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On Modal Logic with an Intuitionistic Base

§ 0. Abstract. A definition of the concept of "Intuitionist Modal Analogue" is presented and motivated through the existence of a theorem preserving translation from ***MIPC*** (see [2]) to a bimodal S^4-S^5 calculus.

§1. This paper is devoted to answering the following formal question: can we find a general criterion that will give us "the" intuitionistic analogue of some of the most usual modal systems? The problem as stated is of a technical nature and therefore the philosophical issues relating to the plausibility of an intuitionistic logic of modality will be in this context ignored.

R. A. BULL in [1], following a suggestion of PRIOR, proposes an extension of the intuitionist propositional calculus (***IC***) with the following rules:

$$R_1 \quad \frac{\alpha\to\beta}{L\alpha\to\beta}; \qquad R_2 \quad \frac{\alpha\to\beta}{\alpha\to M\beta};$$

$$R_3 \quad \frac{\alpha\to\beta}{\alpha\to L\beta}, \quad \text{if } \alpha \text{ is fully modalized;}$$

$$R_4 \quad \frac{\alpha\to\beta}{M\alpha\to\beta}, \quad \text{if } \beta \text{ is fully modalized.}$$

This system — which I call S^5-***IC*** — turns out to be analogous to LEWIS' S^5 in the sense that:

(1) adding the excluded middle to S^5-***IC*** gives a logic equivalent to S^5;

(2) collapsing the modal operators yields Heyting's calculus.

Conditions (1) and (2) are necessary but not sufficient to single out unambiguously the "correct S^5 analogue", for it is easy to find non equivalent systems which satisfy both conditions. Take, for instance, an S^5-thesis which is not derivable in S^5-***IC***, e.g.

$$L(\alpha\vee L\beta) \to L\alpha\vee L\beta.$$

It is easy to verify that S^5-***IC*** on one hand, and S^5-***IC*** plus this formula on the other, are examples of two non equivalent calculi[1] for which both (1) and (2) obtain. Then, we may reasonably ask which of these two systems (or any others so constructed) is to be the "real S^5 analogue".

[1] See also BULL [1].

Actually, there is a result which in a certain sense settles the question of analogy for S^5. R. A. BULL (see [2]) was able to show in fact that there is a translation map $\mathcal{T}$ from S^5-IC formulas to intuitionistic predicative formulas (***I-Pre***) with one free variable such that

$$(3) \qquad \vdash_{S^5\text{-}IC} a \quad \text{iff} \quad \vdash_{I\text{-}Pre} \mathcal{T} a$$

Now it is well known that (3) holds if, in each system, IC is replaced by the ordinary classical calculus. Although condition (3) is discriminatory enough, it cannot be assumed as a general criterion for analogy since, unlike (1) and (2), it depends upon a peculiarity of the S^5 modal operators.

For a more general approach we could follow this other strategy: take IC and simply add a classical modal axiom system[2]. For example, take L as a primitive symbol with the usual definition of M in terms of L and add to IC, the necessity rule and the following axioms:

(i) $La \to a$

(ii) $L(a \to \beta) \to (La \to L\beta)$

(iii) $La \to LLa$

(iv) $L\neg La \vee La$

(v) $La \leftrightarrow \neg M \neg a$

This kind of procedure can obviously be applied to modalities of differing strength but it faces the first objection noted in the beginning of this paper. For, the intuitionistic modal system thus obtained depends, firstly upon the connectives taken as primitive and secondly upon the modal axioms chosen amongst the many alternative systems which are classically — but not intuitionistically — equivalent. For instance, consider the above system to be named I^5, replace (i) by its classical equivalent

(i′) $\neg\neg(La \to a)$,

and call I'^5 the calculus obtained. Then, although we have that $\vdash_{I'^5} \neg\neg(La \to a)$, a simple example shows that *non* $\vdash_{I'^5} La \to a$ and hence $I^5 \neq I'^5$. Again, we find that, in correspondence to a single classical modal system, we have a family of non equivalent intuitionistic modal calculi.

The aim of this paper, then, is to define the concept of "intuitionistic modal analogue" in such a manner as to avoid some of the difficulties illustrated so far. The definition that will be proposed in §2 turns out to be a natural byproduct of a result concerning S^5-IC. What will be shown is the existence, for S^5-IC, of another theorem preserving translation which does not depend upon the specific characteristics of the modal system involved.

