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Lattices Related to Post Algebras and Their Applications to Some Logical Systems

Post algebras were introduced for the first time by Rosenbloom [6] and investigated after that by many autors (for the full bibliography and historical remarks see [4] and [5]). These algebras play such a role for the m-valued logics of E. Post as Boolean algebras for the classical logic. The first standard systems (in the sense of Rasiowa [4]) of m-valued propositional calculi, complete with respect to the class of Post algebras, were constructed by Rousseau [7], [8]. The logic corresponding to these calculi was called by Rousseau classical many-valued logic. In [7] and [8] Rousseau introduced also the notions of intuitionistic many-valued logic and pseudo-Post algebras as a semantic basis of this logic.

In this paper we define a class of generalizations of Post and pseudo-Post algebras (called here D-N-algebras) and a class of relational systems (called here D*-N*-spaces). We give some characterizations of these notions by means of an algebraic version of the notion of forcing. As a consequence we obtain, among others, a prime filters characterization of D-N-algebras, an open sets characterization of D*-N*-spaces, Stone representation theory for D-N-algebras and its dual analogue for D*-N*-spaces.

These results are applied to some logical systems, generalizing classical and intuitionistic m-valued logics of Rousseau. D-N-algebras are used for defining an algebraic semantics for these calculi, while D*-N*-spaces are used for defining a relational (Kripke-style) semantics for them. The standard completeness theorems are proved, as well as some decidability results are obtained.

Among algebras we will consider in this paper, except Post and pseudo-Post algebras, there are two very important, called here quasi-Post and quasi-pseudo-Post algebras respectively. Like Łukasiewicz algebras [2] they do not contain the constant operations $e_0, \ldots, e_{m-1}$. We prove that any quasi-Post (quasi-pseudo-Post) algebra can be embedded in a Post (pseudo-Post) algebra. This gives certain separation theorems for the propositional calculi of Rousseau.

Let us note that another type Kripke-style semantics for the classical m-valued propositional calculi of Rousseau has been given by Dahn [1].

1. Definitions of pseudo-Post and Post algebras. D-N-algebras

DEFINITION A. Let

(1) $\mathfrak{P} = \langle P, \bigvee, \bigwedge, \cup, \cap, \Rightarrow, \neg, D_1, \ldots, D_{m-1}, e_0, \ldots, e_{m-1}\rangle \qquad m \geqslant 2$

be an abstract algebra, where $\bigvee, \bigwedge, e_0, \ldots, e_{m-1}$ are zero-argument operations. $\neg, D_1, \ldots, D_{m-1}$ are one-argument operations and $\cup, \cap, \Rightarrow$ are two--argument operations in P. Following Rousseau we shall say that (1) is a pseudo-Post algebra of order m if it satisfies the following conditions:

(p) $\langle P, \bigvee, \bigwedge, \cup, \cap, \Rightarrow, \neg\rangle$ is a pseudo-Boolean algebra, i.e.

(i) $\langle P, \bigvee, \bigwedge, \cup, \cap\rangle$ is a distributive lattice with the zero element $\bigwedge$ and the unit element $\bigvee$,

(ii) for any $x, a, b \in P$ $x \cap a \leqslant b$ iff $x \leqslant a \Rightarrow b$,

(iii) $\neg a = a \Rightarrow \bigwedge$.

For any $a, b \in P$ and $i = 1, \ldots, m-1$

(D′) $D_i(a \cup b) = D_i(a) \cup D_i(b)$

(D″) $D_i(a \cap b) = D_i(a) \cap D_i(b)$

(R) $D_i(D_j(a)) = D_i(a), \quad j = 1, \ldots, m-1$

(C_1) $D_i(e_j) = \bigvee$ for $i \leqslant j, \quad j = 0, \ldots, m-1$

(C_2) $D_i(e_j) = \bigwedge$ for $i > j, \quad j = 0, \ldots, m-1$

(Mr) $a = (D_1(a) \cap e_1) \cup \ldots \cup (D_{m-1}(a) \cap e_{m-1})$

If we add the following axiom

(B) $\neg D_i(a) \cup D_i(a) = \bigvee, \; i = 1, \ldots, m-1$

then we obtain the definition of Post algebra.

Lemma 1.1. *In any pseudo-Post algebra* (1) *the following conditions are satisfied:*

(2) *If* $a \leqslant b$ *then* $D_i(a) \leqslant D_i(b), \quad i = 1, \ldots, m-1.$

(I) *If* $D_i(a) \leqslant D_i(b), \; i = 1, \ldots, m-1,$ *then* $a \leqslant b$

(M) $D_{i+1}(a) \leqslant D_i(a), \quad i = 1, \ldots, m-2.$

(Lω) $a \leqslant D_1(a)$

(Lα) $D_{m-1}(a) \leqslant a$

(L) $a \cap D_i(b) \leqslant b \cup D_{i+1}(a) \quad i = 1, \ldots m-2.$

Proof. Conditions (2) and (I) folow easily from (D′) and (Mr) respectively. Operating on $a = (D_1(a) \cap e_1) \cup \ldots \cup (D_{m-1}(a) \cap e_{m-1})$ with D_i and applying (D′), (D″), (R), (C_1), and (C_2) we obtain $D_i(a) = D_i(a) \cup \cup D_{i+1}(a) \cup \ldots \cup D_{m-1}(a)$. Thus (M) holds.

