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An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*

I. Introduction.

We display a particular neighbourhood frame $\mathscr{F}$ which models **S4** and prove that the logic **L** determined by this frame is incomplete with respect to the Relational Semantics. Since **L** is determined by a neighbourhood frame which models **S4**, it is clearly an extension of **S4** and is complete with respect to the Neighbourhood Semantics. In a previous paper [4] we showed that there is a logic **L**$'$ which is an extension of **T** complete with respect to the Neighbourhood and incomplete with respect to the Relational Semantics. Gabbay [2] has conjectured that no such extension of **S4** exists. Our present result, then, serves as a counterexample to Gabbay's conjecture.

We are considering the modal propositional language, the language with the classical propositional connectives together with the single unary connective $\Box$ (necessitation). The additional connective $\Diamond$ (possibility) can be defined by $\Diamond A = \sim\Box\sim A$. We define the Relational and Neighbourhood Semantics as in [3], [4] and [5]. A relational frame is a pair $\mathscr{W} = \langle W, < \rangle$ where W is a nonempty set and $<$ is a binary relation on W. An assignment V is a function from the set P of propositional variables to $\mathscr{P}(W)$. V is extended to a function, also called V, from the set of all formulae to $\mathscr{P}(W)$ by $V(\sim A) = W - V(A)$, $V(A \vee B) = V(A) \cup V(B)$, $V(\Box A) = \{w \in W \mid w < v \Rightarrow v \in V(A)\}$. A neighbourhood frame is a pair $\mathscr{F} = \langle U, \mathscr{N} \rangle$ where U is a nonempty set and $\mathscr{N}$ is a function from U to $\mathscr{P}(\mathscr{P}(U))$. (We write $\mathscr{N}_u$ for $\mathscr{N}(u)$.) An assignment V is, as with a relational frame, a function from the set of propositional variables to $\mathscr{P}(U)$ and is extended to a function from the set of formulae to $\mathscr{P}(U)$ by $V(\sim A) = U - V(A)$, $V(A \vee B) = V(A) \cup V(B)$, and $V(\Box A) = \{u \in U \mid V(A) \in \mathscr{N}_u\}$. With both the Relational and Neighbourhood Semantics we write $V(A, u) = \mathsf{T}$ or F according as $u \in V(A)$ or not. We write $\mathscr{F} \models A$ (and say "$\mathscr{F}$ models A") if $V(A, u) = \mathsf{T}$ for each point u and each assignment V. The logic determined by a frame is the set of all formulae modelled by the frame.

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Given a relational frame $\mathscr{W} = \langle W, < \rangle$ we can define a neighbourhood structure on $\mathscr{W}$ by $\mathscr{N}_{\mathscr{W}}(w) = \{N \subset W \mid w < v \Rightarrow v \in N\}$. It is clear that in this

