Roberto Magari

Representation and Duality Theory for Diagonalizable Algebras

(*The algebraization of theories which express Theor*; IV)

Summary. The duality theory established by Halmos in [2] for boolean hemimorphism applies of course to the diagonalizable algebra, because $\nu\tau\nu$ is an hemimorphism.

For commodity in working on diagonalizable algebras we recall the basic facts and give the characteristic conditions on the dual of $\nu\tau\nu$.

Introduction

Let P be the set of the sentences of the Peano arithmetic and T the set of the theorems. It is well known that there exists a formula with one variable, $\dot{T}(x)$, which numerates T and satisfies useful further properties (derivability conditions, Löb's theorem).

This formula gives an unary operation τ in the Lindenbaum algebra of Peano arithmetic. So we can obtain an algebra $\langle A, +, \cdot, \nu, 0, 1, \tau\rangle$ in which:

(τ0) $\langle A, +, \cdot, \nu, 0, 1\rangle$ is a boolean algebra

(τ1) $\tau 1 = 1$

(τ2) $\tau(xy) = \tau x \cdot \tau y$

(τ3) $\tau(\tau x \to x) \leqslant \tau x \quad (x, y \in A,\ x \to y = y + \nu x)$

(τ4) if $x \leqslant y$ then $\tau x \leqslant \tau y \quad (a \leqslant b$ iff $ab = a)$

(τ5) $\tau(x \to y) \leqslant \tau x \to \tau y$

(τ6) $\tau x \leqslant \tau\tau x$

(See also [1], [4], [5])

Previous results (especially the fixed-point theorem in Bernardi [1]) ensure to us that many good properties of the Peano arithmetic (and in general of the theories which express Theor.) possess an adequate counterpart in every "diagonalizable algebra" (i.e. in every system $\langle A, +, \cdot, \nu, 0, 1, \tau\rangle$ which satisfies the previous properties).

In this paper, using the Jonsson-Tarski-Halmos representation theory for hemimorphisms, we show that in every diagonalizable algebra τ is linked with a "relatively founded" partial order in the dual space of $\langle A, +, \cdot, \nu, 0, 1\rangle$.

In § 1 we discuss also the logical relation between various properties of the diagonalizable algebras.

§ 1. We recall that a diagonalizable algebra is a system $\langle A, +, \cdot, \nu, 0, 1, \tau\rangle$ in which:

$\langle A, +, \cdot, \nu, 0, 1\rangle$ is a boolean algebra,
τ is an unary operation on A for which:

$(\tau 1)\quad \tau 1 = 1$
$(\tau 2)\quad \tau(x\cdot y) = \tau x\cdot \tau y \qquad (x, y \in A)$
$(\tau 3)\quad \tau(\tau x \to x) \leqslant \tau x \qquad (a\to b = \nu a + b)$

In every diagonalizable algebra we have of course:

$(\tau 4)\quad$ if $x \leqslant y$ then $\tau x \leqslant \tau y \quad (x, y\in A)(a\leqslant b$ iff $ab = a)$

We have also:

$(\tau 5)\quad \tau(x\to y) \leqslant \tau x \to \tau y$

In fact, using $(\tau 2)$:
$\tau x\cdot \tau(x\to y) = \tau\big(x(\nu x + y)\big) = \tau(x\cdot y) = \tau x\cdot \tau y$ and hence: $\nu\tau x + \tau(x\to y) = \nu\tau x + \tau x\cdot\tau(x\to y) = \nu\tau x + \tau x\cdot \tau y = \nu \tau x + \tau y = \tau x \to \tau y$ and so $(\tau 5)$

$(\tau 6)\quad \tau x \leqslant \tau^2 x \qquad (\tau^{n+1}x = \tau\tau^n x)$

In [5] we have assumed $(\tau 6)$ as axiom, but, as G. SAMBIN has pointed out it is provable from $(\tau 1)$, $(\tau 2)$, $(\tau 3)$. In fact:
$\tau x \leqslant \tau(x + \nu\tau x + \nu\tau^2 x) = \tau\big(x\cdot\tau x + \nu\tau(x\cdot\tau x)\big) = \tau\big(\tau(x\cdot\tau x)\to x\cdot\tau x\big) \leqslant \tau(x\cdot \tau x) \leqslant \tau x\cdot\tau^2 x \leqslant \tau^2 x$. Further, as we have seen in [5]:

$(\tau 7)\quad \tau x + \tau y \leqslant \tau(x+y)$
$(\tau 8)\quad \tau 0 \leqslant \tau x$
$(\tau 9)\quad \tau(\tau x \to x) = \tau x \qquad\qquad (x, y\in A,\ n\in\omega,\ n > 0)$
$(\tau 10)\quad \tau\nu\tau^n x = \tau 0$
$(\tau 11)\quad$ if $\tau x \leqslant x$ then $x = 1$.

