles $(\pi/8,\ \pi/4,\ 5\pi/8)$, $(\pi/4,\ \pi/3,\ 5\pi/12)$ or $(\pi/12,\ \pi/4,\ 2\pi/3)$ tiles the square. Apart from these, only right triangles can tile the square. Moreover, as we show in Theorem 27, if a right triangle with acute angle α tiles the square, then $\tan\alpha$ must be algebraic and such that each of its real conjugates is positive (these numbers are called totally positive). Since $-\sqrt{3}$ is a conjugate of $\tan 60° = \sqrt{3}$, this shows that the triangle with angles 30°, 60°, 90° does not tile the square.

Apart from the three triangles listed above, there is only one more which can tile a rectangle and has no right angle. This is the triangle with angles $(\pi/6,\ \pi/6,\ 2\pi/3)$, which tiles the rectangles of size $a \times b$, where $b/a = r\sqrt{3}$ and r is rational (Theorem 23).

As for tilings of arbitrary polygons, we prove that a triangle can tile a polygon P only if the cotangents of its angles are algebraic over the field generated by the

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coordinates of the vertices of P (Theorem 14). This shows, in particular, that the number of triangles tiling a given polygon is at most countable.

The underlying results to the theorems above are two generalizations of the following theorem by M. Dehn (see [1] or [2], p. 77): if a rectangle R can be decomposed into finitely many non-overlapping squares then (i) the sides of R are commensurable, and (ii) the sides of the squares in the decomposition are also commensurable.

We shall prove that whenever a polygon P is decomposed into finitely many triangles (similar or not), then the coordinates of the vertices of the triangles belong to the field generated by the coordinates of the vertices of P and the cotangents of the angles of the triangles (Theorem 1). For example, if P is a rectangle with commensurable sides then we may assume that the coordinates of the vertices of P are rational. Now suppose that P is decomposed into finitely many isosceles right triangles. Since $\cot 45^\circ = 1$ and $\cot 90^\circ = 0$, our theorem implies that the coordinates of the triangles in the decomposition are all rational. Thus we obtain the second statement of Dehn's theorem by dividing the squares into isosceles right triangles.

We also prove that if a *convex* polygon P is decomposed into finitely many triangles (similar or not), then we can choose the coordinate system in such a way, that the coordinates of the vertices belong to the field generated by the cotangents of the angles of the triangles (Theorem 2). If, for example, P is decomposed into isosceles right triangles, then, in an appropriate coordinate system, the coordinates of its vertices are rational. If P is a rectangle, then we obtain Dehn's theorem.

As a further application of these results, we shall prove that the square cannot be decomposed into finitely many non-overlapping triangles in such a way that the size of every angle of the triangles, when measured in degrees, is an even integer (Corollary 10). This also shows that the triangle with angles 30°, 60°, 90° does not tile the square.

1. Tilings with arbitrary triangles

In this paper all polygons are supposed to be simple and to have all angles different from π. We begin with the generalizations of Dehn's theorem.

Theorem 1. *Suppose that a polygon P is decomposed into the non-overlapping triangles $\Delta_1, \dots, \Delta_N$. Then the coordinates of the vertices of each Δ_j belong to the field generated by the coordinates of the vertices of P and the cotangents of the angles of the triangles Δ_j.*

Theorem 2. *Let the convex polygon K be decomposed into the non-overlapping triangles $\Delta_1, \dots, \Delta_N$ and let F_Δ denote the field generated by the cotangents of the angles of Δ_j $(j = 1, \dots, N)$.*

If K has two vertices with coordinates in F_Δ then the coordinates of each vertex of K and of each triangle Δ_j belong to F_Δ.

The proof of Theorems 1 and 2 is based on the following Lemma.

Lemma 3. *Suppose that a polygon P is decomposed into the non-overlapping triangles $\Delta_1, \dots, \Delta_N$ and let X denote the set of $x-$coordinates of the vertices of*

Δ_j $(j = 1, \ldots, N)$. *Let F be a subfield of the reals such that the slopes of the non-vertical sides of the triangles Δ_j belong to F. Let $x_1 \in X$, $\min X < x_1 < \max X$, and suppose that whenever $V = (x_1, y)$ is a vertex of P then P has a convex angle at V and every open vertical segment containing V intersects the interior of P.*

Then there are points $y_i \in X$ $(i = 1, \ldots, n)$; $z_i \in X$ $(i = 1, \ldots, m)$ and $t_i, s_i \in F \cap \mathbf{R}^+$ such that

$$y_i < x_1 \quad (i = 1, \ldots, n), \quad x_1 < z_i \quad (i = 1, \ldots, m) \tag{1}$$

and

$$\sum_{i=1}^{n} t_i(x_1 - y_i) = \sum_{i=1}^{m} s_i(z_i - x_1). \tag{2}$$

Proof. Since $x_1 \in X$, there is an x_2 such that $V = (x_1, x_2)$ is a vertex of one of the triangles Δ_j. The condition on x_1 easily implies that one of the following statements must hold.

(i) There is a triangle Δ_j with vertices $V = (x_1, x_2)$, $B = (y_1, y_2)$ and $C = (z_1, z_2)$ such that $y_1 < x_1 < z_1$.

(ii) There is a triangle Δ_j having a vertical side on the line $x = x_1$.

If (i) holds then there is a point $D = (x_1, u)$ on the segment BC. Let $\alpha, \beta, \gamma, \delta$ denote the angles DVB, VDB, DVC, VDC, respectively, then the cotangents of these angles belong to F. We have

$$|x_2 - u| = (x_1 - y_1)(\cot\alpha + \cot\beta) = (z_1 - x_1)(\cot\gamma + \cot\delta).$$

This implies

$$t(x_1 - y_1) = s(z_1 - x_1),$$

where $t = (\cot\alpha + \cot\beta)$ and $s = (\cot\gamma + \cot\delta)$, and hence the statement of the Lemma is satisfied with $n = m = 1$.

Next suppose (ii) and let $\ell = \{x_1\} \times [b, c]$ be a maximal vertical segment containing V and contained in the union of the boundaries of the triangles Δ_j. Obviously, (x_1, b) and (x_1, c) are both vertices of some triangles Δ_j. Let L and R denote the sets of all triangles Δ_j which have a side on the segment ℓ and lie to the left and to the right of ℓ, respectively.

It follows from the choice of x_1 that the segment ℓ, apart from its endpoints, lies in the interior of P. Therefore both of L and R cover ℓ (see Figure 1.) We may assume that the triangles Δ_j are indexed in such a way that

$$L = \{\Delta_1, \ldots, \Delta_n\}, \quad R = \{\Delta_{n+1}, \ldots, \Delta_{n+m}\},$$

and there are subdivisions

$$b = u_0 < u_1 < \ldots < u_n = c,$$
$$b = v_0 < v_1 < \ldots < v_m = c$$

such that $\{x_1\} \times [u_{i-1}, u_i]$ is a side of Δ_i for every $i = 1, \ldots, n$ and $\{x_1\} \times [v_{i-1}, v_i]$ is a side of Δ_{n+i} for every $i = 1, \ldots, m$. Let the vertices of Δ_i be $A_i = (y_i, y_i')$, $B_i = (x_1, u_{i-1})$, $C_i = (x_1, u_i)$ $(i = 1, \ldots, n)$ and of Δ_{n+i} be $D_i = (z_i, z_i')$, $E_i = (x_1, v_{i-1})$, $F_i = (x_1, v_i)$ $(i = 1, \ldots, m)$.

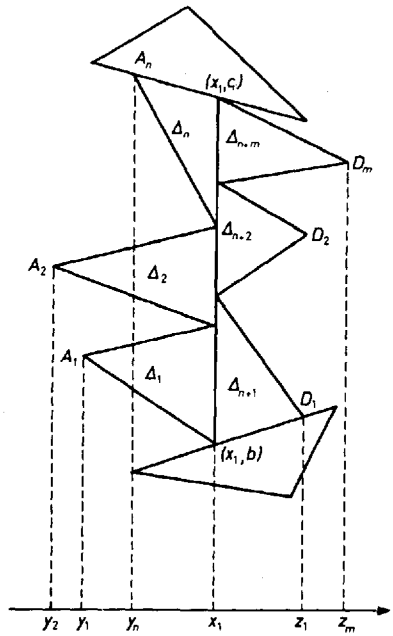

Fig. 1

Then $y_i < x_1$ for every $i = 1, \dots, n$ and $z_i > x_1$ for every $i = 1, \dots, m$. If β_i and γ_i denote the angles of Δ_i at the vertices B_i and C_i then we have $u_i - u_{i-1} = t_i(x_1 - y_i)$, where $t_i = \cot\beta_i + \cot\gamma_i$ $(i = 1, \dots, n)$. Obviously, $t_i > 0$, $t_i \in F$ and $c - b = \sum_{i=1}^{n} t_i(x_1 - y_i)$. We can see in the same way that there are positive numbers $s_i \in F$ such that $c - b = \sum_{i=1}^{m} s_i(z_i - x_1)$. Therefore (1) and (2) are satisfied and this completes the proof of the Lemma. ∎

Proof of Theorem 1. Let F denote the field generated by the coordinates of the vertices of P and by the cotangents of the angles of the triangles Δ_j $(j = 1, \dots, N)$. Obviously, the slopes of the sides of P belong to F. Since the angles between the sides of any of the triangles Δ_j and of P belong to the additive group generated by the angles of the triangles Δ_j and π, it follows that the slopes of the sides of Δ_j belong to F.

Let $x_0 < x_1 < \dots < x_p$ denote the x−coordinates of the vertices of the triangles Δ_j $(j = 1, \dots, N)$. It is enough to prove that each x_i belongs to F. Let L be the linear space over the field F generated by the numbers x_i $(i = 0, \dots, p)$ and let B be a basis of L containing 1. We have to show that $B = \{1\}$. Suppose this is not true and let an element $b_0 \in B \setminus \{1\}$ be selected. Each $x \in L$ can be written uniquely in the form $\sum tb$, where $t \in F$, $b \in B$. Let $c(x)$ denote the coefficient of b_0 in this representation of x. By the definition of B, $c(x_i) \neq 0$ holds for at least one i; we may assume that $c(x_i) > 0$ for some i. Let $m = \max\{c(x_i) : i = 0, \dots, p\}$, then $m > 0$. Let r be the greatest index with $c(x_r) = m$. Then $0 < r < p$. Indeed, x_0 and x_p are the x−coordinates of two vertices of P and hence they belong to F. Thus

$c(x_0) = c(x_p) = 0$ and, consequently, $0 < r < p$. By the same argument, x_r cannot be the x-coordinate of any of the vertices of P. This implies that x_r satisfies the conditions of the Lemma 3 and hence there are numbers t_i, s_i and points y_i and z_i satisfying (1) and (2) with x_r in place of x_1. Since $t_i,\ s_i \in F$, (2) implies

$$\sum_{i=1}^{n} t_i(c(x_r) - c(y_i)) = \sum_{i=1}^{m} s_i(c(z_i) - c(x_r)). \tag{3}$$

By the choice of x_r, the left hand side of (3) is non-negative, while the right hand side is strictly negative. This contradiction completes the proof. ∎

Proof of Theorem 2. By Theorem 1, it is enough to show that the coordinates of the vertices of K belong to F_Δ.