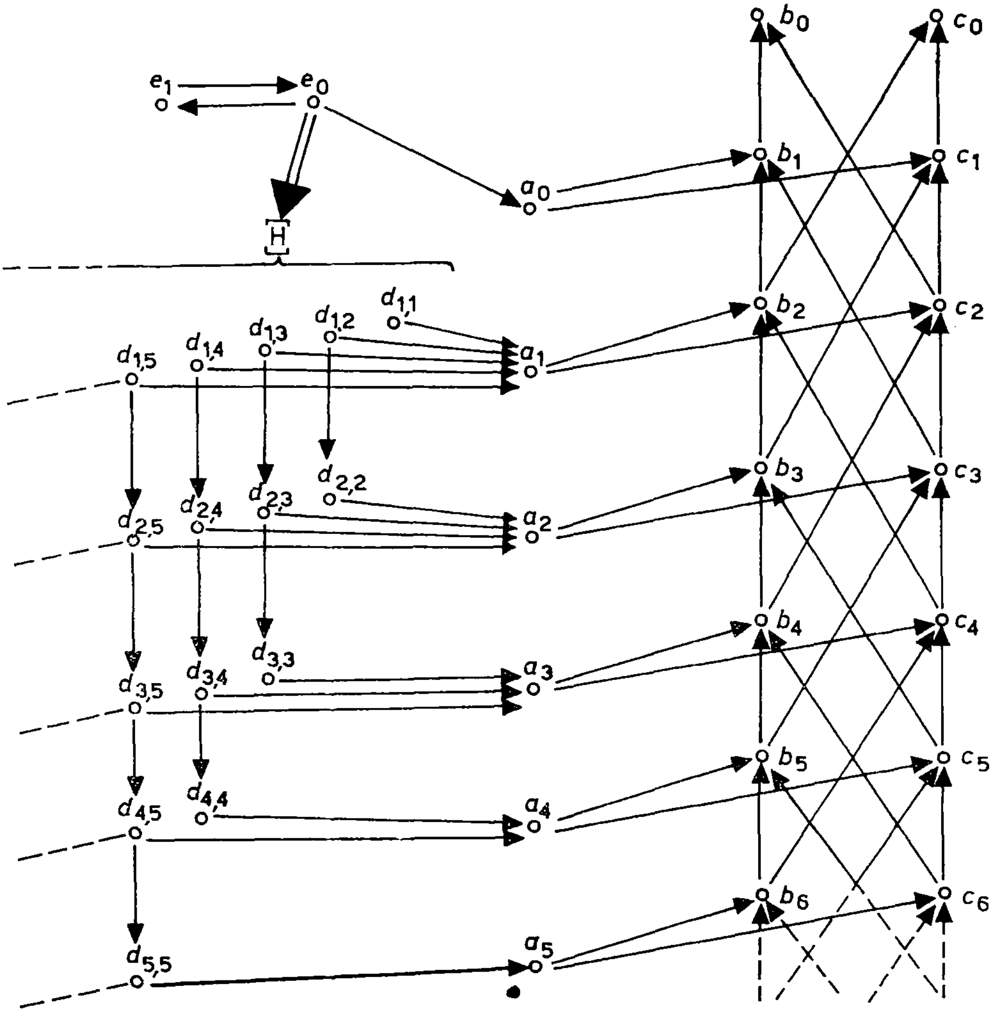


Figure I

case if $w \in W$ then $V(A, w) = \mathsf{T}$ in $\langle W, < \rangle$ iff $V(A, w) = \mathsf{T}$ in $\langle W, \mathscr{N} \rangle$. It is also known that a relational frame $\mathscr{W} = \langle W, < \rangle$ models **S4** if and only if $<$ is reflexive and transitive and that a neighbourhood frame $\mathscr{F} = \langle U, \mathscr{N} \rangle$ models **S4** if and only if for each $u \in U$, (1) $\mathscr{N}_u$ is a filter, (2) $N \in \mathscr{N}_u \Rightarrow u \in N$, and (3) if $N \in \mathscr{N}_u$ there is $N' \subset N$ such that $N' \in \mathscr{N}_u$ and $\forall v \in N'$, $N' \in \mathscr{N}_v$.

II. The neighbourhood frame, $\mathscr{F}$.

Let $\mathscr{F}$ be the neighbourhood frame diagrammed in Figure I to be interpreted as follows. We say that "v is a successor of u" if, as in a relational frame, there is a sequence (possibly of length 0) of single arrows from u to v. If u is anything but e_0 or e_1 then a neighbourhood of u is any set containing all the successors of u. Ξ is a non-principal ultrafilter on the natural numbers. A neighbourhood of e_i ($i = 0$ or 1) is a set containing all successors of e_i (i.e. $e_0, e_1, a_0, b_0, b_1, c_0, c_1$) in addition to $\{d_{1j} | j \in K\}$ for some $K \in \Xi$ and all successors of these d_{1j}. Thus any neighbourhood of e_i contains both e_j's, all a_j's, b_j's, c_j's and all the complete columns of d_{ij}'s corresponding to some $K \in \Xi$. Precisely, $\mathscr{F} = \langle U, N\rangle$ where

$$U = \{e_0, e_1\} \cup \{a_i, b_i, c_i | 0 \leqslant i\} \cup \{d_{ij} | 1 \leqslant i \leqslant j\}$$

for $n \geqslant 0$ $\quad N \in \mathscr{N}_{b_n}$ iff $N \supset \{b_i | i \leqslant n\} \cup \{c_j | j \leqslant n-2\}$

for $n \geqslant 0$ $\quad N \in \mathscr{N}_{c_n}$ iff $N \supset \{c_i | i \leqslant n\} \cup \{b_j | j \leqslant n-2\}$

for $n \geqslant 0$ $\quad N \in \mathscr{N}_{a_n}$ iff $N \supset \{a_n\} \cup \{b_i | i \leqslant n+1\} \cup \{c_i | i \leqslant n+1\}$

for $1 \leqslant i \leqslant j$ $\quad N \in \mathscr{N}_{d_{ij}}$ iff $N \supset \{d_{kj} | i \leqslant k \leqslant j\} \cup \{a_k | i \leqslant k \leqslant j\} \cup \{b_k | k \leqslant i+1\} \cup \{c_k | k \leqslant i+1\}$

for $i = 0,1$ $\quad N \in \mathscr{N}_{e_i}$ iff for some $K \in \Xi$, $N \supset \{d_{jk} | k \in K, 1 \leqslant j \leqslant k\} \cup \{e_0, e_1\} \cup \{a_j, b_j, c_j | j \geqslant 0\}$.