It is useful to observe that if $(\tau 2)$, $(\tau 6)$, $(\tau 11)$ hold then also $(\tau 3)$ holds. In fact, using $(\tau 2)$:

$$\tau x = \tau(x + \tau x)(x + \nu\tau x) = \tau(x+\tau x)\cdot\tau(x + \nu\tau x)$$

and hence, using $(\tau 2)$, $(\tau 4)$ (which follows from $(\tau 2)$): $\tau(\tau x\to x)\to\tau x = \tau x + \nu\tau(x + \nu\tau x) = \tau(x+\tau x)\cdot\tau(x+\nu\tau x) + \nu\tau(x+\nu\tau x) = \tau(x+\tau x) + {} + \nu\tau(x + \nu\tau x)$ and, using $(\tau 6)$ {and $(\tau 5)$ which follows from $(\tau 2)$}:

$$\tau(x+\tau x) + \nu\tau(x+\nu\tau x) \geqslant \tau^2 x + \nu\tau^2(x + \nu\tau x) \geqslant \tau\Big(\big(\tau(x+\nu\tau x) \to \tau x$$

So, from $(\tau 11)$:

$$\tau(\tau x\to x)\to\tau x = 1 \qquad \text{i.e.} \qquad (\tau 3).$$

It is often useful to use, instead of τ, $\sigma = \nu\tau\nu$. Of course for σ the conditions given above become:

$$
\begin{array}{ll}
(\sigma 1) & \sigma 0 = 0 \\
(\sigma 2) & \sigma(x+y) = \sigma x + \sigma y \\
(\sigma 3) & \sigma(x \cdot \nu\sigma x) \geqslant \sigma x \\
(\sigma 4) & \text{if } x \leqslant y \text{ then } \sigma x \leqslant \sigma y \\
(\sigma 5) & \sigma(x \cdot \nu y) \geqslant \sigma x \cdot \nu\sigma y \\
(\sigma 6) & \sigma\sigma x \leqslant \sigma x \qquad\qquad (x,\ y \in A) \\
(\sigma 7) & \sigma(xy) \leqslant \sigma x \cdot \sigma y \\
(\sigma 8) & \sigma^1 \geqslant \sigma x \\
(\sigma 9) & \sigma(x \cdot \nu\sigma x) = \sigma x \\
(\sigma 10) & \sigma\nu\sigma^n x = \sigma 1 \\
(\sigma 11) & \text{if } x \leqslant \sigma x \text{ then } x = 0
\end{array}
$$

σ is an hemimorphism (terminology of [2]: an hemimorphism is a map σ between boolean algebras, for which (σ1), (σ2) hold) for which (σ3) or equivalently (σ6) and (σ11) holds. Terminology:

A τ satisfiyng (τ1), (τ2), (τ3) is a "Löb operator", σ is a "co-Löb operator". Since of course the category of the algebras $\langle A, +, \cdot, \nu, 0, 1, \tau\rangle$ (with algebraic homomorphisms) is isomorphic to the category of the algebras $\langle A, +, \cdot, \nu, 0, 1, \sigma\rangle$ (with algebraic homomorphisms) let us to use in everybody case the name "diagonalizable algebras". If we will be pedantic we speak of "co-diagonalizable algebras".

§ 2. Concrete diagonalizable algebras.

Let R be a binary relation on a set M and let us define a $\sigma: \mathscr{P}(M) \to \mathscr{P}(M)$ putting:

(1) $\sigma X = \{x \in M$: *there exists a* $y \in X$ *for which* $yRx\}$ (writing τ for $\nu\sigma\nu$ we have of course:

(1′) $\tau X = \{x \in M$: *for every* $y \in \nu X$, *non* $(y\mathscr{R}x\}$).