First we suppose that K satisfies the following conditions. *(i)* The origin is a vertex of K and is the lower endpoint of a vertical side of K, and *(ii)* the greatest x-coordinate of the vertices of K is 1. Let $0 = x_0 < x_1 < \ldots < x_p = 1$ be the x-coordinates of the vertices of the triangles Δ_j $(j = 1, \ldots, N)$. The convexity of K easily implies that each of the points x_i $(1 \le i \le p-1)$ satisfies the conditions of the Lemma 3. This implies, by the argument used in the proof of Theorem 1, that $x_i \in F_\Delta$ for every $i = 1, \ldots, p$. In particular, the x-coordinates of the vertices of K belong to F_Δ.

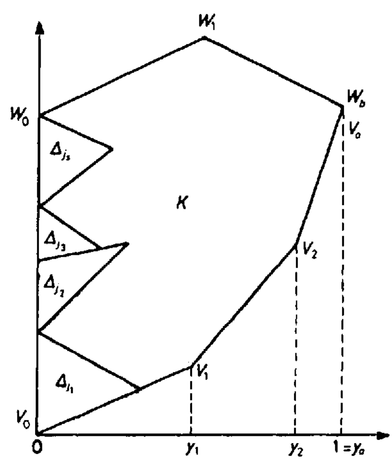

Fig. 2

Since K is convex, its vertices can be listed in two sequences $V_0 = (y_0, z_0), \ldots, V_a = (y_a, z_a)$ and $W_0 = (u_0, v_0), \ldots, W_b = (u_b, v_b)$, where $0 = y_0 < \ldots < y_a = 1$, $0 = u_0 < \ldots < u_b = 1$, and $0 = z_0 < v_0$ (see Figure 2). It is easy to see that the angles between the sides of K and the coordinate axes belong to the additive group generated by the angles of the triangles Δ_j. Therefore the slopes of the sides of K belong to F_Δ. Since $y_0, \ldots, y_a$ belong to F_Δ, this implies that $z_0, \ldots, z_a$ also belong to F_Δ. Let

$\Delta_{j_1}, \ldots, \Delta_{j_s}$ denote those triangles of the decomposition which have a vertical side lying on V_0W_0. Since the x-coordinates of the vertices, and also the slopes of the sides of these triangles are in F_Δ, it follows that the y-coordinates of the vertices of $\Delta_{j_1}, \ldots, \Delta_{j_s}$ also belong to F_Δ. This easily implies that $v_0 \in F_\Delta$. Therefore $v_0, \ldots, v_b$ are in F_Δ as well and hence the Theorem is proved supposing that K satisfies *(i)* and *(ii)*.

Considering the general case, suppose that K has two vertices with coordinates in F_Δ. Using a translation we may assume that one of these vertices is the origin. Since K is convex, there is a rotation R such that $R(K)$ satisfies *(i)* and lies in the half-plane $x \geq 0$. Then there is a homothetic transformation H such that $HR(K)$ satisfies both *(i)* and *(ii)*. Therefore, the coordinates of the vertices of $HR(K)$ belong to F_Δ. By assumption, there is a vertex (x, y) of K such that $x, y \in F_\Delta$, and $x^2 + y^2 \neq 0$. The matrix of HR is of the form

$$HR = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

and hence we have $ax+by \in F_\Delta$ and $-bx+ay \in F_\Delta$. This implies $a, b \in F_\Delta$. Therefore the entries of the matrix of $(HR)^{-1}$ also belong to F_Δ. Since the vertices of K are the images of those of $HR(K)$ under $(HR)^{-1}$, their coordinates belong to F_Δ. This completes the proof of Theorem 2. ∎

Remark 4. The condition of convexity cannot be removed from Theorem 2. Indeed, the hexagon H in Figure 3 is decomposed into two isosceles right triangles and has three vertices with rational coordinates, but the other vertices of H have irrational coordinates.

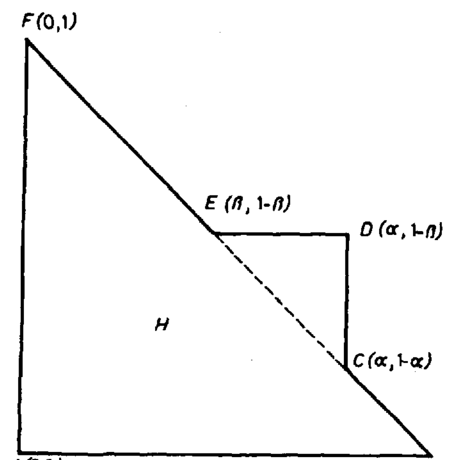

Fig. 3

Our next theorem states that whenever a polygon is decomposed into triangles then the cotangents of the angles of the triangles have to satisfy a certain condition concerning the isomorphisms of the field described in Theorem 1.

Theorem 5. *Let P be a polygon with vertices $V_i = (a_i, b_i)$ $(i = 1, \dots, n)$. Suppose that P is decomposed into the non-overlapping triangles Δ_j $(j = 1, \dots, N)$, and let α_j, β_j, γ_j denote the angles of Δ_j. Let F denote the field generated by the numbers a_i, b_i, $\cot\alpha_j$, $\cot\beta_j$, $\cot\gamma_j$ $(i = 1, \dots, n;\ j = 1, \dots, N)$, and let $\phi: F \to \mathbf{R}$ be a real isomorphism of F that leaves the numbers a_i, b_i, $(i = 1, \dots, n)$ fixed.*
Then there is a j such that at least two of the numbers $\phi(\cot\alpha_j)$, $\phi(\cot\beta_j)$, $\phi(\cot\gamma_j)$ are positive.

Lemma 6. *Let Δ be a triangle with angles α_i $(i = 1, 2, 3)$. Let the vertices of Δ be $V_i = (a_i, b_i)$ $(i = 1, 2, 3)$ and let F be a field containing the numbers a_i, b_i $(i = 1, 2, 3)$. Suppose that $\phi: F \to \mathbf{R}$ is a real isomorphism of F and let the map $\Phi: F \times F \to \mathbf{R}^2$ be defined by*

$$\Phi((x, y)) = (\phi(x), \phi(y)) \quad ((x, y) \in F \times F). \tag{4}$$

Let Δ' denote the triangle with vertices $\Phi(V_i)$ $(i = 1, 2, 3)$, and let α'_i $(i = 1, 2, 3)$ denote the corresponding angles of Δ'. Let $\varepsilon = 1$ if the map Φ does not change the orientation of Δ, and let $\varepsilon = -1$ otherwise.

Then we have $\cot\alpha'_i = \varepsilon\phi(\cot\alpha_i)$ $(i = 1, 2, 3)$.

Proof. Suppose that the order of indices of the vertices (a_i, b_i) corresponds to the positive orientation of Δ. If the angle α_1 is at the vertex (a_1, b_1) and $b_1 \neq b_2$, $b_1 \neq b_3$ then we have

$$\cot\alpha_1 = \frac{[(a_3 - a_1)/(b_3 - b_1)] \cdot [(a_2 - a_1)/(b_2 - b_1)] + 1}{[(a_2 - a_1)/(b_2 - b_1)] - [(a_3 - a_1)/(b_3 - b_1)]} \tag{5}$$

so that $\cot\alpha_1 \in F$. This formula also shows that if $\varepsilon = 1$ then $\cot\alpha'_1 = \phi(\cot\alpha_1)$. On the other hand, if the order of indices corresponds to the negative orientation of Δ, then in the right hand side of (5) the indices 2 and 3 have to be interchanged. This implies that in the case of $\varepsilon = -1$ we have $\cot\alpha'_1 = -\phi(\cot\alpha_1)$. Similar argument applies if $b_1 = b_2$ or $b_1 = b_3$. ∎

Proof of Theorem 5. We may assume that the vertices $V_1, \dots, V_n$ are listed counterclockwise. Let (a_j^i, b_j^i) $(i = 1, 2, 3)$ denote the vertices of Δ_j listed also counterclockwise. By Theorem 1, $a_j^i, b_j^i \in F$ holds for every i and j. Let Δ'_j denote the oriented triangle with vertices $(\phi(a_j^i), \phi(b_j^i))$ $(i = 1, 2, 3)$.

If we consider the boundary $\partial\Delta_j$ of the triangle Δ_j as a cycle (sum of oriented segments), we can see that $\sum_{j=1}^N \partial\Delta_j$ equals the cycle $\Gamma = V_1 \dots V_n V_1$.

Since ϕ is an isomorphism, the map Φ defined by (4) is a collineation on $F \times F$. Therefore the sum of the cycles $\delta_j = \partial\Delta'_j$ equals the cycle Γ, as Φ leaves the vertices V_k fixed. Since the cycle Γ is positively oriented, at least one of the triangles Δ'_j must be positively oriented as well. Indeed, the integrals $I_j = \int_{\delta_j} x\,dy$ and $I = \int_\Gamma x\,dy$ equal the oriented areas of Δ'_j and P, respectively. Since $\sum_{j=1}^N I_j = I$ and I is positive, at least one of the numbers I_j must be positive.

If the orientation of the triangle Δ'_j is positive and α'_j, β'_j, γ'_j denote its angles then, by Lemma 6, we have

$$\cot\alpha'_j = \phi(\cot\alpha_j), \ \cot\beta'_j = \phi(\cot\beta_j), \ \cot\gamma'_j = \phi(\cot\gamma_j).$$

Since at least two of α_j', β_j', γ_j' are acute angles, at least two of their cotangents are positive, and this completes the proof. ∎

Corollary 7. *The square cannot be decomposed into finitely many non-overlapping triangles with angles* 30°, 60° *and* 90°.