It is easy to see that $\mathscr{F}$ models **S4**. Let **L** be the logic determined by $\mathscr{F}$. Then **L** is an extension of **S4**.

III. Some particular formulae.

Consider the following formulae where $p, p_0, p_1, q_0, q_1, r_0, r_1$, are distinct propositional variables, $m \geqslant 0$, and $n \geqslant 1$,

$$\begin{aligned}
B_0 &= q_0 \quad B_1 = q_1\\
C_0 &= r_0 \quad C_1 = r_1\\
B_{m+2} &= \Diamond B_{m+1} \wedge \Diamond C_m \wedge \sim\Diamond C_{m+1}\\
C_{m+2} &= \Diamond C_{m+1} \wedge \Diamond B_m \wedge \sim\Diamond B_{m+1}\\
A_m &= \Diamond B_{m+1} \wedge \Diamond C_{m+1} \wedge \sim\Diamond B_{m+2} \wedge \sim\Diamond C_{m+2}\\
J_1 &= \Box(B_1 \to \Diamond B_0 \wedge \sim\Diamond C_0)\\
J_2 &= \Box(C_1 \to \Diamond C_0 \wedge \sim\Diamond B_0)\\
J_3 &= \Box(B_0 \to \sim\Diamond(B_1 \vee C_1))\\
J_4 &= \Box(C_0 \to \sim\Diamond(C_1 \vee B_1))\\
H_1 &= p \wedge \Box(p \to \Diamond(\sim p \wedge \Diamond p))
\end{aligned}$$

$$\begin{aligned}
D &= J_1 \wedge J_2 \wedge J_3 \wedge J_4\\
E &= H_1 \wedge D \wedge \Diamond A_0\\
F_n &= \sim\Diamond A_{n-1} \wedge \Diamond A_n \wedge \Diamond A_{n+1}\\
G_n &= E \to \Diamond F_n\\
H_2 &= \sim\Big(p_0 \wedge \Box\Big(p_0 \to \Diamond\Big(\sim p_0 \wedge p_1 \wedge \Diamond\big((\sim p_0 \wedge \sim p_1) \wedge \Diamond p_0\big)\Big)\Big)\Big)\\
I &= H_1 \wedge D \wedge \Diamond F_1 \wedge \Diamond F_2 \to \Box(F_1 \to \Diamond F_2)
\end{aligned}$$

We shall show that H_2, I, and all the $G_n (n \geqslant 1)$ are in **L** but that $\sim E$ is not. To do this, we shall show that $\mathscr{F} \models H_2$, $\mathscr{F} \models I$, and $\mathscr{F} \models G_n$ $(n \geqslant 1)$ but that there is an assignment V on $\mathscr{F}$ and $u \in U$ such that $V(E, u) = \mathsf{T}$. We shall then show that if $\mathscr{W}$ is any relational frame modelling **S4** such that $\mathscr{W} \models G_n$ $(n \geqslant 1)$, $\mathscr{W} \models H_2$ and $\mathscr{W} \models I$, then we must have $\mathscr{W} \models \sim E$. It will follow that $\sim E$ is not in **L** but is modelled by any relational frame which models **L**. Therefore **L** is incomplete with respect to the Relational Semantics but complete with respect to the Neighbourhood Semantics.

IV. LEMMA. *In a relational frame* $\mathscr{W} = \langle W, < \rangle$ *modelling* **S4**, *if there is an assignment* V *and a point* $u \in W$ *such that* $V(H_1, u) = \mathsf{T}$ *then there is either an infinite successor-sequence (of distinct points) or a two-cycle accessible from* u. *If* $V(H_2, u) = \mathsf{F}$ *then there is either an infinite successor-sequence or a three-cycle accessible from* u. *(By an* n*-cycle we mean a sequence* $u_1, \ldots, u_n$ *of* n *distinct points such that* $u_1 < \ldots < u_n < u_1$. *Note that if we have an* n*-cycle then we have an* m*-cycle for each* $m \leqslant n$, *by transitivity.)*

The proof is easy and omitted.

V. LEMMA. *If* V *is an assignment on* $\mathscr{F}$ *and* $u \in U$ *such that* $V(H_1, u) = \mathsf{T}$, *then* $u = e_0$ *or* $u = e_1$.

The neighbourhood system of any point v other than e_0 or e_1 is exactly as it is in a relational frame. Thus, for H_1 to be satisfied at any such point v there must be either an infinite sequence or a two-cycle accessible from v. No point other than e_0 or e_1 has this property.