§2. In order to state adequately this new criterion for analogy, we need to introduce a few concepts. First, note that due to the well

[2] Modal intuitionistic systems based upon this idea can be found in [3].

known relationship between the propositional intuitionistic calculus and Lewis' S^4, it can be expected that S^5-IC bears a close structural similarity to a modal sentential calculus with two necessity operators, one of them being a sort of S^4 operator and the other a sort of S^5 operator. Thus, let (S^4, S^5)-C be the "bimodal" calculus with $\neg$, $\rightarrow$, L_1, L_2 as primitive connectives ($M_2 a$ being defined by $\neg L_2 \neg a$) and the following axioms and rules:

(b_0) *classical propositional base*, including Modus Ponens and the usual definitions for $\wedge$ and $\vee$;

(b_1) S^4 *axioms and rules on* L_1:

$$L_1 a \rightarrow a,$$
$$L_1(a \rightarrow \beta) \rightarrow (L_1 a \rightarrow L_1 \beta),$$
$$L_1 a \rightarrow L_1 L_1 a,$$
$$\frac{\vdash a}{\vdash L_1 a};$$

(b_2) S^5 *axioms and rules on* L_2:

$$L_2 a \rightarrow a,$$
$$L_2(a \rightarrow \beta) \rightarrow (L_2 a \rightarrow L_2 \beta),$$
$$M_2 a \rightarrow L_2 M_2 a,$$
$$\frac{\vdash a}{\vdash L_2 a};$$

(b_3) *connecting axioms*:

$$M_2 L_1 a \rightarrow L_1 M_2 a,$$
$$L_1 L_2 a \rightarrow L_2 L_1 a.[3]$$

Now let T be the translation map from S^5-IC to (S^4, S^5)-C which extends the Gödel translation from IC to S^4, in the following manner

for a propositional variable p,

$$T(p) = L_1 p$$

and for arbitrary formulas a and β,

$$T(a \wedge \beta) = Ta \wedge T\beta$$
$$T(a \vee \beta) = Ta \vee T\beta$$
$$T(\neg a) = L_1 \neg T(a)$$
$$T(a \rightarrow \beta) = L_1\big(T(a) \rightarrow T(\beta)\big)$$
$$T(Ma) = M_2 T(a)$$
$$T(La) = L_1 L_2 T(a).$$

Using these notations we shall prove in §§ 3, 4 the following

[3] Actually, the two axioms in (b_3) can be proven to be deductively equivalent. Since the proof of this fact makes use of the full power of the S^5 modality and as we are also interested in capturing a general concept of "bimodal calculus", we include them both. Calculi containing two modalities have been studied by M. Fitting (see [5]), who also considers an (S^4, S^5) system but with connecting axioms different from (b_3).

THEOREM A. *For every formula* α *of* $\boldsymbol{S^5\text{-}IC}$,

$$\vdash_{S^5\text{-}IC}\alpha \quad \textit{iff} \quad \vdash_{(S^4,S^5)\text{-}C} T(\alpha).$$

Now, if $(*)$-$\boldsymbol{C}$ is a classical modal calculus, let $(\boldsymbol{S}^4, *)$-$\boldsymbol{C}$ denote the system obtained by generalizing $(\boldsymbol{S}^4, \boldsymbol{S}^5)$-$\boldsymbol{C}$ in an obvious manner. In particular we have that the well formed formulas of $(\boldsymbol{S}^4, *)$-$\boldsymbol{C}$ are the same as those of $(\boldsymbol{S}^4, \boldsymbol{S}^5)$-$\boldsymbol{C}$. Then Theorem A suggests that it is reasonable to assume as a criterion for a general $(*)$-$\boldsymbol{IC}$ the following definition:[4]

(i) *the language of* $(*)$-$\boldsymbol{IC}$ *is the same as that of* $\boldsymbol{S}^5$-$\boldsymbol{IC}$,

and, as T can rightly be considered a map from the formulas of $(*)$-$\boldsymbol{IC}$ to those of $(\boldsymbol{S}^4, *)$-$\boldsymbol{C}$,

(ii) *the theorems of* $(*)$-$\boldsymbol{IC}$ *are those formulas whose T-translates are theorems of* $(\boldsymbol{S}^4, *)$-$\boldsymbol{C}$.