Proof. Suppose that the unit square has such a decomposition. Since $\cot 30^\circ = \sqrt{3}$ and $\cot 60^\circ = \sqrt{3}/3$, we may apply Theorem 5 with $F = \mathbf{Q}(\sqrt{3})$. Let ϕ be the automorphism of $\mathbf{Q}(\sqrt{3})$ with $\phi(\sqrt{3}) = -\sqrt{3}$. Then ϕ maps $\cot 30^\circ$ and $\cot 60^\circ$ into negative numbers, which contradicts Theorem 5. ∎

Lemma 8. *Let n be a non-zero integer and put $\zeta = e^{\frac{\pi}{2n}i}$. Then*

$$\cot \frac{a}{n}\pi \in \mathbf{Q}(\zeta)$$

for every integer a not divisible by n. In addition, if k is prime to $2n$ then there is an automorphism ϕ of $\mathbf{Q}(\zeta)$ such that

$$\phi(\cot \frac{a}{n}\pi) = (-1)^{(k-1)/2} \cot \frac{ak}{n}\pi \tag{6}$$

for every a not divisible by n.

Proof. We have

$$\cot \frac{a}{n}\pi = i\,\frac{e^{\frac{a}{n}\pi i} + e^{-\frac{a}{n}\pi i}}{e^{\frac{a}{n}\pi i} - e^{-\frac{a}{n}\pi i}}.$$

Now $i = \zeta^n$, and hence

$$\cot \frac{a}{n}\pi = \zeta^n \frac{\zeta^{2a} + \zeta^{-2a}}{\zeta^{2a} - \zeta^{-2a}} \in \mathbf{Q}(\zeta). \tag{7}$$

If $(k, 2n) = 1$ then ζ^k is a primitive $4n^{\text{th}}$ root of unity and hence there is an automorphism ϕ of $\mathbf{Q}(\zeta)$ such that $\phi(\zeta) = \zeta^k$. Then (7) gives

$$\phi\left(\cot \frac{a}{n}\pi\right) = \zeta^{nk}\frac{\zeta^{2ak} + \zeta^{-2ak}}{\zeta^{2ak} - \zeta^{-2ak}} = \zeta^{n(k-1)} \cot \frac{ak}{n}\pi = (-1)^{(k-1)/2} \cot \frac{ak}{n}\pi,$$

since $\zeta^{n(k-1)} = i^{k-1} = (-1)^{(k-1)/2}$. ∎

Theorem 9. *Suppose that each vertex of the polygon P has rational coordinates. Let P be decomposed into non-overlapping triangles $\Delta_1, \dots, \Delta_N$ in such a way that the angles of the triangles Δ_j are all rational multiples of π.*

If the angles in question are $\frac{a_i}{n}\pi$ $(i = 1, \dots, 3N)$, then $4|n$.

Proof. Let F denote the field generated by the numbers $\cot \frac{a_i}{n}\pi$ $(i = 1, \dots, 3N)$. By Lemma 8, $F \subset \mathbf{Q}(\zeta)$. Suppose that $4 \nmid n$. Let $k = 2n + 1$ if n is odd, and let $k = n + 1$ if $n \equiv 2 \pmod 4$. Then $(k, 2n) = 1$ and hence, by Lemma 8, there is an automorphism ϕ of $\mathbf{Q}(\zeta)$ such that

$$\phi(\cot \frac{a_i}{n}\pi) = (-1)^{(k-1)/2} \cot \frac{a_i k}{n}\pi \tag{8}$$

holds for every i. This implies, in particular, that the restriction of ϕ to F is a real isomorphism. As $k \equiv -1 \pmod 4$ and $k \equiv 1 \pmod n$, we have $a_i k \equiv a_i \pmod n$ and hence the right hand side of (8) equals $-\cot \dfrac{a_i}{n}\pi$.

That is, if α is an angle of any of the triangles Δ_j then

$$\phi(\cot\alpha) = -\cot\alpha.$$

This implies, by Theorem 5, that there is a Δ_j such that two of the cotangents of its angles are negative. Since Δ_j has at least two acute angles, this is clearly impossible. This contradiction completes the proof of the Theorem. ∎

Corollary 10. *The square cannot be decomposed into finitely many non-overlapping triangles in such a way that the size of every angle of the triangles, when measured in degrees, is an even integer.*

Proof. Suppose that there is such a decomposition. We may assume that the vertices of the square have rational coordinates. If k is an even integer, $k = 2n$, then $k^\circ = k\pi/180 = n\pi/90$. Since all the angles are of this form, and 90 is not divisible by 4, this contradicts Theorem 9. ∎

Remark 11. We mention here the following theorem by P. Monsky [3]: the square cannot be decomposed into an odd number of non-overlapping triangles of the same area. In spite of the resemblance, there does not seem to be a close connection with Corollary 10. The proof of Monsky's theorem is based on valuation theory.

2. Conjugate tilings

Let P be a polygon with vertices $V_1, \ldots, V_n$ listed counterclockwise, and let P be decomposed into the non-overlapping triangles $\Delta_1, \ldots, \Delta_N$. Let $\mathcal{V}$ denote the set of the vertices of the triangles Δ_j $(j = 1, \ldots, N)$ and let $\Phi: \mathcal{V} \to \mathbf{R}^2$ be a map. If the vertices of the triangle Δ_j are $V_{j,i}$ $(i = 1,2,3)$ then we denote by Δ'_j the (possibly degenerate) triangle with vertices $\Phi(V_{j,i})$ $(i = 1,2,3)$. We say that Φ preserves the orientation of Δ_j if Δ'_j is degenerate, or the orientation of the vertices $\Phi(V_{j,i})$ $(i = 1,2,3)$ is the same as that of $V_{j,i}$ $(i = 1,2,3)$. The map Φ is a collineation if, whenever three points of $\mathcal{V}$ are collinear then their images are also collinear (including the possibility that some of them coincide).

Lemma 12. *Suppose, with the notation above, that Φ is a collineation and preserves the orientation of each of the triangles Δ_j $(j = 1, \ldots, N)$. If the points $\Phi(V_i)$ $(i = 1, \ldots, n)$ are the consecutive vertices of a polygon P' then the triangles Δ'_j $(j = 1, \ldots, N)$ are non-overlapping and constitute a tiling of P'.*

Proof. We give the positive orientation to each triangle Δ_j and denote by ∂_j the sum of the oriented segments of the boundary of Δ_j. Obviously, the sum of the cycles ∂_j $(j = 1, \ldots, N)$ equals the cycle $\partial = V_1, \ldots, V_n, V_1$. Let ∂'_j and ∂' denote the images of ∂_j and ∂ under Φ. Since Φ is a collineation, we have $\sum_{j=1}^N \partial'_j = \partial'$. Now, as ∂'_j is positively oriented for every j (or equals zero if Δ'_j is degenerate), this implies that the triangles Δ'_j are non-overlapping and cover P'. The easiest way to prove this is to use complex line integrals around the cycles ∂'_j. We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ and put

$$I_j(z) = \frac{1}{2\pi i} \int_{\partial'_j} \frac{1}{\xi - z} \, d\xi$$

for every $z \in \mathbf{C} \setminus \partial'_j$. Then $I_j(z) = 0$ or 1 according to whether z is outside or inside Δ'_j. Since

$$\sum_{j=1}^{N} I_j(z) = \frac{1}{2\pi i} \int_{\partial'} \frac{1}{\xi - z} \, d\xi = I(z),$$

and $I(z) = 1$ for every z covered by P' and $I(z) = 0$ for every z outside P', our assertion follows. ∎

Remark 13. The tilings of P and P' in the lemma above need not be topologically isomorphic. Consider the following example. Let the square $ABCD$ be decomposed into the triangles ABF, BCG, CDG, DAE, DEG, EFG as shown in Figure 4. Let Φ map $A, \dots, G$ to $A', \dots, G'$ shown in Figure 5. Then Φ is a collineation and preserves the orientation of the triangles $ABF, \dots, EFG$. Accordingly, the triangles $A'B'F', \dots, E'F'G'$ constitute a tiling of $A'B'C'D'$. However, these tilings are not isomorphic topologically. In fact, even the graphs shown in Figures 4 and 5 are non-isomorphic, as one of them contains a point with valency 5, while the other does not.

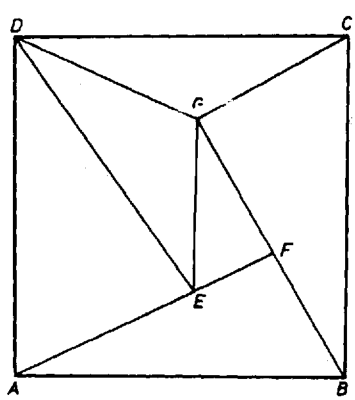

Fig. 4

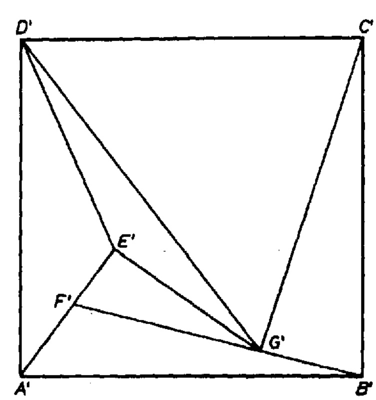

Fig. 5

Let F be a field containing the coordinates of the points of $\mathcal{V}$, let $\phi: F \to \mathbf{R}$ be an isomorphism and let the map Φ be defined by (4); then Φ is a collineation. Suppose that the points $V'_i = \Phi(V_i)$ $(i = 1, \dots, n)$ are the consecutive vertices of a polygon P'. By Lemma 6, Φ preserves the orientation of Δ_j if and only if at least two of the numbers $\phi(\cot \alpha_{j,i})$ $(i = 1, 2, 3)$ are positive, where $\alpha_{j,i}$ $(i = 1, 2, 3)$ denote the angles of Δ_j. If this condition is satisfied for every j then, by the previous

Lemma, we obtain a tiling of P' with the triangles Δ'_j such that the angles $\alpha'_{j,i}$ of Δ'_j satisfy $\cot\alpha'_{j,i} = \phi(\cot\alpha_{j,i})$ $(i = 1, 2, 3)$. If, for every j, at least two of the numbers $\phi(\cot\alpha_{j,i})$ $(i = 1, 2, 3)$ are negative then Φ changes the orientation of the triangles Δ_j. Using a reflection we can reduce this case to the previous one, and obtain a tiling of P' with Δ'_j such that $\cot\alpha'_{j,i} = -\phi(\cot\alpha_{j,i})$ for every $j = 1, \ldots, N$ and $i = 1, 2, 3$.