It follows from (M) and (R) that for any i $D_i(a) \leqslant D_1(a) = D_i(D_1(a))$. So by (I) $a \leqslant D_1(a)$ and (Lω) is fulfilled. In the same way can be proved (Lα).

For (L) we shall verify first the following:

(3) $D_j(a) \cap D_i(b) \leqslant D_j(b) \cup D_{i+1}(a)$ *for* $i = 1, \ldots, m-2, \quad j = 1, \ldots m-1.$

Suppose $j \leqslant i$. Then by (M) $D_i(b) \leqslant D_j(b)$ and (3) holds.

Suppose $j > i$. Then $j \geqslant i+1$ and by (M) $D_j(a) \leqslant D_{i+1}(a)$ and (3) holds. Operating on (3) with D_j and using (2), (D′), (D″), and (R) we obtain $D_j(a) \cap D_j(D_i(b)) \leqslant D_j(b) \cup D_j(D_{i+1}(a))$ and by (D″) and (D′) $D_j(a \cap D_j(b)) \leqslant D_j(b \cup D_{i+1}(a))$. From this we get $a \cap D_i(b) \leqslant b \cup D_{i+1}(a)$ which is (L).

For our later purpose we need a definition of pseudo-Post algebra in which axiom (Mr) is replaced by several simple axioms. It was found that instead of (Mr) we can use (M), (Lω), (La) and (L). So we define:

DEFINITION B. Let $\mathfrak{P}$ be an abstract algebra similar to (1). $\mathfrak{P}$ is said to be a *pseudo-Post algebra* if it satisfies the axioms (p), (D′), (D″), (R), (M), (L), (Lω), (La), (C_1) and (C_2).

LEMMA 1.2. *In any pseudo-Post algebra in the sense of definition B the following conditions are satisfied:*

(i) *if* $D_i(a) \leqslant D_i(b)$ $i = 1, \ldots, j$, $j \leqslant m-1$ *then* $a \leqslant b \cup D_i(b)$,

(ii) *if* $D_i(a) = D_i(b)$ $i = 1, \ldots, m-1$ *then* $a = b$,

(Mr) $a = (D_1(a) \cap e_1) \cup \ldots \cup (D_{m-1}(a) \cap e_{m-1})$

PROOF. (i). By induction. Suppose $j = 1$. Then by (Lω) $a \leqslant D_1(a) \leqslant D_1(b) = b \cup D_1(b)$. Let $j = k+1 \leqslant m-1$ and by inductive hypothesis $a \leqslant b \cup D_k(b)$. Since $D_{k+1}(a) \leqslant D_{k+1}(b)$ by (L) we get $a = a \cap a \leqslant (a \cap b) \cup \cup (a \cap D_k(b)) = b \cup (b \cup D_{k+1}(a)) \leqslant b \cup D_{k+1}(b)$.

(ii). Putting $j = m-1$ in (i) and using (La) we obtain (ii).

(Mr). Observe that from (M) we obtain $D_i(a) = D_i(a) \cup \ldots \cup D_{m-1}(a)$. Applying to this equality (C_1), (C_2), (R), (D′), (D″) we have: $D_i(a) = D_i(D_1(a)) \cap D_i(e_1) \cup \ldots \cup D_i(D_i(a)) \cap D_i(e_i) \cup \ldots \cup D_i(D_{m-1}(a)) \cap D_i(e_{m-1}) = D_i(D_1(a) \cap e_1 \cup \ldots \cup D_i(a) \cap e_i \cup \ldots \cup D_{m-1}(a) \cap e_{m-1})$. By (ii) we obtain $a = D_1(a) \cap e_1 \cup \ldots \cup D_{m-1}(a) \cap e_{m-1}$.

THEOREM 1.3. *Definitions A and B for pseudo-Post algebras are equivalent.*

PROOF. By lemma 1.1. and lemma 1.2.

Let us mention that the axioms (C_1) and (C_2) in the definition B may be replaced by the following two axioms:

(C_1') $D_i(e_i) = \bigvee$ $i = 1, \ldots, m-1$,

(C_2') $D_{i+1}(e_i) = \bigwedge$ $i = 0, \ldots, m-2$.

Now we turn to define some notions. Let $\mathfrak{P} = \langle P, \bigvee, \bigwedge, \cup, \cap, \Rightarrow, \neg, D_1, \ldots, D_{m-1} \rangle$ be an abstract algebra in which all operations have the same number of arguments as in (1), perhaps containing also the constants $e_0, \ldots, e_{m-1}$. We shall call $\mathfrak{P}$ a *D-algebra* if the axioms (p), (D′), (D″) and the following two are satisfied:

(D_0') $D_i(\bigwedge) = \bigwedge$ $i = 1, \ldots, m-1$,

(D_0'') $D_i(\bigvee) = \bigvee$ $i = 1, \ldots, m-1$.