VI. LEMMA. *If* V_0 *is an assignment on* $\mathscr{F}$ *such that* $V_0(g_i) = \{b_i\}$; $V_0(r_i) = \{c_i\}$, $V_0(p) = \{e_0\}$, $(i = 0,1)$ *then*

$$\begin{aligned}
V_0(B_n) &= \{b_n\},\ V_0(C_n) = \{c_n\} \quad (n \geqslant 0)\\
V_0(A_0) &= \{a_0\},\ V_0(A_n) = \{a_n, d_{nn}\} \quad (n \geqslant 1)\\
V_0(E) &= \{e_0\}.
\end{aligned}$$

Thus $V_0(E, e_0) = \mathsf{T}$ *and so* $\mathscr{F}$ *non* $\models \sim E$. *Hence* $\sim E$ *is not in* **L**.

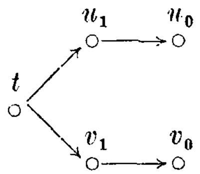
The proof is strightforward and omitted.

VII. LEMMA. $V(G_n, u) = \mathsf{T}\ \forall n \geqslant 1, \forall u \in U, \forall$ *assignments* V *on* $\mathscr{F}$.

If u is not e_0 or e_1 then (by V) $V(E, u) = \mathsf{F}$ so $V(G_n, u) = \mathsf{T}$. Without loss assume that u is e_0 and that $V(E, e_0) = \mathsf{T}$. (e_0 and e_1 have the same set of neighbourhoods and so have the same formulae valid in them.)

Since $V(E, e_0) = \mathsf{T}$, $V(D, e_0) = \mathsf{T}$. Therefore there is $K \in \Xi$ such that $V\big((B_1 \to \Diamond B_0 \wedge \sim\Diamond C_0) \wedge (C_1 \to \Diamond C_0 \wedge \sim\Diamond B_0) \wedge (B_0 \to \sim\Diamond (B_1 \vee C_1)) \wedge (C_0 \to \to \sim\Diamond (C_1 \vee B_1)), v\big) = \mathsf{T}\ \forall v \in S = \{d_{jk} | k \in K, 1 \leqslant j \leqslant k\} \cup \{e_0, e_1\} \cup \{a_j, b_j, c_j | j \geqslant 0\}$. Note that $S \in \mathscr{N}_{e_0}$. Since $V(E, e_0) = \mathsf{T}$, $V(\Diamond A_0, e_0) = \mathsf{T}$ and so we must have $t \in S$ such that $V(A_0, t) = \mathsf{T}$. Choose one such t.

(a) First suppose that t is not e_0 or e_1. Then the neighbourhood system of t is as in a relational frame. Now $V(\Diamond B_1 \wedge \Diamond C_1, t) = \mathsf{T}$ and since $t \in S$, we must have points u_1, v_1 in $S - \{e_0, e_1\}$ such that $t < u_1$, $t < v_1$, $V(B_1, u_1) = V(C_1, v_1) = \mathsf{T}$. Now since $u_1, v_1 \in S - \{e_0, e_1\}$ we must have $u_0, v_0 \in S - - \{e_0, e_1\}$ such that $u_1 < u_0$, $v_1 < v_0$, $V(B_0, u_0) = V(C_0, v_0) = \mathsf{T}$. And, further, we must have $u_1 \nless v_0$, $v_1 \nless u_0$, $u_0 \nless v_1$, $u_0 \nless u_1$, $v_0 \nless u_1$, $v_0 \nless v_1$. Thus we have



and

this can only occur if $u_0 = b_m$, $v_0 = c_m$, $u_1 = b_{m+1}$, $v_1 = c_{m+1}$ (or the preceding with b's and c's interchanged). It will then follow, whether or not we chose $t = a_m$, that $V(A_0, a_m) = \mathsf{T}$. In fact, it is easy to see that we must have $V(B_0) = \{b_m\}$, $V(B_1) = \{b_{m+1}\}$, $V(C_0) = \{c_m\}$, $V(C_1) = \{c_{m+1}\}$, and it follows easily that if $m = 0$ $V(A_0) = \{a_0\}$ and if $m > 0$ $V(A_0) = \{a_m, d_{mm}\}$; and further that for $j \geqslant 1$, $V(B_j) = \{b_{m+j}\}$, $V(C_j) = \{c_{m+j}\}$ and $V(A_j) = \{a_{m+j}, d_{m+j\, m+j}\}$ (or we may have had the preceding with all the b's and c's interchanged). Then for all $n \geqslant 1$ and $j \geqslant 1$ we have $V(\sim\Diamond A_{n-1} \wedge \Diamond A_n \wedge \Diamond A_{n+1}, d_{m+n\, m+n+j}) = \mathsf{T}$. Therefore $V(F_n, d_{m+n\, m+n+j}) = \mathsf{T}\ \forall j \geqslant 1$ and so $V(\Diamond F_n, e_0) = \mathsf{T}$. So we have $V(G_n, e_0) = \mathsf{T}\ \ \forall n \geqslant 1$.