It is well known that σ is an hemimorphism i.e. satisfies (σ1), (σ2).

Let us call *concrete algebras* (c.a.) every system $\langle M, R, A\rangle$ where A is a field of subsets of M and R is a binary relation on M. The "abstract" of a c.a. $\langle M, R, A\rangle$ will be $\langle A, \cup, \cap, ', \emptyset, M, \sigma\rangle$ where σ is defined by (1). If $A = \mathscr{P}(M)$ we will speak of the *complete* abstract of $\langle M, R\rangle$.

LEMMA 1. *Let M, R be as above and let A be an abstract. Let us consider the conditions*:

(i) *R is transitive*

(ii) *R is founded (i.e. no sequence $(x_n)_{n\in\omega}$ with $x_{n+1}Rx_n$ exists or equivalently every non void subset of M admit a minimal element)*

(iii) *R is a (strict) partial order founded.*

Then:

(i)$\to$(σ6) (ii)$\to$(σ11) (iii)$\to$(σ3)

PROOF. (i)$\to$(6). Let R be transitive and $X \subseteq M$. If $x \in \sigma\sigma X$ then there exist y, z with $z \in X$, zRy, yRx. So zRx and $x \in \sigma X$.

(ii)$\to$(σ11). Let be $X \subseteq M$ and $X \neq \emptyset$. Let x be minimal in X. Then $x \notin \sigma X$ and $\sigma X \not\supseteq X$.

(iii)$\to$(σ3). Obvious because (iii) is equivalent to the conjunction of (i) (ii) and (σ3) (upon σ2) is equivalent to the conjunction of (σ6), (σ11)

COROLLARY 1. *If R is a (strict) partial order founded then every abstract of $\langle M, R\rangle$ is a diagonalizable algebra.*

Now suppose $A = \mathscr{P}(M)$. We have:

LEMMA 2. *If A is the complete abstract of $\langle M, R\rangle$ then*:

(σ6)$\to$(i) (σ11)$\to$(ii) (σ3)$\to$(iii)

PROOF. (σ6)$\to$(i). Let be xRy, yRz and $x \in X$. So $z \in \sigma\sigma X \subseteq \sigma X$, xRz.
(σ11)$\to$(ii) Let be $X \neq \emptyset$. Then $\sigma X \not\supseteq X$ i.e. X has minimal elements.
(σ3)$\to$(iii). Obvious from the recalled equivalences.

In general the implications (σ6)$\to$(i), (σ11)$\to$(ii) fail. Suppose for example $M = \{z_n\colon n \in \omega\} \cup \{a\} \cup \{y_n\colon n \in \omega\}$ with:

$$
\begin{array}{lll}
z_i R z_j & \text{when } i < j & \\
i_i R y_j & & (i, j \in \omega) \\
z_i R y_j & & (i, j \in \omega) \\
z_i R a & & \\
y_i R y_j & \text{when } j < i & \\
a R y_j & \text{when } i \neq 0 &
\end{array}
$$

and A be the algebra of the finite subsets of M excluding a and of the co-finite subsets of M including a.

It is easy to see that A becomes a diagonalizable algebra but R is not transitive nor founded.

§ 2. Representation.

Let A be a diagonalizable algebra. Is there a "representation" of A, *i.e.* a couple $\langle M, R\rangle$ with a monomorphism of A in the complete abstract of $\langle M, R\rangle$?

An adfirmative answer can be found in the general theory of duality for hemimorphism established in [2] §§ 8, 9[(1)], but let us start with some simple preliminary remarks.

LEMMA 3. *Let $\langle M, R, A\rangle$ a c. a. and define a new relation on M putting:*

(2) *$y < x$ iff, for every $x \in A$, if $y \in X$ then $x \in \sigma X$.*

Let σ^* the operator linked with $<$. Then $\sigma^* = \sigma$ (where σ is the operator linked with R) and $<$ is the maximum of the relations linked with σ.

PROOF. Of course $R \subseteq <$ and therefore for every $X \in A$ $\sigma X \subseteq \sigma^* X$. Let be $X \in A$ and $x \in \sigma^* X$. Then there exists an $y \in X$ with $y < x$ and, by (2), $x \in \sigma X$.