In each case we may call this tiling of P' a *conjugate* to the original tiling of P.

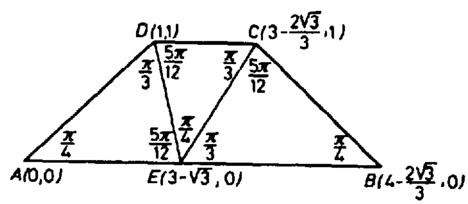

Fig. 6

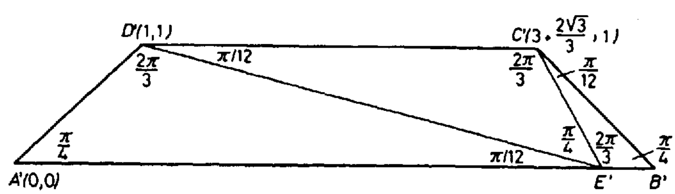

Fig. 7

Consider the following example. Let P be the trapezoid with vertices $A(0,0)$, $B((12-2\sqrt{3})/3, 0)$, $C((9-2\sqrt{3})/3, 1)$, $D(1,1)$. Then P can be tiled with three triangles of angles $\pi/4$, $\pi/3$, $5\pi/12$ (see Figure 6). The coordinates of the vertices of this decomposition belong to $\mathbf{Q}(\sqrt{3})$. Let ϕ be the automorphism of $\mathbf{Q}(\sqrt{3})$ satisfying $\phi(\sqrt{3}) = -\sqrt{3}$. The image P' of P under the map Φ is shown on Figure 7. Since $\phi\left(\cot\frac{\pi}{4}\right) = 1 = \cot\frac{\pi}{4}$, $\phi\left(\cot\frac{\pi}{3}\right) = \phi\left(\frac{\sqrt{3}}{3}\right) = -\frac{\sqrt{3}}{3} = \cot\frac{2\pi}{3}$, $\phi\left(\cot\frac{5\pi}{12}\right) = \phi(2-\sqrt{3}) = 2+\sqrt{3} = \cot\frac{\pi}{12}$, the argument above applies and we obtain a tiling of P' with three triangles of angles $\pi/12$, $\pi/4$, $2\pi/3$. This tiling, shown on Figure 7, is the conjugate of the tiling of Figure 6.

3. Tilings of polygons with similar triangles

We say that a triangle Δ tiles the polygon P if P can be decomposed into finitely many non-overlapping triangles similar to Δ.

Theorem 14. *Let Δ be a triangle with angles α, β and γ. If Δ tiles a polygon P then $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are algebraic over the field F generated by the coordinates of the vertices of P.*

Proof. We fix a tiling of P with triangles similar to Δ. First we suppose that $\cot\alpha$ is algebraic over F and prove that in this case $\cot\beta$ and $\cot\gamma$ are also algebraic over F.

Assume that $\cot\beta$ is transcendental over F; then $\cot\beta$ is transcendental over $F_1 = F(\cot\alpha)$ as well. Let U denote the set of those real numbers which are transcendental over F_1 and let $u \in U$ be arbitrary. Then there exists an isomorphism $\phi\colon F_1(\cot\beta) \to \mathbf{R}$ such that ϕ leaves the elements of F_1 fixed and $\phi(\cot\beta) = u$. Since $\gamma = \pi - \alpha - \beta$, we have

$$\cot\gamma = -\cot(\alpha+\beta) = \frac{1-\cot\alpha\cot\beta}{\cot\alpha+\cot\beta} \in F_1(\cot\beta)$$

and

$$\phi(\cot\gamma) = \frac{1-\cot\alpha\cdot u}{\cot\alpha+u}.$$

Since U is everywhere dense in $\mathbf{R}$, we can choose u in such a way that u and $(1-\cot\alpha\cdot u)/(\cot\alpha+u)$ are both negative. However, the existence of such an isomorphism contradicts Theorem 5. Therefore, if one of the numbers $\cot\alpha$, $\cot\beta$, $\cot\gamma$ is algebraic over F then they are all algebraic.

Suppose now that $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are all transcendental over F. Let V be a vertex of P at which P has a convex angle δ. Then there are non-negative integers n, k, m such that $n\alpha+k\beta+m\gamma = \delta$. Since $\delta < \pi$, at least one of n, k, m is zero; we may assume that $m = 0$. If $k = 0$ then we have $n\alpha = \delta$. Since $\cot\delta \in F$, this implies that $\cot\alpha$ is algebraic over F which is impossible. Therefore $k \neq 0$ and, similarly, $n \neq 0$.

We fix a real number u such that $u < \cot\delta$ and u is transcendental over F. Let $\bar F$ denote the algebraic closure of F. Since $\cot\alpha$ and u are both transcendental over $\bar F$, there is an isomorphism ϕ of $\mathbf{C}$ into itself such that ϕ leaves the elements of $\bar F$ fixed and $\phi(\cot\alpha) = u$. We show that $\phi(\cot\beta)$ and $\phi(\cot\gamma)$ are real.

Let $x = e^{i\alpha}$, $y = e^{i\beta}$, $z = e^{i\delta}$; then $n\alpha + k\beta = \delta$ gives $x^n y^k = z$. Since $\cot\delta = i(z+z^{-1})/(z-z^{-1}) \in F$, we have $z \in \bar F$ and thus

$$(\phi(x))^n(\phi(y))^k = z. \tag{9}$$

For every $c \in \mathbf{C}$, $i(c+c^{-1})/(c-c^{-1})$ is real if and only if $|c| = 1$ and $c \neq \pm 1$. Therefore, by

$$i\frac{\phi(x)+\phi(x)^{-1}}{\phi(x)-\phi(x)^{-1}} = \phi(\cot\alpha) = u \in \mathbf{R},$$

we have $|\phi(x)| = 1 = |z|$ and thus, by (9), $|\phi(y)| = 1$. Hence

$$\phi(\cot\beta) = i\frac{\phi(y)+\phi(y)^{-1}}{\phi(y)-\phi(y)^{-1}} \in \mathbf{R}.$$

This gives then $\phi(\cot\gamma) = -\phi(\cot(\alpha+\beta)) \in \mathbf{R}$.

This implies that the restriction of ϕ to the field $F(\cot\alpha, \cot\beta, \cot\gamma)$ is real and hence, by Theorem 5, at least two of the numbers $\phi(\cot\alpha)$, $\phi(\cot\beta)$, $\phi(\cot\gamma)$ are positive. Let α', β', $\gamma' \in (0,\pi)$ be such that

$$\phi(\cot\alpha) = \cot\alpha',\ \phi(\cot\beta) = \cot\beta',\ \phi(\cot\gamma) = \cot\gamma'.$$

Applying the results of the previous section we find that the isomorphism ϕ induces a tiling of P with triangles of angles α', β', γ'. At the vertex V this tiling has $n+k$ triangles with angles α' and β', respectively. Therefore $n\alpha' + k\beta' = \delta$, from which we obtain $\alpha' < \delta$ and $\cot\alpha' > \cot\delta$. However, this contradicts $\cot\alpha' = \phi(\cot\alpha) = u < \cot\delta$, completing the proof. ∎

Corollary 15. *For every polygon P, the set of triangles which tile P is countable.*

Remark 16. The set of triangles tiling a given polygon can be infinite. For example, if P is a parallelogram, then we can divide P into nk congruent parallelograms and then, dividing each of these small parallelograms into two triangles, we obtain a tiling of P with $2nk$ congruent triangles.

In order to formulate further necessary conditions on tilings with similar triangles, we shall need the following definitions.

Definition 17. *Let P be a polygon with angles $\delta_1, \dots, \delta_n$ and let*

$$L_P = \left\{ \sum_{i=1}^{n} r_i\delta_i\colon\ r_i \in \mathbf{Q}\ (i = 1, \dots, n) \right\}.$$

We denote by d_P the dimension of L_P as a linear space over $\mathbf{Q}$, and put

$$r_P = d_P - 1.$$

Definition 18. *Let P be a polygon. Suppose that the polygon P' is similar to P and has at least two vertices with rational coordinates. Then we denote by F_P the field generated by the coordinates of the vertices of P'. (It is easy to check that F_P does not depend on P', only on P.) The trancendence degree of F_P over $\mathbf{Q}$ will be denoted by t_P.*

Lemma 19. *For every triangle Δ we have $t_\Delta \le r_\Delta \le 2$.*

Proof. Let the angles of Δ be α, β, γ. If the vertices of Δ with angles α, β are placed at $(0,0)$ and $(1,0)$, respectively, then the coordinates of the third vertex of Δ are $x = \cos\alpha\sin\beta/\sin\gamma$ and $y = \sin\alpha\sin\beta/\sin\gamma$.

If $r_\Delta = 0$ then α, β, γ are rational multiples of π. In this case x, y are algebraic numbers and hence $t_\Delta = 0$.

Suppose next $r_\Delta = 1$. Then two of the angles of Δ generate L_Δ. We may suppose that these are α and β, and then $r\alpha + s\beta = \pi$ with some $r, s \in \mathbf{Q}$. This easily implies that any two of the numbers $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\sin\gamma$ are algebraically dependent and, consequently, $t_\Delta \le 1$.

As $t_\Delta \le 2$ and $r_\Delta \le 2$ hold in every triangle, the proof is complete. ∎

In the sequel we shall only consider similar tilings of *convex* polygons.

Theorem 20. *Suppose that the triangle Δ tiles the convex polygon P. Then we have*
(i) $F_P \subset F_\Delta$,
(ii) $t_P = t_\Delta \le r_\Delta \le 2$,
(iii) $r_P \le r_\Delta$, *and*
(iv) if $r_P = 0$ *then* $r_\Delta \le 1$.

Proof. We may assume that P has two vertices with rational coordinates. Let α, β, γ denote the angles of Δ. By Theorem 2, we have $F_P \subset \mathbf{Q}(\cot\alpha, \cot\beta, \cot\gamma)$. Since $\cot\alpha$, $\cot\beta$, $\cot\gamma$ belong to F_Δ, this implies $F_P \subset F_\Delta$.

As we saw in the proof of Lemma 19, $F_\Delta = \mathbf{Q}(\cos\alpha\sin\beta/\sin\gamma, \sin\alpha\sin\beta/\sin\gamma)$. Then, by Theorem 14, F_Δ is an algebraic extension of F_P and hence $t_P = t_\Delta$. Now (ii) follows from Lemma 19.