(b) Now suppose that $t = e_j$ ($j = 0$ or 1). Without loss we may assume that $t = e_0$. Again we have $V(\Diamond B_1 \wedge \Diamond C_1, t) = \mathsf{T}$. First suppose that $V(B_1, e_i) = \mathsf{T}$ for $i = 0$ or 1. Then since e_i has the same set of neighbourhoods as e_0, $V(\Diamond C_1, e_i) = \mathsf{T}$ also. So $V(B_1 \wedge \Diamond C_1, e_i) = \mathsf{T}$, contradicting the fact that $e_i \in S$. Similarly, if $V(C_1, e_i) = \mathsf{T}$. Therefore $V(B_1)$ and $V(C_1)$ must both be disjoint from $\{e_0, e_1\}$.

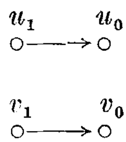
We now consider four subcases: (i) $V(B_1) \subset \{d_{ij} | i \leqslant j\}$ and $V(C_1) \not\subset \{d_{ij} | i \leqslant j\}$, (ii) $V(C_1) \subset \{d_{ij} | i \leqslant j\}$ and $V(B_1) \not\subset \{d_{ij} | i \leqslant j\}$, (iii) $V(B_1) \subset \{d_{ij} | i \leqslant j\}$ and $V(C_1) \subset \{d_{ij} | i \leqslant j\}$, and (iv) $V(B_1) \not\subset \{d_{ij} | i \leqslant j\}$ and $V(C_1) \not\subset \{d_{ij} | i \leqslant j\}$.

(i) In this case there is $d_{mn} \in S$ such that $V(B_1, d_{mn}) = \mathsf{T}$. Now if $V(C_1, b_k) = \mathsf{T}$ for any k then, since $b_k \in S$, $V(\Diamond C_0, b_k) = \mathsf{T}$ and so there is a v with $b_k < v$ such that $V(C_0, v) = \mathsf{T}$. But then $d_{mn} < b_k < v$ and so $V(\Diamond C_0, d_{mn}) = \mathsf{T}$. We then have $V(B_1 \wedge \Diamond C_0, d_{mn}) = \mathsf{T}$ contradicting the fact that $d_{mn} \in S$. Similarly, if $V(C_1, c_k) = \mathsf{T}$ for any k. So since $V(C_1) \not\subset \{d_{ij} | i < j\}$, we must have $V(C_1, a_k) = \mathsf{T}$ for some k. But then $a_k \in S$ and so there is a u_0 with $V(C_0, u_0) = \mathsf{T}$ and $a_k < u_0$ and $u_0 \nless a_k$. This means that $u_0 = b_h$ or c_h for some $h \leqslant k+1$. But then $V(B_1 \wedge \Diamond C_0, d_{mn}) = \mathsf{T}$ again, contradicting d_{mn} in S. So case (i) cannot occur.

(ii) This case is similar to case (i) and so cannot occur either.

(iii) In this case we let $S_1 = \{n | d_{in} \in V(B_1)$ *for some* $i \leqslant n\}$, $S_2 = \{n | d_{in} \in V(C_1)$ *for some* $i \leqslant n\}$, $S_3 = \{u | \exists i \leqslant j \leqslant n$ such that $d_{in} \in V(B_1)$, $d_{jn} \in V(C_1)\}$, $S_4 = \{u | \exists i \leqslant j \leqslant n$ *such that* $d_{in} \in V(C_1)$, $d_{jn} \in V(B_1)\}$. Then $S_1 \in \Xi$ and $S_2 \in \Xi$. Also $S_1 \cap S_2 = S_3 \cup S_4$ and so, since Ξ is an ultrafilter, either $S_3 \in \Xi$ or $S_4 \in \Xi$. Without loss assume that $S_3 \in \Xi$. Then $S_3 \cap K \in \Xi$ (where K is the set of column numbers of d's in S—see second paragraph of VII). Let $n \in S_3 \cap K$. Let k and h be such that $k \leqslant h \leqslant n$ and $V(B_1, d_{kn}) = \mathsf{T}$ and $V(C_1, d_{hn}) = \mathsf{T}$. Then we have a v_0 with $V(C_0, v_0) = \mathsf{T}$ and $d_{hn} < v_0$. But then $d_{kn} < v_0$ also and so $V(B_1 \wedge \Diamond C_0, d_{kn}) = \mathsf{T}$ contradicting the fact that $d_{kn} \in S$.