It is easy to see that in any pseudo-Post algebra (D_0') and (D_0'') are satisfied. Thus any pseudo-Post algebra is a D-algebra.

Let Ax be the set of axioms (R), (M), (Lω), (La), (L), (C_1), (C_2), (B) and (J), where (J) is the following axiom

(J) $\quad D_i\big(D_i(a) \Rightarrow a\big) = \bigvee, \quad i = 1, \ldots, m-1.$

Let N be a subset of Ax and $\mathfrak{P}$ a D-algebra. We shall say that $\mathfrak{P}$ is a D-N-algebra if all of the axioms of N are satisfied.

Later we shall show that axiom (J) is satisfied in any Post algebra. So D-N-algebras are generalizations of Post algebras.

A D-N-algebra is said to be a *quasi-Post algebra* if $N = Ax \setminus \{C_1, C_2, J\}$, a *quasi-pseudo-Post algebra* if $N = Ax \setminus \{C_1, C_2, B\}$. In both cases $\mathfrak{P}$ does not contain the constant operations $e_0, \ldots, e_{m-1}$.

2. D*-N*-spaces

We shall examine D-N-algebras by means of some relational systems which we shall call here D*-N-*spaces. Let

(4) $\quad \boldsymbol{S} = \langle S, \subset, d_1, \ldots, d_{m-1}, E_0, \ldots E_{m-1}\rangle \quad m \geqslant 2$

be a relational system where $\subset$ is an ordering relation in S, $d_1, \ldots, d_{m-1}$ are one-argument operations in S and $E_0, \ldots, E_{m-1}$ are subsets of S. We shall say that $\boldsymbol{S}$ is a *D^*-space of order m* if the following conditions are satisfied for any $x, y \in S$:

(D^*) $\quad$ If $x \subset y$ then $d_i x \subset d_i y \quad i = 1, \ldots, m-1,$

(E^*) $\quad$ if $x \in E_i$ and $x \subset y$ then $y \in E_i \quad i = 1, \ldots, m-1.$

We shall call *D^*-spaces* (without subsets E_i) systems $\boldsymbol{S} = \langle S, \subset, d_1, \ldots, d_{m-1}\rangle$ $m \geqslant 2$, where operations have the same number of arguments as in (4) and satisfying axiom (D^*).

Let Ax^* be the set of the following conditions, corresponding to the conditions of the set Ax:

(R^*) $\quad d_i d_j x = d_i x \quad i, j = 1, \ldots, m-1.$

(M^*) $\quad d_{i+1} x \subset d_i x \quad i = 1, \ldots, m-2.$

(L^*_ω) $\quad x \subset d_1 x.$

(L^*_a) $\quad d_{m-1} x \subset x.$

(L^*) $\quad d_i x \subset x$ or $x \subset d_{i+1} x \qquad i = 1, ., \ldots, m-2.$

(B^*) $\quad$ If $x \subset y$ then $d_i y \subset d_i x \quad i = 1, \ldots, m-1.$

(J^*) $\quad$ If $d_i x \subset y$ then $d_i y \subset y \quad i = 1, \ldots, m-1.$

(C_1^*) $\quad$ If $i \leqslant j$ then $d_i x \in E_j \quad i = 1, \ldots m-1, \ j = 0, \ldots, m-1.$

(C_2^*) $\quad$ If $i > j$ then $d_i x \notin E_j \quad i = 1, \ldots m-1, \ j = 0, \ldots, m-1.$

Let N^* be a subset of Ax^*. We shall say that the system (4) is a D^*-N^*-space if it is a D^*-space and all of the axioms from N^* are satisfied in S.

A D^*-N^*-space is said to be

(i) a *Post space* if $N^* = Ax^* \setminus \{J^*\}$,
(ii) a *pseudo-Post space* if $N^* = Ax^* \setminus \{J^*, B^*\}$,
(iii) a *quasi-Post space* if $N^* = Ax^* \setminus \{J^*, C_1^*, C_2^*\}$,
(iv) a *quasi-pseudo-Post space* if $N^* = Ax \setminus \{B^*, C_1^*, C_2^*\}$.

Now we shall give an example of a D^*-N^*-space.

THEOREM 2.1. *Let $\mathfrak{P}$ be a D-N-algebra and $S(P)$ be the set of all prime filters in $\mathfrak{P}$. Define*

(5) $d_i \nabla = \{a \in P / D_i(a) \in \nabla\}, \quad \nabla \subset P, \quad i = 1, \ldots, m-1,$

(6) $E_i(P) = \{\nabla \in S(P) / e_i \in \nabla\}, \quad i = 0, \ldots, m-1.$

Then the system

(7) $S(\mathfrak{P}) = \langle S(P), \subseteq, d_1, \ldots, d_{m-1}, E_0(P), \ldots, E_{m-1}(P)\rangle$

where $\subseteq$ is the set-theoretical inclusion is a D^*-N^*-space, where N^* is the set of conditions corresponding to these of the set N.