Since each angle of P is a linear combination of α, β, γ with integer coefficients, we have $L_P \subset L_\Delta$ and (iii).

Finally, suppose $r_P = 0$. Then each angle of P is a rational multiple of π. Let δ be an angle of P, then $\delta = a\alpha + b\beta + c\gamma$ with non-negative integers a, b, c. Since $\delta < \pi$, one of a, b, c must be zero. We may suppose that $c = 0$. Let $\delta = r\pi$ $(r \in \mathbf{Q})$, then $a\alpha + b\beta = r(\alpha+\beta+\gamma)$ and hence $\dim(L_\Delta) \le 2$. Thus $r_\Delta \le 1$, which proves (iv). ∎

Theorem 21. *Let Δ be a triangle with angles $\alpha \le \beta \le \gamma$, and let P be a convex polygon with angles δ_i $(i = 1, \dots, n)$. If Δ tiles P then either*
(i) there is an i such that $\delta_i \in \{\alpha, \beta, \gamma, \pi-\alpha, \pi-\beta, \pi-\gamma, 2\gamma\}$, *or*
(ii) $r_P = r_\Delta \le 1$.

Proof. Let P be decomposed into the triangles $\Delta_1, \dots, \Delta_N$ such that each Δ_j is similar to Δ. Let V be a vertex of one of the triangles Δ_j and let $\{\Delta_{j_1}, \dots, \Delta_{j_k}\}$ be the set of triangles having V as a vertex. Let α_{j_s} denote the angle of Δ_{j_s} at V $(s = 1, \dots, k)$. If V is a vertex of P with angle δ_i then we have

$$\sum_{s=1}^{k} \alpha_{j_s} = \delta_i. \tag{10}$$

If V is not a vertex of P then we have either

$$\sum_{s=1}^{k} \alpha_{j_s} = 2\pi \tag{11}$$

or

$$\sum_{s=1}^{k} \alpha_{j_s} = \pi. \tag{12}$$

In fact, (11) holds if V is in the interior of P and whenever V is on the boundary of a triangle Δ_j then necessarily V is a vertex of Δ_j; in all other cases we have (12).

Since $3\gamma \ge 2\gamma + \beta \ge 2\gamma + \alpha \ge \alpha + \beta + \gamma = \pi$ and $\gamma + 2\beta \ge \gamma + \beta + \alpha = \pi$, each of the equations (10) must take the form of one of the following equations:

$$2\gamma = \delta_i,\ \beta + \gamma = \delta_i,\ k\alpha + \gamma = \delta_i\ (k \ge 0) \quad \text{and} \quad k\alpha + m\beta = \delta_i\ (k, m \ge 0).$$

If any of the equations $2\gamma = \delta_i$, $\beta + \gamma = \delta_i$, $\gamma = \delta_i$, $\alpha + \gamma = \delta_i$, $\alpha = \delta_i$, $\beta = \delta_i$, $\alpha + \beta = \delta_i$ occurs among the equations (10), then (i) is true.

Therefore we may assume that each of the equations (10) is of the form

$$k_i\alpha + \gamma = \delta_i \qquad (k_i \geq 2) \tag{13}$$

or

$$k_i\alpha + m_i\beta = \delta_i \qquad (\max(k_i, m_i) \geq 2). \tag{14}$$

Now we suppose that neither of (i) and (ii) is true. Since $r_P \leq r_\Delta \leq 2$ by Theorem 20, this implies that either $r_\Delta = 2$ or $r_P = 0$ and $r_\Delta = 1$.

First we consider the case when $r_\Delta = 2$. Then α, β, γ are linearly independent over $\mathbf{Q}$. Hence, if

$$p\alpha + q\beta + r\gamma = v\pi \tag{15}$$

holds, where p, q, r, v are integers, then $p = q = r = v$.

Consequently, the left hand sides of the equations (12) and (11) are of the form $\alpha+\beta+\gamma$ and $2\alpha+2\beta+2\gamma$, respectively. Since the left hand sides of the equations (10), (11), (12) contain N α's, β's and γ's (as they involve all the angles of the triangles $\Delta_1, \ldots, \Delta_N$), this implies that the left hand sides of the equations (10) contain the same number of α's, β's and γ's. However, each of the equations (13) contains more α's than γ's. Hence the total number of α's in the equations (10) is greater than the number of γ's unless both are zero. In this case, however, the number of β's will be different. This shows that $r_\Delta = 2$ is impossible.

Next we suppose that $r_P = 0$ and $r_\Delta = 1$. Then $\delta_1, , \ldots, \delta_n$ are rational multiples of π. We distinguish between two cases.

Case 1: equations of the form (13) do occur among the equations (10). Then the total number of α's is strictly greater than the number of γ's in the equations (10) and hence there must be an equation (15) among the equations (11) or (12) such that $p < r$ (and $v = 1$ or $v = 2$.) Subtracting $r\alpha + r\beta + r\gamma = r\pi$ from (15) we obtain

$$(p - r)\alpha + (q - r)\beta = (v - r)\pi. \tag{16}$$

Let $k\alpha + \gamma = \delta$, $k \geq 2$ be an equation of the form (13). Subtracting $\alpha + \beta + \gamma = \pi$ we obtain

$$(k - 1)\alpha - \beta = \delta - \pi. \tag{17}$$

If (16) and (17) determine α and β then, as δ is a rational multiple of π, α and β are also rational multiples of π. This implies $r_\Delta = 0$ which is not the case. Therefore the equations (16) and (17) are not independent; that is, there is a number c such that

$$c(k - 1) = p - r, \quad -c = q - r, \quad c(\delta - \pi) = (v - r)\pi.$$

Since $p < r$ and $k \geq 2$, we have $c < 0$, $q > r$, $v > r$. Hence, by $v \leq 2$ and $p < r$, we obtain $p = 0$, $r = 1$, $v = 2$. This gives

$$\pi = (v - r)\pi = c(\delta - \pi) = -(\delta - \pi)/(k - 1)$$

and $(k - 1)\pi = \pi - \delta$, which is impossible.

Case 2: each of the equations (10) is of the form (14). Let

$$kα + mβ = δ \tag{18}$$

be one of these equations. Then either $k \geq \max(m, 2)$ or $m \geq \max(k, 2)$. Suppose the former. (The latter can be treated similarly.) Then following the argument of the previous case we find an equation of the form (15) with $p < r$. If p, q, r are all positive then we can subtract $\alpha + \beta + \gamma = \pi$ from (15) so that we may assume $\min(p, q) = 0$. As in the previous case, we find that the equations (18) and (16) cannot be independent and hence there is a number c such that

$$ck = p - r, \;\; cm = q - r, \;\; c\delta = (v - r)\pi.$$

Then $c < 0$ and, by $m \leq k$, we obtain $p \leq q$ and $v < r$. This gives, by $\min(p, q) = 0$, $p = 0$. We also have

$$(r - v)\pi = -c\delta = \frac{r}{k}\delta < \frac{r}{k}\pi,$$

and $k(r - v) < r$. Since $v = 1$ or 2, $v < r$, $k \geq 2$, this implies $k = 2$, $v = 2$, $r = 3$. Thus $c = -3/2$ and $\delta = 2\pi/3$.

Also, $-3m/2 = q - 3$ and $m \leq k = 2$ yield $m = 0$, $q = 3$ or $m = 2$, $q = 0$. In the first case we have $2\alpha = \delta = 2\pi/3$, $\alpha = \pi/3$ which is impossible since, in this case, $\delta = \pi - \alpha$ and we assumed that (i) does not hold. In the second case (15) implies $3\gamma = 2\pi$ and $\gamma = 2\pi/3 = \delta$ contradicting the same assumption. This contradiction completes the proof. ∎

4. Tilings of rectangles with similar triangles

In this section our aim is to find those triangles which tile a rectangle. Suppose that the angles of Δ are α, β, γ and Δ tiles a rectangle P. If γ is the largest angle of Δ then, by Theorem 21, one of the following statements is true.

(i) $\frac{\pi}{2} \in \{\alpha, \beta, \gamma, \pi - \alpha, \pi - \beta, \pi - \gamma, 2\gamma\}$;

(ii) $r_P = r_\Delta$.

Since $2\gamma \geq 2\pi/3 > \pi/2$, (i) implies that Δ is a right triangle. On the other hand, as $r_P = 0$, (ii) implies $r_\Delta = 0$. We have proved the following.

Lemma 22. *If a triangle Δ tiles a rectangle then either Δ is a right triangle or the angles of Δ are rational multiples of π.*

Since every right triangle tiles a rectangle, we may confine our attention to those triangles which tile a rectangle, have no right angle and satisfy $r_\Delta = 0$. Our first aim is to prove that one of the angles of these triangles is $\pi/6$ or $\pi/4$.

Let

$$\alpha = \frac{a}{n}\pi, \quad \beta = \frac{b}{n}\pi, \quad \gamma = \frac{c}{n}\pi$$

where a, b, c, n are positive integers and $a + b + c = n$. Multiplying all these numbers by 2, we may assume that n is even. By Lemma 8, the cotangents of α, β, γ belong to $\mathbf{Q}(\zeta)$, where ζ is the first $4n^{\text{th}}$ root of unity. Suppose that Δ tiles the rectangle P with vertices $V_1(0,0)$, $V_2(1,0)$, $V_3(1,y)$, $V_4(0,y)$. Then, by Theorem 2,

$$y \in \mathbf{Q}(\cot \frac{a}{n}\pi, \cot \frac{b}{n}\pi, \cot \frac{c}{n}\pi) \stackrel{\text{def}}{=} F.$$

Let k be an integer prime to n. Then, by Lemma 8, there is an automorphism ϕ of $\mathbf{Q}(\zeta)$ such that

$$\text{(19)} \quad \phi\left(\cot \frac{a}{n}\pi\right) = \eta \cot \frac{ak}{n}\pi, \quad \phi\left(\cot \frac{b}{n}\pi\right) = \eta \cot \frac{bk}{n}\pi, \quad \phi\left(\cot \frac{c}{n}\pi\right) = \eta \cot \frac{ck}{n}\pi$$

where $\eta = (-1)^{(k-1)/2}$. Then ϕ is real on F and hence $\phi(y) = y' \in \mathbf{R}$. Let the map Φ be defined by

$$\Phi((x_1, x_2)) = (\phi(x_1), \phi(x_2)) \qquad (x_1, x_2 \in F).$$

Using the notation of Section 2, we find that the points V_1, V_2, $V_3'(1,y')$, $V_4(0,y')$ are the consecutive vertices of a rectangle (listed counterclockwise or clockwise depending on the sign of y'). Let $\varepsilon = 1$ if Φ preserves the orientation of Δ and let $\varepsilon = -1$ otherwise. Since each triangle in the tiling of P is similar to Δ, it follows from the results of Section 2 that Φ induces a tiling of P' with triangles of angles α', β', γ' such that

$$\text{(20)} \qquad \cot \alpha' = \varepsilon\eta \cot \frac{ak}{n}\pi, \quad \cot \beta' = \varepsilon\eta \cot \frac{bk}{n}\pi, \quad \cot \gamma' = \varepsilon\eta \cot \frac{ck}{n}\pi.$$