(iv) In this case there are u_1, v_1 in $\{a_i, b_i, c_i | i \geqslant 0\}$ such that $V(B_1, u_1) = V(C_1, v_1) = \mathsf{T}$. Then also there are u_0, v_0 such that $V(B_0, u_0) = V(C_0, v_0) = \mathsf{T}$ and $u_1 < u_0$, $u_1 \nless v_0$, $v_1 < v_0$, $v_1 \nless u_0$, $u_0 \nless u_1$, $u_0 < u_1$, $v_0 \nless v_1$, $v_0 \nless u_1$. Thus we have



with the only accessibility between the four points being that indicated in this diagram. This can only occur if $u_0 = b_m$, $v_0 = c_m$, $u_1 = b_{m+1}$, $v_1 = c_{m+1}$ (or the preceding with b's and c's interchanged). But now we are back to the situation in case (a), and so $V(G_n, e_0) = \mathsf{T}\ \forall n \geqslant 1$.

Thus $\mathscr{F} \models G_n \forall n \geqslant 1$.

VIII. Lemma. $\mathscr{F} \models H_2$.

We see that if u is other than e_0 or e_1 then the neighbourhood system of u is as in a relational frame, and so by IV we know that $V(H_2, u) = \mathsf{F}$

implies that either an infinite successor-sequence or a three-cycle is accessible from u. Since this is not the case, $V(H_2, u) = \mathsf{T}$ for any u other then e_0 or e_1.

Suppose $V(H_2, e_0) = \mathsf{F}$. Then $e_0 \in V(p_0)$. Let N be the set $V\big(p_0 \rightarrow \Diamond\big(\sim p_0 \wedge p_1 \wedge \Diamond((\sim p_0 \wedge \sim p_1) \wedge \Diamond p_0)\big)\big)$. Then $N \in \mathscr{N}_{e_0}$. Thus $V\big(\sim p_0 \wedge p_1 \wedge \wedge \Diamond(\sim p_0 \wedge \sim p_1 \wedge \Diamond p_0)\big) \cap N \neq \emptyset$ and $V(\sim p_0 \wedge \sim p_1 \wedge \Diamond p_0) \cap N \neq \emptyset$. Since $V\big(\sim p_0 \wedge p_1 \wedge \Diamond(\sim p_0 \wedge \sim p_1 \wedge \Diamond p_0)\big)$ and $V(\sim p_0 \wedge \sim p_1 \wedge \Diamond p_0)$ are obviously disjoint, and both disjoint from $V(p_0)$, one of them must include a point u in N other than e_0 or e_1. If this point is in $V(\sim p_0 \wedge \sim p_1 \wedge \Diamond p_0)$ then there is a point v in $N - \{e_0, e_1\}$ such that $u < v$, $v \in V(p_0)$. But then $N \in \mathscr{N}_v$ and so $V(\sim H_2, v) = \mathsf{T}$ and so an infinite sequence or three-cycle is accessible from v — an impossibility. If the point u is in $V\big(\sim p_0 \wedge p_1 \wedge \wedge \Diamond(\sim p_0 \wedge \sim p_1 \wedge \Diamond p_0)\big)$ then there is a point u' in $N - \{e_0, e_1\}$ such that $u < u'$ and $u' \in V(\sim p_0 \wedge \sim p_1 \wedge \Diamond p_0)$, reducing us to the case just dealt with.

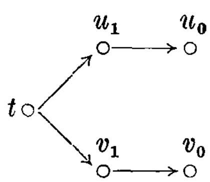
Similarly, if $V(H_2, e_1) = \mathsf{F}$. Therefore $\mathscr{F} \models H_2$.

IX. Lemma. $\mathscr{F} \models I$

If u is other than e_0 or e_1 then $V(H_1, u) = \mathsf{F}$ and so $V(I, u) = \mathsf{T}$. So we may assume without loss that u is e_0 and that $V(H_1 \wedge D \wedge \Diamond F_1 \wedge \wedge \Diamond F_2, e_0) = \mathsf{T}$.

Now, since $V(D, e_0) = \mathsf{T}$, there is a set $K \in \Xi$ such that $V\big((B_1 \rightarrow \rightarrow \Diamond B_0 \wedge \sim\Diamond C_0) \wedge (C_1 \rightarrow \Diamond C_0 \wedge \sim\Diamond B_0) \wedge (B_0 \rightarrow \sim\Diamond(B_1 \vee C_1)) \wedge (C_0 \rightarrow \sim\Diamond(B_1 \vee \vee C_1)), v\big) = \mathsf{T}\ \forall v \in S$ where S is the set $\{d_{jk} | k \in K, 1 \leqslant j \leqslant k\} \cup \{e_0, e_1\} \cup \{a_j, b_j, c_j | j \geqslant 0\}$. Note that $S \in \mathscr{N}_{e_0}$. Since $V(\Diamond F_1, e_0) = \mathsf{T}$ we must have $w \in S$ such that $V(F_1, w) = \mathsf{T}$. If w is e_0 or e_1, then since $V(\Diamond F_2, e_0) = \mathsf{T}$ we must also have $V(\Diamond F_2, w) = \mathsf{T}$.