PROOF. It is easy to see, using only the axioms for D-algebra that the set $S(P)$ is closed with respect to the operations d_i. Axioms (D^*) and (E^*) are true on account of (5) and (6). For the remaining part of the theorem we shall use the following well known lemma (see [3]):

LEMMA 2.2. *Let $\mathfrak{P}$ be a pseudo-Boolean algebra and $S(P)$ be the set of all prime filters in $\mathfrak{P}$. Then for any $\nabla_1 \in S(P)$ and $a, b \in P$ we have:*

(i) $a \Rightarrow b \in \nabla_1 \equiv (\forall \nabla_2 \in S(P))((\nabla_1 \subseteq \nabla_2 \ \& \ a \in \nabla_2) \rightarrow b \in \nabla_2)$
(ii) $\neg a \in \nabla_1 \equiv (\forall \nabla_2 \in S(P))(\nabla_1 \subseteq \nabla_2 \rightarrow a \notin \nabla_2)$
(iii) $(\forall \nabla \in S(P))(a \in \nabla \equiv b \in \nabla) \rightarrow (a = b)$

Let $X^* \in N^*$. As an example we shall take only the case $X^* = B^*$. We have to prove that for any $\nabla_1, \nabla_2 \in S(P)$: $\nabla_1 \subseteq \nabla_2$ implies $d_i\nabla_2 \subseteq d_i\nabla_1$. Suppose $\nabla_1 \subseteq \nabla_2$ and for the contrary $d_i\nabla_2 \nsubseteq d_i\nabla_1$. Then for some $a \in d_i\nabla_2$, $a \notin d_i\nabla_1$, i.e. $D_i(a) \in \nabla_2$ and $D_i(a) \notin \nabla_1$. By axiom (B) $\neg D_i(a) \in \nabla_1$. Since $\nabla_1 \subseteq \nabla_2$ by lemma 2.2 (ii) we have $D_i(a) \notin \nabla_2$ contrary to $D_i(a) \in \nabla_2$.

We shall call the system (7) *D^*-N^*-space of sets over the D-N-algebra* $\mathfrak{P}$.

THEOREM 2.3. *In any pseudo-Post space*

(i) $x \in E_i$ *iff* $d_i x \subset x \quad i = 1, \ldots, m-1. \quad E_0 = \varnothing.$

(ii) $x \notin E_i$ *iff* $x \subset d_{i+1}x \quad i = 0, \ldots, m-2. \quad E_{m-1} = S.$

(C^*) $d_i x$ *non* $\subset d_j y$ *for* $i < j \quad i = 1, \ldots, m-1, \quad j = 1, \ldots, m-1.$

PROOF. (i) Suppose that $x \in E_i$ but $d_i x \not\subset x$ for some x.

Case 1. $i < m-1$. Then by (L^*) $x \subset d_{i+1}x$ and by (E^*) $d_{i+1}x \in E_i$ which contradicts (C_2^*).

Case 2. $i = m-1$. Then by assumption $d_{m-1}x \not\subset x$ which contradicts (L_α^*). Hence $x \in E_i$ implies $d_i x \subset x$.

For the converse suppose $d_i x \subset x$. By (C_1^*) $d_i x \in E_i$ and by (E^*) $x \in E_i$ which completes the proof of (i).

In the same way one can prove (ii).

For (C^*) suppose the contrary : $d_i x \subset d_j y$ for some x, y and $i < j$. By (C_1^*) $d_i x \in E_i$ and by (E^*) $d_j y \in E_i$ which contradicts (C_2^*).

Using this theorem we can eliminate subsets E_i in the definition of Post-space and pseudo-Post space. Namely we have:

THEOREM 2.4. *Let* $\mathbf{S}$ *be a* D^**-space (without subsets* E_i*) and let us define*

(8) $$E_0 = \varnothing, \quad E_i = \{x \in S / d_i x \subset x\} \quad i = 1, \ldots, m-1.$$

Then: (i) $\mathbf{S}$ *is a pseudo-Post space if and only if the axioms* (R^*), (M^*), (L_α^*), (L_ω^*), (L^*) *and* (C^*) *are satisfied.*

(ii) $\mathbf{S}$ *is a Post space if and only if the axioms* (R^*), (M^*), (L_α^*), (L_ω^*), (L^*), (C^*) *and* (B^*) *are satisfied.*

PROOF. In one direction this follows from theorem 2.3. For the converse direction we have to verify (C_1^*), (C_2^*) and (E^*). For (C_1^*) let $i \leqslant j$. Then by (M^*) $d_j x \subset d_i x$ which by (R^*) is equivalent to $d_j d_i x \subset d_i x$. By (8) $d_i x \in E_j$. In the same way one can verify (C_2^*) and (E^*).

The proof of (ii) follows from (i).

THEOREM 2.5. *In any pseudo-Post (Post, quasi-Post) space the axiom* (J^*) *holds.*

PROOF. (For pseudo-Post and quasi-Post spaces). Let $d_i x \subset y$ and suppose $d_i y \not\subset y$. If $i = m-1$ then this contradicts (L_α^*). Let $i < m-1$. Then by (L^*) $y \subset d_{i+1} y$ and by the assumption we get $d_i x \subset d_{i+1} y$ which contradicts (C^*).

In case of quasi-Post spaces $d_i x \subset y$ implies $d_i y \subset d_i x$ which get $d_i y \subset y$.