We note that $\varepsilon = 1$ if and only if Φ preserves the orientation of P, and hence $\varepsilon = \operatorname{sgn} y'$. Now (20) implies that we have either

$$\text{(21)} \qquad \alpha' = \left\{\frac{ak}{n}\right\}\pi, \qquad \beta' = \left\{\frac{bk}{n}\right\}\pi, \qquad \gamma' = \left\{\frac{ck}{n}\right\}\pi,$$

or

$$\text{(22)} \quad \alpha' = \left(1 - \left\{\frac{ak}{n}\right\}\right)\pi, \quad \beta' = \left(1 - \left\{\frac{bk}{n}\right\}\right)\pi, \quad \gamma' = \left(1 - \left\{\frac{ck}{n}\right\}\right)\pi.$$

Suppose that the origin is the vertex of $p + q + r$ triangles of the tiling of P with angles α, β, γ, respectively; then $p\alpha + q\beta + r\gamma = \pi/2$. If Δ has no right angle then this implies $r \le 1$ and $\max(p,q) \ge 2$. By symmetry we may assume $p \ge 2$. This implies $\alpha \le \pi/4$. Also, the conjugate tiling induced by Φ has p triangles with angles α' at the origin. Therefore $p\alpha' \le \pi/2$ and hence $\alpha' \le \pi/4$. This yields, by (21) and (22) that either $\{ak/n\} \le 1/4$ or $\{ak/n\} \ge 3/4$, and this has to be valid whenever k is prime to n.

We prove that this condition implies $a/n = 1/6$ or $a/n = 1/4$.

Let $(a,n) = d$, $a = da_1$ and $n = dn_1$. If k is prime to n_1 and i is the product of those prime divisors of n which do not divide k (or $i = 1$ if there is no such a prime) then $k + in_1$ is prime to n. Hence we have either

$$\left\{\frac{a_1 k}{n_1}\right\} = \left\{\frac{a(k + in_1)}{n}\right\} \le \frac{1}{4},$$

or

$$\left\{\frac{a_1k}{n_1}\right\} = \left\{\frac{a(k+in_1)}{n}\right\} \geq \frac{3}{4}.$$

If k runs through the reduced residue system mod n_1 then so does a_1k, as $(a_1, n_1) = 1$. Therefore either $\{k/n_1\} \leq 1/4$, or $\{k/n_1\} \geq 3/4$ holds for every k prime to n_1. That is,

$$(n_1, k) > 1 \text{ for every } \frac{n_1}{4} < k < \frac{3n_1}{4}. \tag{23}$$

The only integers satisfying this condition are $n_1 = 1,\ 4,\ 6$. Indeed, if $n_1 > 4$ then, by "Bertrand's postulate", there is a prime p such that $n_1/4 < p < n_1/2$. By (23), this implies $p|n_1$ and thus $n_1 = 3p$. If $p < i < 2p$ then (23) gives $3|i$. Therefore $p = 2$ and $n_1 = 6$. Now it follows from $0 < a_1 \leq n_1/4$ and $(a_1, n_1) = 1$ that $a_1 = 1$ and hence $a/n = a_1/n_1 = 1/4$ or $a/n = 1/6$, as we stated.

Thus we have proved

$$\{\alpha, \beta\} \cap \left\{\frac{\pi}{6}, \frac{\pi}{4}\right\} \neq \emptyset. \tag{24}$$

In the following argument we shall use the notation of Theorem 21. Since Δ has no right angle, each of the equations (10) is of the form (13) or (14) with $\delta_i = \pi/2$. If (13) actually occurs among the equations, then

$$\beta = \pi - (\alpha + \gamma) > \pi - (k_i\alpha + \gamma) = \pi/2,$$

contradicting the fact that γ is the largest angle of Δ.

Hence each of the equations (10) is of the form (14). Since the total number of α's, β's and γ's in the equations (10)-(12) is N, this implies that at least one of the equations (11) and (12) is of the form (15) with $p + q < 2r$ (and $v = 1$ or 2). Subtracting, if necessary, $\alpha + \beta + \gamma = \pi$ from (15), we may assume that $\min(p, q, r) = 0$. This does not affect the validity of $p + q < 2r$ and hence we have $\min(p, q) = 0 < r$.

In the sequel we shall assume $q = 0$. (If $p = 0$ then we can interchange the roles of α and β; the condition $\alpha \leq \beta$ will not be used in the argument.) Thus we have

$$p\alpha + r\gamma = v\pi \tag{25}$$

where $0 \leq p < 2r$ and $v = 1, 2$.

Suppose first $v = 1$. Since, by (24), $\gamma > \pi/3$, this implies $r \leq 2$. If $r = 1$ then $p \leq 1$ and $p\alpha + r\gamma \leq \alpha + \gamma < \pi$, a contradiction. If $r = 2$, $p = 0$ then $\gamma = \pi/2$, and if $r = 2$, $p > 1$ then $p\alpha + r\gamma > \alpha + 2\gamma \geq \pi$, both impossible. If $r = 2$, $p = 1$ then $p\alpha + r\gamma = \alpha + 2\gamma > \alpha + \beta + \gamma = \pi$ unless $\gamma = \beta$. In this case (24) implies $\alpha = \pi/6$ or $\alpha = \pi/4$, and we obtain

$$(\alpha, \beta, \gamma) = \left(\frac{\pi}{6}, \frac{5\pi}{12}, \frac{5\pi}{12}\right) \text{ or } \left(\frac{\pi}{4}, \frac{3\pi}{8}, \frac{3\pi}{8}\right).$$

Thus, for the triple $(a/n,\ b/n,\ c/n)$ we get

$$\left(\frac{1}{6}, \frac{5}{12}, \frac{5}{12}\right) \text{ or } \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right). \tag{26}$$

Next suppose $v = 2$. Then, by (25), $\gamma = (2\pi - p\alpha)/r$. If $\alpha = \pi/6$ or $\alpha = \pi/4$ then, taking $0 \le p < 2r \le 10$ and $\gamma \ge \beta$ into consideration, we obtain the following triples for $(a/n,\ b/n,\ c/n)$:

$$\begin{array}{ll} & \left(\frac{1}{6},\frac{1}{12},\frac{3}{4}\right), \quad \left(\frac{1}{6},\frac{1}{6},\frac{2}{3}\right), \quad \left(\frac{1}{6},\frac{2}{9},\frac{11}{18}\right), \quad \left(\frac{1}{6},\frac{5}{18},\frac{5}{9}\right), \\ (27) & \left(\frac{1}{6},\frac{7}{18},\frac{4}{9}\right), \quad \left(\frac{1}{6},\frac{3}{8},\frac{11}{24}\right), \quad \left(\frac{1}{4},\frac{1}{8},\frac{5}{8}\right), \quad \left(\frac{1}{4},\frac{1}{12},\frac{2}{3}\right), \\ & \left(\frac{1}{4},\frac{1}{6},\frac{7}{12}\right), \quad \left(\frac{1}{4},\frac{1}{3},\frac{5}{12}\right), \quad \left(\frac{1}{4},\frac{5}{16},\frac{7}{16}\right), \quad \left(\frac{1}{4},\frac{7}{20},\frac{2}{5}\right). \end{array}$$

If $\beta = \pi/6$ or $\beta = \pi/4$ then, using $(r-p)\gamma = (2-p)\pi + p\beta$ and $\gamma \ge \alpha$, it is easy to check that no new triples arise.

This proves that whenever a triangle Δ with angles $a\pi/n,\ b\pi/n,\ c\pi/n$ tiles a rectangle and does not have a right angle then $(a/n,\ b/n, c/n)$ is one of the triples listed in (26) and (27). Now some of these triples can be discarded by the following argument.

If k is prime to n then there is a conjugate tiling with a triangle Δ' with angles $\alpha',\ \beta', \gamma'$ satisfying (21) or (22). Since Δ' does not have a right angle and also tiles a rectangle, the corresponding triple also has to be listed in (26) or (27). That is, we can exclude the triple $(a/n,\ b/n,\ c/n)$ if there is a k such that $(k,n) = 1$ and neither $(\{ak/n\},\{bk/n\},\{ck/n\})$ nor $(1-\{ak/n\}, 1-\{bk/n\}, 1-\{ck/n\})$ is listed in (26) or (27). In this way the following triples can be ruled out:

$$\begin{array}{ll} & \left(\frac{1}{6},\frac{5}{12},\frac{5}{12}\right) : k=5; \quad \left(\frac{1}{4},\frac{3}{8},\frac{3}{8}\right) : k=3; \quad \left(\frac{1}{6},\frac{2}{9},\frac{11}{18}\right) : k=5; \\ (28) & \left(\frac{1}{6},\frac{5}{18},\frac{5}{9}\right) : k=7; \quad \left(\frac{1}{6},\frac{7}{18},\frac{4}{9}\right) : k=5; \quad \left(\frac{1}{6},\frac{3}{8},\frac{11}{24}\right) : k=5; \\ & \left(\frac{1}{4},\frac{5}{16},\frac{7}{16}\right) : k=3; \quad \left(\frac{1}{4},\frac{7}{20},\frac{2}{5}\right) : k=3. \end{array}$$

Next we show that the triples $(1/6, 1/4, 7/12)$ and $(1/12, 1/6, 3/4)$ are also impossible. Suppose that the triangle Δ with angles $\alpha = \pi/6,\ \beta = \pi/4,\ \gamma = 7\pi/12$ tiles a rectangle. Consider the conjugate tiling corresponding to $k = 5$. Since $\cot(5\pi/6)$ and $\cot(35\pi/12)$ are negative, it follows from Lemma 6 that the map Φ changes the orientation of Δ. With the notation of (20) this means $\varepsilon = -1$. Since $\eta = (-1)^{(5-1)/2} = 1$, this implies that the angles of the conjugate tiling satisfy (22) and thus $\alpha' = \pi/6,\ \beta' = 3\pi/4,\ \gamma' = \pi/12$.