§3. The remaining two sections of this paper are devoted to the proof of theorem A, the latter being divided into two halves, theorem A_1 and theorem A_2. As theorem A_1 can be proven semantically[5] once completeness for $\boldsymbol{S}^5$-$\boldsymbol{IC}$ and $(\boldsymbol{S}^4, \boldsymbol{S}^5)$-$\boldsymbol{C}$ are seen to hold, we proceed to define the structures with respect to which each of these calculi are complete.

Given a *Heyting algebra* A and two operators K, I on A, we say that the triple $(A; K, I)$ is a *monadic Heyting algebra* ($\boldsymbol{HM}$), if:

(i) K, I are monotone,

(ii) $KIx \leqslant Ix$, $IKx \geqslant Kx$, for any $x \in A$,

(iii) $K[A]$, the range of K, is a subalgebra of A,

(iv) $Ix \leqslant x \leqslant Kx$, $(x \in A)$.

Using these notations and definitions one can prove:

THEOREM 1. *Given a Heyting algebra* A *and* $K, I\colon A \to A$, *the following are equivalent*[6]:

(i) $(A; K, I)$ *is a monadic Heyting algebra*;

(ii) *there is a subalgebra* B *of* A *such that, for every* $x \in A$,

$$Kx = \min\{y \in B\colon x \leqslant y\} \quad \textit{and} \quad Ix = \max\{y \in B\colon y \leqslant x\}.$$

[4] Obviously some restrictions on the class of cacluli to which this criterion applies are needed. For example, $(*)$-$\boldsymbol{C}$ must be a modal system having the full propositional calculus as its base. Moreover, it should be required that $(*)$-$\boldsymbol{IC}$ be a sub-system of $(*)$-$\boldsymbol{C}$ such that adding the excluded middle to the former system yields the latter one. It can be checked that these conditions are satisfied by those modal calculi studied in [7] and [8] in which the rule of Substitution of Equivalents holds.

[5] Actually it is possible to prove theorem A_1 by induction on the length of the proofs in $\boldsymbol{S}^5$-$\boldsymbol{IC}$. Although such a straightforward syntactical argument is available, the algebraic tools involved in the above semantic proof provide a deeper insight into some of the ideas connected with an intuitionistic concept of modality.

[6] The full proof of theorem 1 is to be found in [4].

THEOREM 2. *The S^5-IC theorems are precisely those formulas which are true*[7] *in all monadic Heyting algebras.*

PROOF. Use theorem 1 and the proof in [1], §§2, 3.

Now we define a *bimodal algebra* as a triple $(B; I_1, I_2)$, where B is a Boolean algebra, I_1 a topological interior operator and I_2 a universal quantifier on B[8] such that:

(i) $I_1 I_2 x \leqslant I_2 I_1 x$

(ii) $K_2 I_1 x \leqslant I_1 K_2 x,\qquad (x \in B)$

where $K_2 x =_{df} -I_2 - x$.

It is easy to verify that the following completeness result holds:

THEOREM 3. *A formula a is a theorem of (S^4, S^5)-C iff a is true in all bimodal algebras.*

Using theorems 2 and 3 we can proceed to prove theorem A_1 in the following equivalent form:

THEOREM A_1'. *For each formula a of S^5-IC, if a is true in every monadic Heyting algebra, then $T(a)$ is true in every bimodal algebra.*

Before we get to the proof of A_1', we give some definitions and prove a few other theorems. Let us recall the well known fact that if $\mathscr{B} = (B; I_1, I_2)$ is a bimodal algebra, the set of those elements of B which are open with respect to I_1 is a Heyting algebra (with the same unit element as B). Thus, if we define

$$(1)\qquad \begin{cases} A = \{x \in B:\ x \leqslant I_1 x\}; \\ K = K_2 \restriction A; \\ I = I_1 I_2 \restriction A, \end{cases}$$

we can show that

THEOREM 4. *$(A; K, I)$ is a monadic Heyting algebra.*

PROOF. (a) Note that by definition I is an operator on A. Furthermore, for $x \in A$, $Kx = K_2 I_1 x \leqslant I_1 K_2 x = I_1 K x$ and so $Kx \in A$.