INFERENCE. *Every pseudo-Post (quasi-Post, Post) space is a quasi--pseudo-Post space.*

THEOREM 2.6. *In any quasi-pseudo-Post space* $\mathbf{S}$ *the following holds:*

(i) *for any* $x \in S$ *there exists* i $(1 \leqslant i \leqslant m-1)$ *such that* $x = d_i x$;

(ii) $d_i x \subset y \rightarrow (\exists z \in S)(\exists_j)(x \subset z \ \&\ y = d_j z \ \&\ j \leqslant i)$.

PROOF. (i). Let $M_x = \{j \,/ 1 \leqslant j \leqslant m-1$ and $d_j x \subset x\}$. By (L_α^*) $m-1 \in M_x$, so $M_x \neq \varnothing$. Let i be the minimal element of M_x.

Case 1: $i = 1$. By assumption $d_1 x \subset x$, by (L_ω^*) $x \subset d_1 x$, so $d_1 x = x$.

Case 2: $i > 1$. Since $d_i x \subset x$ and since i is a minimal element of M_x $i-1 \notin M_x$ and $d_{i-1}\, x$ *non* $\subset x$. Then by (L^*) we get $x \subset d_i x$. Thus $d_i x = x$.

(ii). Suppose $d_i x \subset y$. By (i) there exist k and l such that $x = d_k x$ and $y = d_l y$. Put $z = d_k y$. Then $d_i x \subset y$ implies $x = d_k x = d_k d_i x$

$\subset d_k y = z$, so $x \subset z$. We have also $d_l z = d_l d_k y = d_l y = y$. If $l \leqslant i$ then put $j = l$ and the theorem holds. If $l > i$ then by (M^*) $y = d_l y \subset d_i y$. By (J^*) $d_i x \subset y$ implies $d_i y \subset y$. Hence $y = d_i y$. In this case put $j = i$ and the theorem also holds.

Now we turn to construct some examples of pseudo-Post and Post spaces. Let $\langle A, \leqslant \rangle$ be an ordered set. Define

$$S = \{\langle x, i\rangle \,/ x \in A,\ 1 = i = m-1\} \quad m \geqslant 2$$

$$d_i\langle x, j\rangle = \langle x, i\rangle \quad i = 1, \ldots, m-1$$

$$\langle x, i\rangle \subset \langle y, j\rangle \ \text{ iff } \ x \leqslant y \ \text{ and } \ j \leqslant i.$$

THEOREM 2.7. *The system* $\mathbf{S}$, *thus defined, is a pseudo-Post space of order* m, *and if the ordering* $\leqslant$ *is identity relation then the system is a Post space of order* m.

PROOF. Use theorem 2.4.

3. D-N-algebras of sets

THEOREM 3.1. *Let* $\mathbf{S} = \langle S, \subset, d_1, \ldots, d_{m-1}, E_0, \ldots, E_{m-1}\rangle$ $m \geqslant 2$ *be a* D^*-N^*-*space. A subset* A *of* S *is said to be open if it satisfies the condition*

(9) $\quad (\forall x, y \in S)(x \subset y \ \& \ x \in A \to y \in A)$

Let $P(S)$ *be the set of all open subsets of* S. *Define for any* $A, B \subseteq S$

(10) $\quad \bigvee = S, \bigwedge = \varnothing,$

(11) $\quad A \cup B = A \bigcup B = \{x \in S / x \in A \ \text{or} \ x \in B\},$

(12) $\quad A \cap B = A \bigcap B = \{x \in S / x \in A \ \text{and} \ x \in B\},$

(13) $\quad A \Rightarrow B = \{x \in S / (\forall y \in S)((x \subset y \ \& \ y \in A) \to y \in B),$

(14) $\quad \neg A = \{x \in S / (\forall y \in S)(x \subset y \to y \notin A),$

(15) $\quad D_i(A) = \{x \in S / d_i x \in A\} \quad i = 1, \ldots, m-1,$

(16) $\quad e_i = E_i \quad i = 0, \ldots, m-1.$

Then the system

(17) $\quad \mathfrak{P}(\mathbf{S}) = \langle P(S), \bigvee, \bigwedge, \cup, \cap, \Rightarrow, \neg, D_1, \ldots, D_{m-1}, e_0, \ldots, e_{m-1}\rangle$

with operations defined by (10), ..., (16) *is a D-N-algebra, where* N *is the set of axioms, corresponding to these of the set* N^*.

PROOF. It is well known that the set $P(S)$ is closed with respect to the operations defined by (10), ..., (14) and that the system $\langle P(S), \bigvee, \bigwedge, \cup, \cap, \Rightarrow, \neg\rangle$ is a pseudo-Boolean algebra (see [3]). By axiom (E^*) we have that $E_i \in P(S)$ $i = 1, \ldots, m-1$. Suppose $A \in P(S)$, $x \subset y$ and $x \in D_i(A)$. Then by (D^*) $d_i x \subset d_i y$, by (15) $d_i x \in A$ and by (9) $d_i y \in A$. Thus by (15) $y \in D_i(A)$ which shows that the set $P(S)$ is closed under the operations D_i. The proof that (17) is a D-algebra is left to the reader.