Suppose that the tiling consists of N triangles, and let $P,\ Q,\ R$ denote the total number of α's β's and γ's in the equations (11) and (12).

Let $p\alpha + q\beta + r\gamma = \pi/2$ be any of the equations (10). Switching to the conjugate tiling we can see that $p\alpha' + q\beta' + r\gamma' = \pi/2$ also has to be valid. With the given values of the angles involved, the only possibility is $p = 3,\ q = r = 0$. Hence we have

$$(29) \qquad N = P + 12, \quad Q = R = N.$$

Now let $p\alpha + q\beta + r\gamma = v\pi$ be any of the equations (11) or (12), where $v = 1$ or $v = 2$. Then the conjugate tiling gives $p\alpha' + q\beta' + r\gamma' = v'\pi$, where $v' = 1$ or $v' = 2$. The only possible triples satisfying these conditions are $(6,0,0)$, $(12,0,0)$, $(5,0,2)$, $(1,1,1)$, $(7,1,1)$, $(0,1,3)$, $(3,2,0)$, $(2,2,2)$. Since each of these triples satisfies $3q \le 2p+r$, we have $3Q \le 2P+R$. By (29), this implies $3N \le 2(N-12)+N$, which is impossible.

This shows that the triple $(1/6, 1/4, 7/12)$ does not give a tiling. The same is true for $(1/12, 1/6, 3/4)$, since any tiling by the latter would produce a conjugate tiling (with k=5) by the former.

Summing up: these two triples and those of (28) can be deleted from the lists (26) and (27). There are four remaining triples and hence the following theorem is proved.

Theorem 23. *If a triangle Δ tiles a rectangle then either Δ is a right triangle, or its angles are given by one of the following triples:*

$$\left(\frac{\pi}{6},\frac{\pi}{6},\frac{2\pi}{3}\right),\ \left(\frac{\pi}{8},\frac{\pi}{4},\frac{5\pi}{8}\right),\ \left(\frac{\pi}{4},\frac{\pi}{3},\frac{5\pi}{12}\right),\ \left(\frac{\pi}{12},\frac{\pi}{4},\frac{2\pi}{3}\right). \tag{30}$$ ∎

Corollary 24. *If a rectangle of size $a \times b$ can be tiled with a triangle which has no right angle, then $b/a \in \mathbf{Q}(\sqrt{2}) \cup \mathbf{Q}(\sqrt{3})$. In particular, only a countable number of rectangles can have this property.*

Proof. This is an immediate consequence of Theorems 2, 23 and the relations $\cot\frac{\pi}{8} = \sqrt{2}+1$, $\cot\frac{\pi}{12} = \sqrt{3}+2$. ∎

5. Examples

In this section we show that each of the triangles listed in (30) tiles a rectangle. As figure 8 shows, the triangle with angles $\pi/6$, $\pi/6$, $2\pi/3$ tiles the rectangle of size $1 \times \sqrt{3}$.

In the sequel we shall prove that each of the remaining three triangles tiles the square. Consider first the triangle Δ_1 of angles $\alpha = \pi/8$, $\beta = \pi/4$, $\gamma = 5\pi/8$.

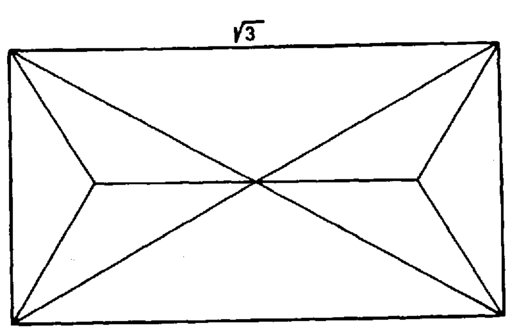

Fig. 8

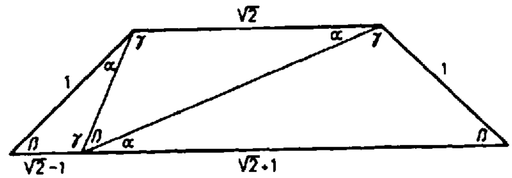

Fig. 9

We offer two tilings of the square with Δ_1. The first contains several hundred triangles, but hardly needs any calculation. The second uses 74 triangles and is based on a particular construction.

If a triangle Δ tiles a polygon P then we shall denote this fact by $P \in \mathbf{T}(\Delta)$.

Let $P(a, b)$ denote the parallelogram of angle $\pi/4$ and sides a and b. Since $\sin\gamma/\sin\alpha = \sqrt{2}+1$, $P(1, \sqrt{2}+1)$ can be divided into two triangles similar to Δ_1. If r is a positive rational number then $P(1, r(\sqrt{2}+1))$ can be decomposed into finitely many parallelograms similar to $P(1, \sqrt{2}+1)$ and hence $P(1, r(\sqrt{2}+1)) \in \mathbf{T}(\Delta_1)$.

Let $T(a, b)$ denote the symmetrical trapezoid of angle $\pi/4$ with leg a and shorter base b. As Figure 9 shows, $T(1, \sqrt{2}) \in \mathbf{T}(\Delta_1)$. Since $P(1, 2a + \sqrt{2})$ can be decomposed into two trapezoids of size $T(1, a)$, this implies $P(1, 3\sqrt{2}) \in \mathbf{T}(\Delta_1)$ and hence $P(1, r\sqrt{2}) \in \mathbf{T}(\Delta_1)$ for every $r \in \mathbf{Q}$, $r > 0$.

If a, b are positive rational numbers with $a > b + 1$, then

$$T(1, a\sqrt{2}+b) = P(1, (a-1-b)\sqrt{2}) \cup P(1, b(\sqrt{2}+1)) \cup T(1, \sqrt{2}) \in \mathbf{T}(\Delta_1).$$

Next we prove that $T(1, a\sqrt{2}+b) \in \mathbf{T}(\Delta_1)$ whenever $a, b \in \mathbf{Q}$ and $a > b > 0$. Indeed, let n be a positive integer with $1/n < a-b$. We divide $T(1, a\sqrt{2}+b)$ into n trapezoids by $n-1$ equidistant lines parallel to the bases, and obtain the decomposition

$$T(1, a\sqrt{2}+b) = \bigcup_{i=0}^{n-1} T\left(\frac{1}{n}, \left(a + \frac{i}{n}\right)\sqrt{2} + b\right).$$

For every $i \geq 0$, $T\left(\frac{1}{n}, \left(a + \frac{i}{n}\right)\sqrt{2} + b\right)$ is similar to $T(1, (na+i)\sqrt{2} + nb)$ and hence, as $na + i > nb + 1$, can be tiled with Δ_1. Therefore $T(1, a\sqrt{2}+b) \in \mathbf{T}(\Delta_1)$, as we stated.

Now Figure 10 shows a decomposition of the isosceles right triangle Δ_0 into two triangles similar to Δ_1 and three trapezoids of size $T(\sqrt{2}+4, 4\sqrt{2}+4)$, $T(4\sqrt{2}+3, 2\sqrt{2}+4)$ and $T(\sqrt{2}+3, \sqrt{2}+1)$, respectively. Since

$$\frac{4\sqrt{2}+4}{\sqrt{2}+4} = \frac{6\sqrt{2}+4}{7}, \quad \frac{2\sqrt{2}+4}{4\sqrt{2}+3} = \frac{10\sqrt{2}+4}{23}, \quad \frac{\sqrt{2}+1}{\sqrt{2}+3} = \frac{2\sqrt{2}+1}{7},$$

the argument above proves that these trapezoids can be tiled with Δ_1. Therefore $\Delta_0 \in \mathbf{T}(\Delta_1)$ and thus the square can be tiled with Δ_1.

The second tiling is shown on Figures 11 and 12. Figure 11 shows $T(2, 2\sqrt{2}+2) \in \mathbf{T}(\Delta_1)$, and Figure 12 gives a tiling of Δ_0 using four triangles similar to Δ_1, a trapezoid $T(1, \sqrt{2})$, and three trapezoids similar to $T(2, 2\sqrt{2}+2)$.

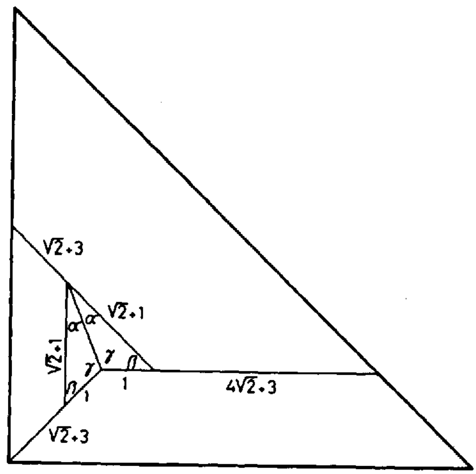

Fig. 10

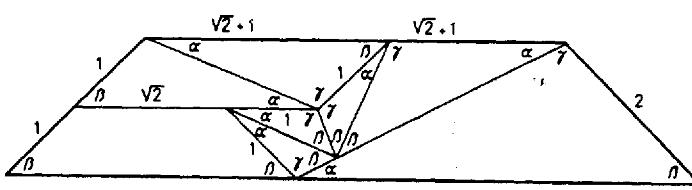

Fig. 11

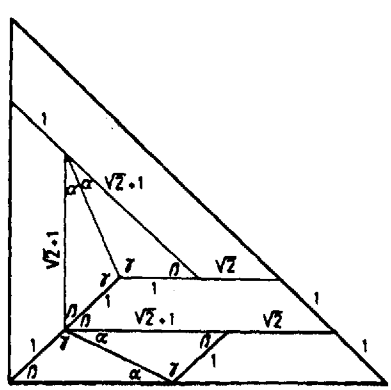

Fig. 12

Remark 25. It follows from Theorem 2 that a rectangle R of size $a \times b$ can be tiled with Δ_1 only if $b/a \in \mathbf{Q}(\sqrt{2})$. This condition is still not sufficient. If $b/a = p + q\sqrt{2}$

where $p, q \in \mathbf{Q}$, then another necessary condition is that

$$p > |q|\sqrt{2}. \tag{31}$$

Indeed, we may assume that the vertices of R are $V_1(0,0)$, $V_2(1,0)$, $V_3(1,y)$, $V_4(0,y)$, where $y = p + q\sqrt{2} > 0$. Since $\cot\frac{5\pi}{4}$ and $\cot\frac{25\pi}{8}$ are positive, the map Φ of the conjugate tiling corresponding to $k = 5$ does not change the orientation of Δ_1. This implies that Φ does not change the orientation of R; that is, $y' = p - q\sqrt{2} > 0$ and hence (31) must hold.