LEMMA 4. *Let $M, <, A, \sigma$ as over and let ϱ be the relation on M defined by:*

(3) *$x \varrho y$ iff, for every $X \in A$, X does not separe x and y.*

Then in a natural way we have a representation of $\langle A, \sigma\rangle$ by $\langle M/\varrho, </\varrho\rangle$.

PROOF. Obvious because if $x_1 \varrho x_2$, $y_1 \varrho y_2$, $x_1 < y_1$ then, using (2), we found $x_2 < y_2$.

Now let us recall the basic facts on duality theory for hemimorphism established in P. R. HALMOS [2][1]. Let A, B be boolean algebras and X, Y be its dual spaces. Think A, B as the algebras of the continuous functions from $A(B)$ to 2 equipped with the discrete topology. Then for every hemimorphism f (normal and additive map) from A to B the "dual" is the binary relation f^* from Y to X defined by:

(4) $y\ f^*\ x$ iff, for every $p \in A$, $p\ x \leqslant (f\ p)\ y$

This dual is a "boolean" relation i.e. the inverse image of a clopen (closed and open) of X is a clopen of Y and the direct image a point of Y is a closed set of X.

The "dual" R^* of a boolean relation from Y to X is the map from A to B defined by:

(5) $(R^* p) y = \overset{2}{\bigvee} \{px \colon yRx\}$.

P. R. HALMOS shows that:

PROPOSITION 1. *If f is an hemimorphism then f^* is a boolean relation and $f^{**} = f$. If R is a boolean relation then R^* is an hemimorphism and $R^{**} = R$.*

Now we have from prop. 1, lemma 3, lemma 4:

THEOREM 1. *If A is a boolean algebra and σ is an hemimorphism ($\sigma\colon A \to A$), then $\langle A, \sigma\rangle$ admits a representation in $\langle X, \sigma^{*-1}\rangle$ and every representation of $\langle A, \sigma\rangle$ can be reduced via lemma 3, lemma 4 to a represent-*

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[1] This theory extends the theory developped in [3] whch should be sufficient for our goals. See also [7].

ation in $\langle Z, \sigma^{*-1}\rangle$ where Z is a suitable dense subspace of X. In particular every diagonalizable algebra is representable.

PROOF. Obvious.

Let us observe that, roughly speaking, σ^* is precisely the maximum of the lemma 2.

It is also useful to observe that in general *not for every dense subspace* Z of X there exists an R with a representation of A on $\langle Z, R\rangle$.

EX. 2. Let A be the algebra of the finite or co-finite subsets of ω and define σ putting:

$$\sigma X = \begin{cases} \emptyset & \text{if } X \text{ is finite} \\ \{0\} & \text{if } X \text{ is co-finite.} \end{cases}$$

It is easy to see that σ is a (dual) Löb operation (and further $\sigma(X\cap Y) = \sigma X \cap \sigma Y$).

As dual space of A we can assume $\omega+1$ when A is represented by the algebra of the finite subsets of ω and of the co-finite subsets of $\omega+1$ including ω as element.

ω is obviously the only dense subspace and if an R exists for which σ is representable in $\langle \omega, R\rangle$ then, by lemma 3, σ is representable in $\langle \omega, \sigma *^{-1}_{|\omega}\rangle$ i.e. in $\langle \omega, <\rangle$ where: $x<y$ iff, for every $X\in A$, if $x\in X$ then $y\in\sigma X$ i.e. the void relation. Of course σ does not coincide with the σ' linked with $<$.

§ 4. Duality.

We have:

THEOREM 2. *Let A, σ, X as above, $< \; = \sigma^{*-1}$. Then:*

(i): ($\sigma 6$) *holds iff $<$ is transitive.*

(ii): ($\sigma 11$) *holds iff $<$ is "relatively" founded i.e. every clopen has minimal elements.*

(iii): ($\sigma 3$) *holds iff $<$ is transitive and relatively founded.*

PROOF. Obvious.

Now let A_1, A_2, boolean algebras, σ_1, σ_2 hemimorphisms on A_1, A_2, X_1, X_2 the dual spaces of A_1, A_2.