(b) We prove that the operators defined in (1) satisfy (i)–(iv) of the definition of ***HM***. Condition (i) follows from (1), since K_2, I_2 and

[7] As usual (see [7]), a formula $a \in F$ (where F is the algebra of formulas of S^5-IC) is said to be *true* in a monadic Heyting algebra A, if every homomorphism $e: F \to A$ carries a into the unit element of A. A similar definition holds for (S^4, S^5)-C and bimodal algebras introduced below.

[8] A *universal quantifier* I on a Boolean algebra B is an operator on B such that for every $x, y \in B$, $Ix \leqslant x$, $I1 = 1$ and $I(x \cup Iy) = Ix \cup Iy$ (see [6]).

I_1 are all monotone. For (ii), if $x \in A$, then $KIx = K_2I_1I_2x \leqslant I_1K_2I_2x = I_1I_2x = Ix$ and, using part (a), $Kx = K_2x = I_1K_2x = I_1I_2K_2x = IKx$.

As for (iii), if $u, v \in K[A]$, i.e. $u = K_2x$ and $v = K_2y$ (for some x, y in A), then:

$$u \cup v = K_2x \cup K_2y = K_2(x \cup y) \in K[A]; \qquad u \cap v = K_2x \cap K_2y = $$
$$= K_2(x \cap K_2y) \in K[A];$$

$$u \Rightarrow y = I_1(-K_2x \cup K_2y) = I_1(K_2 - K_2x \cup K_2y) =$$
$$= I_1K_2(-K_2x \cup K_2y) \in K[A],$$

since $I_1K_2 = KI_1K_2$; $0 \in K[A]$, since $0 \in A$ and $K_20 = 0$. Hence, $K[A]$ is a subalgebra of A.

Finally, I_1 being monotone, inequalities (iv) follow from definition (1) and similar inequalities for I_1, I_2 and K_2.

From now on, $\mathscr{A}_{\mathscr{B}}$ will denote the monadic Heyting algebra which, according to theorem 4, is associated with a bimodal algebra $\mathscr{B}$. Using these notations we prove

LEMMA 1. *Let $\mathscr{B}$ be a bimodal algebra, let F_0 be the algebra of formulas of (S^4, S^5)-C and F the algebra of formulas of S^5-IC. If V_0 is the set of propositional variables and $v\colon F_0 \to \mathscr{B}$, $w\colon F \to \mathscr{A}_{\mathscr{B}}$ are two homomorphisms such that*

(2) $\qquad w \restriction V_0 = I_1 v \restriction V_0;$

then

(3) $\qquad w(a) = v(Ta) \qquad\qquad (a \in F).$

PROOF. By induction on the height of a:

(i) If p is a propositional variable, given (1), the definition of T and the fact that v is a homomorphism, we have $v(Tp) = v(L_1p) = I_1v(p) = w(p)$.

(ii) By the definition of T and the hypothesis on v, $vT(a \vee \beta) = vT(a) \cup vT(\beta)$; then using the inductive hypothesis, w being a homomorphism, $vT(a) \cup vT(\beta) = w(a) \cup w(\beta) = w(a \vee \beta)$.

(iii) Similarly for the case of $a \wedge \beta$.

(iv) For reasons similar to those in (ii), $vT(\neg\beta) = v(L_1 \neg T(\beta)) = I_1v(\neg T(\beta)) = I_1 - vT(\beta) = I_1 - w(\beta)$, but, by definition of $\mathscr{A}_{\mathscr{B}}$, $I_1 - w(\beta) = w(\neg\beta)$.

(v) Similarly for the case of $a \to \beta$.

(vi) $v(T(M\beta)) = v(M_2T(\beta)) = K_2vT(\beta) = K_2w(\beta)$ by definition of T and the inductive hypothesis. But $w(\beta) \in \mathscr{A}_{\mathscr{B}}$, so $K_2w(\beta) = Kw(\beta) = w(M\beta)$.

(vii) $v(T(L\beta)) = v(L_1L_2T(\beta)) = I_1I_2vT(\beta) = I_1I_2w(\beta) = Iw(\beta) = w(L\beta)$ for reasons similar to those in (vi).