For the remaining part of the theorem suppose $X^* \in N^*$. As an example we shall take the case $X^* = B^*$. We have to prove (B): $D_i(A) \cup \neg D_i(A) = \bigvee = S$. Suppose $x \notin \neg D_i(A)$. Then by (14) there exists $y \in S$ such that $x \subset y$ and $y \in D_i(A)$. By (B^*) we have $d_i y \subset d_i x$, by (15) $d_i y \in A$ and since $A \in P(S)$ $d_i x \in A$. So by (15) $x \in D_i(A)$, which completes the proof.

For any D^*-N^*-space S we shall call the algebra $\mathfrak{P}(S)$ a *D-N-algebra of sets over* S.

THEOREM 3.2. *If S is a quasi-pseudo-Post (pseudo-Post, quasi-Post, Post) space then $\mathfrak{P}(S)$ is a quasi-pseudo-Post (pseudo-Post, quasi-Post, Post) algebra. Moreover in all four cases the following equations hold:*

(18) $D_i(A \Rightarrow B) = (D_1(A) \Rightarrow D_1(B)) \cap \ldots \cap (D_i(A) \Rightarrow D_i(B))$ $i = 1, \ldots, m-1$.

(19) $D_i(\neg A) = \neg D_1(A)$ $i = 1, \ldots, m-1$.

PROOF. The first part of the theorem follows from the theorem 3.1. For the second part suppose $x \notin D_i(A \Rightarrow B)$, i.e. $d_i x \notin A \Rightarrow B$. By (13) there exists y such that $d_i x \subset y$, $y \in A$ but $y \notin B$. By theorem 2.6. (ii) there exists $z \in S$ such that $x \subset z$ and $y = d_k z$ for some $k \leqslant i$. So $d_k z \in A$ and $d_k z \notin B$, i.e. $z \in D_k(A)$, $z \notin D_k(B)$. Since $x \subset z$ by (13) we have that $x \notin D_k(A) \Rightarrow D_k(B)$. Thus $x \notin (D_1(A) \Rightarrow D_1(B)) \cap \ldots \cap (D_i(A) \Rightarrow D_i(B))$. This shows that $(D_1(A) \Rightarrow D_1(B)) \cap \ldots \cap (D_i(A) \Rightarrow D_i(B)) \subseteq D_i(A \Rightarrow B)$. The convers inclusion is easy to prove. The condition (19) follows from (18) by putting $B = \varnothing$.

4. Forcing. Characterizations of D-N-algebras and D*-N*-spaces

Let $\mathfrak{P}$ be an algebra similar to D-algebra, S be a relational system similar to D^*-space and $\Vdash \subseteq S \times P$. Instead of $(x, a) \in \Vdash$ where $x \in S$ and $a \in P$ we shall write $x \Vdash a$ and read "x forces a". We shall say that the relation $\Vdash$ is a *forcing* from S to P if the following conditions are satisfied for any $x, x_1, x_2 \in S$ and $a, b \in P$:

(F1) $x \Vdash \bigvee$,

(F2) x *non* $\Vdash \bigwedge$,

(F3) $x \Vdash a \cap b$ iff $x \Vdash a$ and $x \Vdash b$,

(F4) $x \Vdash a \cup b$ iff $x \Vdash a$ or $x \Vdash b$,

(F5) $x \Vdash a \Rightarrow b$ iff $(\forall y \in S)((x \subset y \ \& \ y \Vdash a) \to y \Vdash b)$,

(F6) $x \Vdash \neg a$ iff $(\forall y \in S)(x \subset y \to y$ *non* $\Vdash a)$,

(F7) $x \Vdash D_i(a)$ iff $d_i x \Vdash a$, $i = 1, \ldots, m-1$,

(F8) $x \Vdash e_i$ iff $x \in E_i$, $i = 0, \ldots, m-1$,

(F9) $(x_1 \subset x_2) \to (\forall a \in P)(x_1 \Vdash a \to x_2 \Vdash a)$.

The forcing $\Vdash$ is said to be *r-strong* if it saitsfies the condition

(Fr) $(\forall z \in S)(z \Vdash a \equiv z \Vdash b) \to (a = b)$.

The forcing $\Vdash$ is said to be *l-strong* if it satisfies the conditions

(Fl1) $(\forall a \in P)(x_1 \Vdash a \to x_2 \Vdash a) \to (x_1 \subset x_2)$,

(Fl2) $(\forall a \in P)(x_1 \Vdash a \equiv x_2 \Vdash a) \to (x_1 = x_2)$.

We say that $\Vdash$ is a *strong forcing* if it is simultaneously r-strong and l-strong one.