As HALMOS recalls in [2], f is an homomorphisms from A_1 to A_2 iff f^* is a (continuous) function with domain X. Now we have:

LEMMA 5. *f is an homomorphism from $\langle A_1, \sigma_1\rangle$ to $\langle A_2, \sigma_2\rangle$ iff f^* is a (continuous) function with domain x for which:*

(S) $$(f^*y, x) \in \sigma_1^* \text{ iff there exists } z \text{ with } f^*z = x \text{ and } (y, z) \in \sigma_2^*.$$

Let us to call *strict* an homomorphism between relational structures which satisfies (S) (or analogue).

PROOF. From the theorem 9 in [2] we have:

$$(f\sigma_1)^* = \sigma_1^* f^* \qquad (\sigma_2 f)^* = f^* \sigma_2^*$$

so the equality $f\sigma_1 = \sigma_2 f$ (i.e.: f is an homomorphism from $\langle A_1, \sigma_1\rangle$ to $\langle A_2, \sigma_2\rangle$) holds iff the equality $\sigma_1^* f^* = f^* \sigma_2^*$ holds. The lemma follows. So we have:

THEOREM 3. *The category C of the boolean algebras equipped with an hemimorphism (and of the algebraic homomorphisms) is equivalent to the category S of the Stone spaces equipped with a binary boolean relation (and of the continuous mappings which are strict homomorphism for the relational structures). In particular the full subcategory of C given by the diagonalizable algebras is equivalent to the full subcategory of S given by the objects which satisfies the condition:*

$$R^{-1} \text{ is transitive and relatively founded.}$$

PROOF. Obvious.

Now let us observe that, when we have equivalent categories it is often convenient to construct a category of more rich objects and morphism which form a category equivalent to the given categories. In this case we can consider the structures:

$$\langle X, A, \tau, \sigma, R, \circ\rangle \quad \text{where:}$$

X is a Stone space
A is a boolean algebra
$\circ$: $A \times X \to 2$
τ, σ are unary operations on A $\quad \tau = \nu\sigma\nu \quad \sigma = \nu\tau\nu$
R is a boolean relation on X
the map $\lambda x[p\circ x]$ is continuous for every $p \in A$ (where 2 has the discrete topology)
the map $\lambda p[p\circ x]$ is in *Hom* $(A, 2)$ (where 2 has the structure of two-elements simple algebra)

every ultrafilter F of A has exactly one point $x \in X$ for which, for every $p \in F$, $px = 1$

every clopen Z of X has exactly a $p \in A$ for which $p \cdot x = 1$ iff $x \in Z$

R^{-1} is transitive and relatively founded

$$(\sigma p) \circ x = 1 \text{ iff there exists an } y \text{ for which } py = 1 \text{ and } xRy$$

$$(\tau p) \circ x = 0 \text{ iff there exists an } y \text{ for which } py = 0 \text{ and } xRy$$

xRy iff, for every $p \in A$, iff $py = 1$ then $(\sigma p)y = 1$, i.e. for every $p \in A$, iff $py = 0$ then $(\tau p)y = 0$

(τ1), (τ2), (τ3), (σ1), (σ2), (σ3) holds

A *morphism* from $\langle X_1, A_1, \tau_1, \sigma_1, R_1, \circ\rangle$ to $\langle X_2, A_2, \tau_2, \sigma_2, R_2, \circ\rangle$ is a couple (f, f^*) for which

f is an homomorphism from $\langle A_1, \tau_1, \sigma_1\rangle$ to $\langle A_2, \tau_2, \sigma_2\rangle$

f^* is a continuous map from X_2 to X_1 which is a strict homomorphism from the relational structures $\langle X_2, R_2\rangle$, $\langle X_1, R_1\rangle$

With this *redundant* but useful definition the duality theory becomes the adfirmation that $\langle A, \tau\rangle$, $\langle A, \sigma\rangle$, $\langle X, R\rangle$ uniquely determine the structure and that $f(f^*)$ uniquely determine $f^*(f)$. (these "uniquely" are intended, obviously, cum grano salis).

Finally let us observe that often if we consider an hemimorphism type on Boolean algebra, the same conditions which holds for the binary relation naturally induced in the space of the atoms when the algebra is complete and atomic, hold for the dual of the hemimorphism in the dual space. So relatively to (σ6) the condition given in lemmas 1, 2 is the same that in theor. 2, analoguous situation for quantifier (the dual is an equivalence relation) and so on. This *is not true* for (σ11), (σ3).

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SIENA UNIVERSITY
SIENA, ITALY

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