Now suppose w is other than e_0 or e_1. Then the neighbourhood system of w is as in relational frames. Since $V(F_1, w) = \mathsf{T}$ there must be a point t such that $w < t$ and $V(A_1, t) = \mathsf{F}$. Then, in turn, there must be u_1, v_1 such that $t < u_1$, $t < v_1$, $V(B_1, u_1) = V(C_1, v_1) = \mathsf{T}$, and u_0, v_0 such that $u_1 < u_0$, $v_1 < v_0$, $V(B_0, u_0) = V(C_0, v_0) = \mathsf{T}$. Since $w \in S$, t_1, u_1, u_0, v_1, v_0 must all be in S and so $u_1 \nless v_0$, $v_1 \nless u_0$; therefore, we have



and, as previously, we must have $u_0 = b_i$, $v_0 = c_i$, $u_1 = b_{i+1}$, $v_1 = c_{i+1}$ (or the preceding with b's and c's interchanged). In fact, as before, we can see that $V(B_j) = \{b_{i+j}\}$ and $V(C_j) = \{c_{i+j}\}$ (or $V(B_j) = \{c_{i+j}\}$ and

$V(B_j) = \{b_{i+j}\})$ for $j \geqslant 0$, and so $V(A_j) = \{a_{i+j}, d_{i+j\, i+j}\}$. We see that in this case, then, $V(F_j) = \{d_{i+j\, h} | h > i+j\}$ for $j \geqslant 1$. So for $h \geqslant i+3$ we have $V(F_1 \wedge \Diamond F_2, d_{i+1\, h}) = \mathsf{T}$ and for $u \in \{d_{i+1\, k} | k \geqslant i+1\}$ we have $V(F_1, u) = \mathsf{F}$. So in this case we have $V(F_1 \rightarrow \Diamond F_2, u) = \mathsf{T}$ for each u other than $d_{i+1\, i+1}$ or $d_{i+1\, i+2}$. So for every $w \in S$ other than $d_{i+1\, i+1}$ or $d_{i+1\, i+2}$ we have $V(\mathscr{F}_1 \rightarrow \Diamond F_2, w) = \mathsf{T}$. Since $S - \{d_{i+1\, i+1}, d_{i+1\, i+2}\} \in \mathscr{N}_{e_0}$ we have $V(\Box(F_1 \rightarrow \Diamond F_2), e_0) = \mathsf{T}$ and so $V(I, e_0) = \mathsf{T}$. Thus $\mathscr{F} \models I$.

X. We have seen now that the logic **L** determined by the neighbourhood frame $\mathscr{F}$ includes the formulae H_2, I and G_n for each $n \geqslant 1$, but does not include the formula $\sim E$. It remains only to show that each relational frame modelling **L** models $\sim E$. We shall show that each relational frame modelling **S4**, H_2, I and all G_n for $n \geqslant 1$ models $\sim E$. Our approach will be to show that a relational frame failing to model $\sim E$ but modelling **S4**, I, and G_n for each $n \geqslant 1$ must fail to model H_2.

Suppose that $\mathscr{W} = \langle W, < \rangle$ is a relational frame such that $\mathscr{W} \models$ **S4**, $\mathscr{W} \models G_n$ for each $n \geqslant 1$, $\mathscr{W} \models I$ but $\mathscr{W}$ *non* $\models \sim E$. Then there is a point $w \in W$ and a V on $\mathscr{W}$ such that $V(E, w) = \mathsf{T}$. For a formula A let A^n be as in [5], namely, the result of replacing all occurrences of subformulae $B_i(C_i)$ with B_{i+n} (respectively C_{i+n}) $(i = 0,1)$ in A.

CLAIM 1: $B_n^m = B_{m+n}$; $C_n^m = C_{m+n}$; $A_n^m = A_{m+n}$ $(m, n \geqslant 0)$; $F_n^n = F_{m+n}$ $(m \geqslant 0, n \geqslant 1)$. The proof by induction of a similar claim is given in [5] and so will not be given here.

CLAIM 2: *For* $m \geqslant 0$, **S4** $\vdash D^m \rightarrow D^{m+1}$; *thus if* $m' \geqslant m$, **S4** $\vdash D^m \rightarrow D^{m'}$. The proof of a similar claim to this also appears in [5] and so will not be repeated here.