We do not know if this condition is sufficient for $R \in \mathbf{T}(\Delta_1)$. We remark that $p > 2q + 2 > 0$ is a sufficient condition. Indeed, let $p' = (p-2)/2$, then $p' > q > 0$ and hence $T(1, p'\sqrt{2} + q) \in \mathbf{T}(\Delta_1)$. Thus $T = T(\sqrt{2}, 2p' + q\sqrt{2}) \in \mathbf{T}(\Delta_1)$, and we can complete T by two isosceles right triangles to a rectangle of size $1 \times (p + q\sqrt{2})$.

Now we turn to the triangle Δ_2 of angles $\alpha = \pi/4$, $\beta = \pi/3$, $\gamma = 5\pi/12$. Since

$$\sin\frac{5\pi}{12} \Big/ \sin\frac{\pi}{3} = \frac{\sqrt{2}}{2}\left(1 + \frac{\sqrt{3}}{3}\right),$$

$P(1, \sqrt{2}(3+\sqrt{3})/6)$ can be decomposed into two triangles similar to Δ_2.

Hence $P(1, r\sqrt{2}(3+\sqrt{3})) \in \mathbf{T}(\Delta_2)$ for every $r \in \mathbf{Q}$, $r > 0$.

As Figure 6 shows, $T(1, \sqrt{2}(3-\sqrt{3})/3) \in \mathbf{T}(\Delta_2)$. Since $P(1, 2a + \sqrt{2})$ can be decomposed into two $T(1,a)$'s, this implies $P(1, \sqrt{2}(9-2\sqrt{3})/3) \in \mathbf{T}(\Delta_2)$. Therefore

$$P(1, 3\sqrt{2}) = P(1, 2\sqrt{2}(3+\sqrt{3})/3) \cup P(1, \sqrt{2}(3-2\sqrt{3})/3) \in \mathbf{T}(\Delta_2)$$

and thus $P(1, r\sqrt{2}) \in \mathbf{T}(\Delta_2)$ for every $r \in \mathbf{Q}$, $r > 0$.

If a, b are rational numbers such that $a > b + 2 \geq 2$ then

$$T\left(1, \sqrt{2}\left(a + b\frac{\sqrt{3}}{3}\right)\right) \in \mathbf{T}(\Delta_2).$$

Indeed,

$$T\left(1, \sqrt{2}\left(a + b\frac{\sqrt{3}}{3}\right)\right) =$$

$$P\left(1, (b+1)\sqrt{2}\left(1+\frac{\sqrt{3}}{3}\right)\right) \cup P(1, (a-b-2)\sqrt{2}) \cup T\left(1, \sqrt{2}\left(1-\frac{\sqrt{3}}{3}\right)\right) \in \mathbf{T}(\Delta_2).$$

This implies, in the same way as in the case of Δ_1, that $T\left(1, \sqrt{2}\left(a + b\frac{\sqrt{3}}{3}\right)\right) \in \mathbf{T}(\Delta_2)$ whenever $a, b \in \mathbf{Q}$, $a > b \geq 0$.

Figure 13 shows that the isosceles right triangle Δ_0 can be decomposed into four triangles similar to Δ_2, and three trapezoids of size $T\left(x + 1, \frac{\sqrt{6}}{3} + y\right)$,

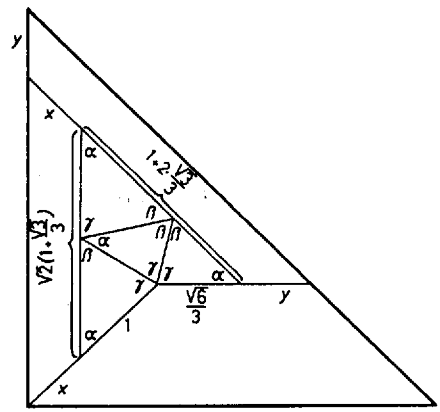

Fig. 13

$T\left(y, 1+2\frac{\sqrt{3}}{3}+x\right)$, $T\left(x, \sqrt{2}+\frac{\sqrt{6}}{3}\right)$ respectively, where we choose $x = 1+\frac{\sqrt{3}}{3}$ and $y = \sqrt{2}(34+12\sqrt{3})/3$.

Since

$$\frac{\frac{\sqrt{6}}{3}+y}{1+x} = \sqrt{2}\,\frac{165+44\sqrt{3}}{33}, \qquad \frac{1+2\frac{\sqrt{3}}{3}+x}{y} = \sqrt{2}\,\frac{96+30\sqrt{3}}{1448},$$

and

$$\frac{\sqrt{2}+\frac{\sqrt{6}}{3}}{x} = \sqrt{2},$$

these trapezoids can be tiled with Δ_2. Therefore the isosceles right triangle and the square also can be tiled with Δ_2.

If we tile the unit square with Δ_2, then the conjugate tiling corresponding to $k = 5$ will produce a tiling of the square with the triangle Δ_3 of angles $\pi/12$, $\pi/4$, $2\pi/3$.

Remark 26. The map Φ of the conjugate tiling corresponding to $k = 5$ maps Δ_2 and Δ_3 into each other, without changing their orientation. This implies that a rectangle R of size $a \times b$ can be tiled with Δ_2 or Δ_3 only if $b/a = p + q\sqrt{3}$, where $p > |q|\sqrt{3}$. We do not know whether this condition is sufficient for $R \in \mathbf{T}(\Delta_2)$.

6. Tiling the square with similar triangles

In this section we address the following question: which triangles tile the square? By Theorem 23, such a triangle is either a right triangle or its angles are given by (30). As we saw in the previous section, the second, third and fourth of the triangles

(30) do tile the square. The first, however, does not tile, since 6 is not divisible by 4 (see Theorem 9).

Thus we may confine our attention to right triangles. Let $\Delta(\alpha)$ denote the right triangle with acute angles α and $\frac{\pi}{2} - \alpha$.

Theorem 27. *If $\Delta(\alpha)$ tiles the square then $\cot\alpha$ is a totally positive algebraic number. That is, it is algebraic, and each of its real conjugates is positive.*

Proof. We may assume that $\Delta(\alpha)$ tiles the unit square. By Theorem 14, this implies that $\cot\alpha$ is algebraic. Let u be a real conjugate of $\cot\alpha$ and let ϕ be an isomorphism of $\mathbf{Q}(\cot\alpha)$ such that $\phi(\cot\alpha) = u$. Then, by Theorem 5, at least two of the numbers

$$\phi\left(\cot\frac{\pi}{2}\right) = 0, \quad \phi(\cot\alpha) = u, \quad \phi\left(\cot\left(\frac{\pi}{2} - \alpha\right)\right) = \frac{1}{u}$$

are positive and hence $u > 0$. ∎

We do not know whether or not the triangles satisfying the condition of Theorem 27 tile the square. We show, however, that this condition is strong enough to determine those right triangles which tile the square and whose angles are rational multiples of π.

Lemma 28. *Let a, n be integers with $0 < a < n$. If $\cot(\frac{a}{n}\pi)$ is totally positive then*

$$\frac{a}{n} \in \left\{\frac{1}{4}, \frac{1}{12}, \frac{5}{12}\right\}.$$

Proof. We may assume that a and n are coprime. Let

$$P_n = \{k \in \mathbf{Z}\colon (k,n) = 1, \ k \equiv 1 \pmod 4\}$$

and

$$Q_n = \{k \in \mathbf{Z}\colon (k,n) = 1, \ k \equiv -1 \pmod 4\}.$$

By Lemma 8, the numbers $\cot(k\pi/n)$ $(k \in P_n)$ are conjugates of $\cot(\pi/n)$, and the numbers $\cot(k\pi/n)$ $(k \in Q_n)$ are conjugates of $\cot(-\pi/n)$. Since $\cot(-\pi/n) \le 0$ and $\cot(a\pi/n)$ is totally positive, it follows that the numbers $a + in$ $(i \in \mathbf{Z})$ are not congruent to -1 mod 4. This easily implies that $4|n$ and $a \in P_n$. Consequently, $\cot(k\pi/n) > 0$ holds for every $k \in P_n$.

If $q = 4i-1$ is prime and $q < n/2$, then $n-q \equiv 1 \pmod 4$ and $\cot((n-q)\pi/n) < 0$. Thus $n - q \notin P_n$ and hence $q|n$. That is, n is divisible by every prime which is of the form $4i - 1$ and is less than $n/2$.

Let $3 = q_1 < q_2 < \ldots$ be the sequence of primes of the form $4i - 1$, and let $q_s < n/2 < q_{s+1}$. Then we have

$$4q_1q_2\ldots q_s \le n < 2q_{s+1}.$$

Now one of the numbers $q_1\ldots q_s + 2$ and $q_1\ldots q_s + 4$ is of the form $4i - 1$ and hence has a prime factor of this form. Therefore $q_{s+1} \le q_1\ldots q_s + 4$ from which we obtain $4q_1\ldots q_s < 2q_1\ldots q_s + 8, \ \ q_1\ldots q_s < 4, \ \ s \le 1, \ \ n < 14$. Since $n = 8$ does not have the required property, we have $n = 4$ or $n = 12$. Finally, $a \in P_n$ gives the assertion of the Theorem. ∎

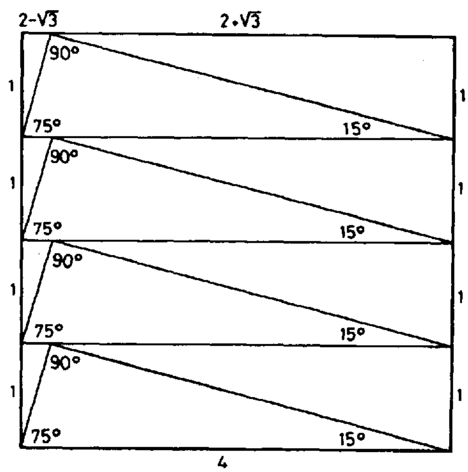

Fig. 14

Now suppose that $\Delta(\alpha)$ tiles the square and α is a rational multiple of π. Then it follows from Theorem 27 and Lemma 28 that the angles of $\Delta(\alpha)$ are 45°, 45°, 90°, or 15°, 75°, 90°. Both triangles tile the square. As for the second, we refer to Figure 14.

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