Let $\mathscr{B}$, F_0, F and Γ_0 be as in lemma 1. We have then:

LEMMA 2. *For every homomorphism* $v: F_0 \to \mathscr{B}$, *there exists a homomorphism* $w: F \to \mathscr{A}_{\mathscr{B}}$ *such that* (3) *holds.*

PROOF. Suppose that v is a homomorphism from F_0 to $\mathscr{B}$. Let $\bar{v} = v \restriction \Gamma_0$; then $I_1 \bar{v}: \Gamma_0 \to \mathscr{A}_{\mathscr{B}}$. Since F is free on Γ_0[9], there is a (unique) homomorphism $w: F \to \mathscr{A}_{\mathscr{B}}$ such that $w \restriction \Gamma_0 = I_1 \bar{v} = I_1 v \restriction \Gamma_0$. Claim now follows from lemma 1.

LEMMA 3. *For every homomorphism* $w: F \to \mathscr{A}_{\mathscr{B}}$ *there exists a homomorphism* $v: F_0 \to \mathscr{B}$ *such that* (3) *holds.*

PROOF. Let $w: F \to \mathscr{A}_{\mathscr{B}}$. Define $\bar{w} = w \restriction \Gamma_0$; then $\bar{w} = I_1 \bar{w}$ can be considered as a map from Γ_0 to $\mathscr{B}$ and, F_0 being free on Γ_0, let $v: F_0 \to \mathscr{B}$ be the homomorphism extending it. Then $I_1 v \restriction \Gamma_0 = I_1 \bar{w} = \bar{w} = w \restriction \Gamma_0$ and lemma 1 yields the desired result.

From the above lemmas we can now prove:

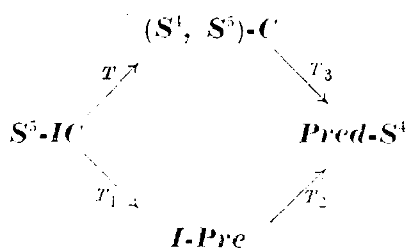
THEOREM 5. *Let* $\mathscr{B} = (B; I_1, I_2)$ *be a bimodal algebra. For every formula* a *of* $\boldsymbol{S^5}$-$\boldsymbol{IC}$, $T(a)$ *is true in* $\mathscr{B}$ *iff* a *is true in* $\mathscr{A}_{\mathscr{B}}$.

PROOF. Suppose that $T(a)$ is true in $\mathscr{B}$. Let $w: F \to \mathscr{A}_{\mathscr{B}}$; we want to prove that $w(a) = 1$. Now, by lemma 2, there is a $v: F_0 \to \mathscr{B}$ such that $v(Ta) = w(a)$; hence, $w(a) = v(Ta) = 1$[10]. The converse follows similarly using lemma 3.

And finally for the PROOF OF THEOREM A_1'

Let $\mathscr{B}$ be any bimodal algebra and let a be a formula of $\boldsymbol{S^5}$-$\boldsymbol{IC}$, true in each monadic Heyting algebra. In particular, then, a is true in $\mathscr{A}_{\mathscr{B}}$ and hence $T(a)$ is true in $\mathscr{B}$, by theorem 5.

§4. The remaining part of theorem A can be proven[11] through the use of a combination of translations. Consider accordingly the following diagram and definitions:



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[9] In the class of algebras which are similar to the monadic Heyting algebras. Note also that F_0 is free on Γ_0 in the class of algebras similar to the bimodal algebras.

[10] Recall that $\mathscr{B}$ and $\mathscr{A}_{\mathscr{B}}$ have the same unit element.

[11] To complete the proof of theorem A through means similar to those used for theorem A_1', the following "Representation theorem" would do: *For each monadic Heyting algebra* $\mathscr{A}$, *there is a bimodal algebra* $\mathscr{B}$ *such that* $\mathscr{A} = \mathscr{A}_{\mathscr{B}}$. I have not as yet obtained any result in this direction.

DEFINITION of T_1: Replace (in a $1-1$ fashion) propositional variables by predicates of some one fixed variable x, leave non-modal connectives unaltered, replace L, M by $\forall x$ and $\exists x$ respectively[12].

DEFINITION of T_2: Take the Gödel translation[13].