Since $V(E, w) = \mathsf{T}$ and since $\mathscr{W} \vdash G_1$ we have $V(\Diamond F_1, w) = \mathsf{T}$. Let u_1 be such that $w < u_1$ and $V(F_1, u_1) = \mathsf{T}$. We shall construct an infinite sequence $u_1, u_2, \ldots$ such that for $i < j$

$$u_i \neq u_j$$

$$u_i < u_j$$

$$V(F_i, u_i) = \mathsf{T}$$

$$w < u_i.$$

We already have u_1 satisfying the conditions. Suppose we have $u_1, \ldots, u_n$ satisfying the conditions. Now, since there are no occurrences in H_1 of r_0, r_1, g_0, g_1 we have $H_1^{n-1} = H_1$ and so $V(H_1^{n-1}, w) = \mathsf{T}$. Since $\mathscr{W} \models$ **S4**, we have, by Claim 2, $V(D^{n-1}, w) = \mathsf{T}$. Since $V(F_n, u_n) = \mathsf{T}$ and $w < u_n$, we have $V(\Diamond F_n, w) = \mathsf{T}$, and so $V(\Diamond F_1^{n-1}, w) = \mathsf{T}$ by Claim 1. Since

$V(E, w) = \mathsf{T}$ and $\mathscr{W} \vDash G_{n+1}$, we have $V(\Diamond F_{n+1}, w) = \mathsf{T}$ and so $V(\Diamond F_2^{n-1}, w) = \mathsf{T}$. Thus $V(D_1^{n-1} \wedge D_2^{n-1} \wedge \Diamond F_1^{n-1} \wedge \Diamond F_2^{n-1}, w) = \mathsf{T}$. But I^{n-1} is a substituted case of I and $w \vDash I$, so $V(I^{n-1}, w) = \mathsf{T}$. Therefore $V\big(\Box(F^{n-1} \to \Diamond F^{n-1}), w\big) = \mathsf{T}$; i.e., $V\big(\Box(F_n \to \Diamond F_{n+1}), w\big) = \mathsf{T}$. So there is u_{n+1} such that $u_n < u_{n+1}$ and $V(F_{n+1}, u_{n+1}) = \mathsf{T}$. Pick one such u_{n+1}. Then $w < u_n < u_{n+1}$ and so $w < u_{n+1}$. It remains to show that for $i \leqslant n$, $u_i \neq u_{n+1}$. If $i \leqslant n$ we have $u_i < u_n$ and $V(F_n, u_n) = \mathsf{T}$ and so $V(\Diamond F_n, u_i) = \mathsf{T}$. Thus $V\big(\Diamond(\Diamond A_n), u_i\big) = \mathsf{T}$ and so $V(\Diamond A_n, u_i) = \mathsf{T}$. But $V(F_{n+1}, u_{n+1}) = \mathsf{T}$ and so $V(\sim\Diamond A_{n+1-1}, u_{n+1}) = \mathsf{T}$ and hence $V(\Diamond A_n, u_{n+1}) = \mathsf{F}$. Thus $u_{n+1} \neq u_i$. So we have our infinite sequence.

If we now let $V(p_0) = \{w\} \cup \{u_{3n+1} \mid n \geqslant 0\}$, $V(p_1) = \{u_{3n+2} \mid n \geqslant 0\}$, then we see that $V(\sim H_2, w) = \mathsf{T}$ and so $\mathscr{W}$ *non* $\vDash H_2$. Thus if a relational frame models H_2, I, and G_n for every $n \geqslant 1$, then it must model $\sim E$. So **L** is incomplete with respect to the Relational Semantics and the proof is complete.

XI. Conclusions.

We say that two semantics for a language are equivalent if they determine the same logics, or in other words, if every logic complete with respect to one is also complete with respect to the other. We have just seen that even with respect to extensions of **S4** the Neighbourhood and Relational Semantics are not equivalent.

Of course, the two semantics are equivalent if we restrict ourselves to extensions of **S4.3.** Bull [1] has shown that all normal extensions of **S4.3** have the finite model property. If **L** is the logic determined by a neighbourhood frame for **S4.3** then **L** is normal and so is complete with respect to finite structures. Since it is known that every finite structure can actually be viewed as a relational frame, **L** must be complete with respect to the Relational Semantics.

However, we have yet to find a solution to the problem of finding the real distinction between the two semantics we have been studying, of finding a non-semantic characterization for the logics complete with respect to each. Both in the present paper and in [4], in order to show that our logic was incomplete with respect to the Relational Semantics we had to use an infinite number of formulae of the logic. While we made no attempt to axiomatize the logic, the need to refer to infinitely many formulae from the logic leads us to make (tentatively) the conjecture that if **L** is a modal propositional logic which is an extension of **T** and which is complete with respect to the Neighbourhood Semantics but incomplete with respect to the Relational Semantics then **L** is not finitely axiomatizable.

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