THEOREM 4.1. *Let $\mathfrak{P}$ be an algebra similar to D-algebra, $\boldsymbol{S}$ be a relational system similar to D*-space, and $\Vdash$ a forcing from S to P. Then the following holds:*

(i) *If $\Vdash$ is a l-strong forcing then $\boldsymbol{S}$ is a D*-space.*

(ii) *If $\Vdash$ is a r-strong forcing then $\mathfrak{P}$ is a D-algebra.*

(iii) *If $\Vdash$ is a strong forcing then the following two conditions are equivalent:*

(iii1) *$\boldsymbol{S}$ is a D^*-N^*-space,*

(iii2) *$\mathfrak{P}$ is a D-N-algebra.*

PROOF. (i). It follows from (Fl1) and (Fl2) that the relation $\subset$ is an ordering in S. Suppose $x \subset y$, $a \in P$ and $d_i x \Vdash a$. Then by (F7) $x \Vdash D_i(a)$, by (F9) $y \Vdash D_i(a)$ and again by (F7) $d_i y \Vdash a$. Since a is an arbitrary element of P then by (Fl1) $d_i x \subset d_i y$ and axiom (D*) holds. Suppose now $x \in E_i$ and $x \subset y$. Then by (F8) $x \Vdash e_i$ and again by (F8) $y \in E_i$. Thus (E*) is fulfilled and $\boldsymbol{S}$ is a D*-space.

(ii). First we shall prove that $\mathfrak{P}$ is a pseudo-Boolean algebra. We shall use the following axioms for pseudo-Boolean algebra [4]:

(dl1) $a \cap (a \cup b) = a$, (dl2) $a \cap (b \cup c) = (c \cap a) \cup (b \cap a)$,

(pb1) $(a \Rightarrow a) \cap b = b$, (pb2) $a \cap (a \Rightarrow b) = a \cap b$,

(pb3) $(a \Rightarrow b) \cap b = b$, (pb4) $(a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \cap c)$,

(pb5) $a \Rightarrow \neg b = b \Rightarrow \neg a$, (pb6) $\neg (a \Rightarrow a) \cup b = b$.

As an example we shall verify (pb2). Let $x \in S$ and $x \Vdash a \cap (a \Rightarrow b)$. Then by (F3) $x \Vdash a$ and $x \Vdash a \Rightarrow b$. Since $x \subset x$ then by (F5) $x \Vdash b$ and by (F3) $x \Vdash a \cap b$. For the converse implication suppose $x \Vdash a \cap b$ and $x \subset y$. By (F3) we have $x \Vdash a$, $x \Vdash b$ and by (F9) $y \Vdash b$. Then by (F5) $x \Vdash a \Rightarrow b$ and by (F3) $x \Vdash a \cap (a \Rightarrow b)$. Thus $x \Vdash a \cap (a \Rightarrow b)$ iff $x \Rightarrow a \cap b$ and by (Fr) $a \cap (a \Rightarrow b) = a \cap b$.

In the same way one can verify the axioms (D$'$), (D$''$), (D$_0'$) and (D$_0''$). Hence $\mathfrak{P}$ is a D-algebra.

(iii). (iii1)$\to$(iii2). Suppose $\boldsymbol{S}$ is a D*-N*-space and let X$^* \in$ N*. We have to prove that X $\in$ N. As an example let X* = J*, so we have to prove that (J): $D_i(D_i(a) \Rightarrow a) = \bigvee$ holds. Suppose $D_i(D_i(a) \Rightarrow a) \neq \bigvee$ for some $a \in P$. Then by (Fr) and (F1) x *non* $\Vdash D_i(D_i(a) \Rightarrow a)$ for some $x \in S$. So by (F7) $d_i x$ *non* $\Vdash D_i(a) \Rightarrow a$ and by (F5) there exists $y \in S$ such that $d_i x \subset y$,

$y \Vdash D_i(a)$ but y *non* $\Vdash a$. By (F7) $d_i y \Vdash a$ and by (J^*) $d_i y \subset y$. Applying (F9) we obtain $y \Vdash a$ contrary to y *non* $\Vdash a$. Thus (J) holds and by (ii) $\mathfrak{P}$ is a D-N-algebra.

(iii2)$\to$(iii1). Suppose $\mathfrak{P}$ is a D-N-algebra. We shall treat of only the case (J)$\in$N. We have to prove that (J^*) holds in $\boldsymbol{S}$. Since $D_i(D_i(a) \Rightarrow a) = \bigvee$ by (F1) and (F7) we have $d_i x \Vdash D_i(a) \Rightarrow a$ for any $x \in S$. Suppose now $d_i x \subset y$ and $d_i y \Vdash a$. By (F9) and (F7) we have $y \Vdash D_i(a) \Rightarrow a$ and $y \Vdash D_i(a)$. Since $y \subset y$ then by (F5) $y \Vdash a$. Thus $d_i x \subset y$ implies: for any $a \in P$ if $d_i y \Vdash a$ then $y \Vdash a$. By (F11) this gives: "if $d_i x \subset y$ then $d_i y \subset y$" which is (J^*). This completes the proof. The proof for other axioms is analogous.