DEFINITION of T_3 (inductively):

(i) Like T_1 for propositional variables;

(ii) $T_3(\alpha \vee \beta) = T_3\alpha \vee T_3\beta$;

(iii) $T_3(\neg \alpha) = \neg T_3\alpha$;

(iv) $T_3(L_1\alpha) = LT_3\alpha$;

(v) $T_3(L_2\alpha) = (\forall x)T_3\alpha$.

With these notations and symbols we have the following theorems.

THEOREM 6. *For every formula* α *of* $\boldsymbol{S^5\text{-}IC}$, $\vdash_{S^5\text{-}IC}\alpha$ *iff* $\vdash_{I\text{-}Pre}T_1\alpha$

PROOF. To be found in [2].

THEOREM 7. *For every formula* γ *of* ***I-Pre***, $\vdash_{I\text{-}Pre}\gamma$ *iff* $\vdash_{Pred\text{-}S^4}T_2\gamma$

PROOF. See [9].

THEOREM 8. *For every formula* β *of* $(\boldsymbol{S^4}, \boldsymbol{S^5})$-***IC***, *if* $\vdash_{(S^4,S^5)\text{-}C}\beta$ *then* $\vdash_{Pred\text{-}S^4}T_3\beta$.

PROOF. It is sufficient to show that for every axiom β of $(\boldsymbol{S^4}, \boldsymbol{S^5})$-***C***, $\vdash_{Pred\text{-}S^4}T_3\beta$, and for each rule of inference, say $\frac{\beta}{\beta'}$, of $(\boldsymbol{S^4}, \boldsymbol{S^5})$-***C***, if $\vdash_{Pred\text{-}S^4}T_3\beta$ then $\vdash_{Pred\text{-}S^4}T_3\beta'$. The easy proof rests on some well known features of ***Pred-S***^{4}.

THEOREM 9. $T_3T\alpha = T_2T_1\alpha$, *for any formula* α *in* $\boldsymbol{S^5\text{-}IC}$.

PROOF. Straightforward, by induction on the height of the formulas of $\boldsymbol{S^5\text{-}IC}$.

And finally:

THEOREM A_2. *For each formula* α *of* $\boldsymbol{S^5\text{-}IC}$, *if* $\vdash_{(S^4,S^5)-C}T\alpha$, *then* $\vdash_{S^5\text{-}IC}\alpha$.

PROOF. Let $\vdash_{(S^4,S^5)\text{-}C}T\alpha$. Then, by theorem 8, $T_3T\alpha$ is a thesis of ***Pred-S***^{4}, hence $\vdash_{Pred\text{-}S^4}T_2T_1\alpha$, by theorem 9. Using theorem 7, $\vdash_{I\text{-}Pre}T_1\alpha$, whence $\vdash_{S^5\text{-}IC}\alpha$ by theorem 6.

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[12] See BULL [2]. Note that T_1 is the translation map $\mathcal{T}$ to which we referred in §1.

[13] See for instance [9], pp. 484–485.

References

[1] R. A. Bull, *A modal extension of intuitionistic logic*, ***Notre Dame Journal of Formal Logic***, Vol. 6, No. 2 (1965), pp. 142–146.

[2] —, *MIPC as the formalization of an intuitionist concept of modality*, ***Journal of Symbolic Logic***, Vol. 31, No. 4 (1966), pp. 609–616.

[3] —, *Some modal calculi based on IC*, in *Formal systems and recursive functions* (eds. J. N. Crossley and A. E. Dummet), North-Holland, 1966.

[4] G. Fischer Servi, *Un'algebraizzazione del calcolo intuizionista monadico*, to appear in ***Matematiche*** (Catania).

[5] M. Fitting, *Logics with several modal operators*, ***Theoria*** 35 (1969), pp. 259–266.

[6] P. R. Halmos, *Algebraic logic*, New York, 1962.

[7] E. J. Lemmon, *Algebraic semantics for modal logics* I, ***Journal of Symbolic Logic***, Vol. 31, No. 1 (1966), pp. 46–65.

[8] —, *Algebraic semantics for modal logics* II, ***Journal of Symbolic Logic***, Vol. 31, No. 2 (1966), pp. 191–218.

[9] H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, PWN, Warsaw, 1963.

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