THEOREM 4.2. *Let $\mathfrak{P}$ be an algebra similar to D-algebra. Then the following two conditions are equivalent:*

(i) *$\mathfrak{P}$ is a D-N-algebra*
(ii) *There exists a D^*-N^*-space $\boldsymbol{S}$ and a strong forcing $\Vdash$ from S to P.*

PROOF. (i)$\to$(ii). Take $\boldsymbol{S}$ to be the D^*-N^*-space of sets $\boldsymbol{S}(P)$ over $\mathfrak{P}$ defined in theorem 2.1, where $S(P)$ was the set of all prime filters in $\mathfrak{P}$. Define for any $\nabla \in S(P)$ and $a \in P$: $\nabla \Vdash a$ iff $a \in \nabla$. It is easy to see, using lemma 2.2 that this relation is a strong forcing from $S(P)$ to P.

The proof of (ii)$\to$(i) follows from theorem 4.1.

THEOREM 4.3. *Let $\boldsymbol{S}$ be a system similar to D^*-space. Then the following two conditions are equivalent:*

(i) *$\boldsymbol{S}$ is D^*-N^*-space.*
(ii) *There exist a D-N-algebra $\mathfrak{P}$ and a strong forcing $\Vdash$ from S to P.*

PROOF. (i)$\to$(ii) Take $\mathfrak{P}$ to be the D-N-algebra of sets $\mathfrak{P}(\boldsymbol{S})$ over $\boldsymbol{S}$, defined in theorem 3.1, where $P(S)$ was the set of all open subsets of S. Define for any $x \in S$ and $A \in P(S)$: $x \Vdash A$ iff $x \in A$. Then it is easily seen that $\Vdash$ is a strong forcing from S to $P(S)$. The proof of (ii)$\to$(i) follows from theorem 4.1.

Theorems 4.2 and 4.3 can be formulated without using the notion of forcing. Namely we have:

THEOREM 4.4. (Prime filters characterization of D-N-algebras). *Let $\mathfrak{P}$ be an algebra similar to D-algebra. Then the following two conditions are equivalent:*

(i) *$\mathfrak{P}$ is a D-N-algebra.*
(ii) *There exists a system $\langle S(P), \subseteq, d_1, \ldots, d_{m-1}, E_o(P), \ldots, E_{m-1}(P)\rangle$ similar to D^*-space and satisfying the following conditions:*
(ii1) *$S(P)$ is a non empty set of subsets of P,*
(ii2) *each condition of the set N^* (corresponding to N) is satisfied,*
(ii3) *for any ∇, ∇_1 $\nabla_2 \in S(P)$ and $a, b \in P$ the following hold:*

(f1) $\bigvee \in \nabla$

(f2) $\bigwedge \notin \nabla$

(f3) $a \cap b \in \nabla$ iff $a \in \nabla$ and $b \in \nabla$

(f4) $a \cup b \in \nabla$ iff $a \in \nabla$ or $b \in \nabla$

(f5) $a \Rightarrow b \in \nabla$ iff $(\forall \nabla' \in S(P))((\nabla \subseteq \nabla' \ \& \ a \in \nabla') \rightarrow b \in \nabla')$

(f6) $\neg a \in \nabla$ iff $(\forall \nabla' \in S(P))(\nabla \subseteq \nabla' \rightarrow a \notin \nabla')$

(f7) $D_i(a) \in \nabla$ iff $a \in d_i \nabla, \quad i = 1, \ldots, m-1$

(f8) $e_i \in \nabla$ iff $\nabla \in E_i, \quad i = 0, \ldots, m-1$

(f9) $(\nabla_1 \subseteq \nabla_2) \rightarrow (\forall a \in P)(a \in \nabla_1 \rightarrow a \in \nabla_2)$

(fr) $(\forall \nabla' \in S(P))(a \in \nabla' \equiv b \in \nabla') \rightarrow (a = b)$

(f11) $(\forall a \in P)(a \in \nabla_1 \rightarrow a \in \nabla_2) \rightarrow (\nabla_1 \subseteq \nabla_2)$

(f12) $(\forall a \in P)(a \in \nabla_1 \equiv a \in \nabla_2) \rightarrow (\nabla_1 = \nabla_2)$.

PROOF. (ii)$\rightarrow$(i). Suppose (ii) holds. Then for any $\nabla \in S(P)$ and $a \in P$ define: $\nabla \Vdash a$ iff $a \in \nabla$. Then by (f1), ..., (f12) we obtain that $\Vdash$ is a strong forcing from $S(P)$ to P. By (ii2) and theorem 4.1. (i) $\mathbf{S}(\mathfrak{P})$ is a D*-N*-space, and by theorem 4.1. (iii) $\mathfrak{P}$ is a D-N-algebra. It is easy to see from (f1)-(f4) that the set $S(P)$ is the set of all prime filters in $\mathfrak{P}$, from (f9) and (f11) that the relation $\subseteq$ is the set-theoretical inclusion, and that (f7) and (f8) coincide with the definitions (5) and (6) respectively.

(i)$\rightarrow$(ii). Suppose now $\mathfrak{P}$ is a D-N-algebra and take the system $\mathbf{S}(\mathfrak{P})$ to be the D*-N*-space of sets over $\mathfrak{P}$ as in theorem 2.1, where $S(P)$ was the set of all prime filters in $\mathfrak{P}$. Then the condition (ii2) is fulfilled. Since the elements of $S(P)$ are prime filters we have (f1)-(f4) and by lemma 2.2 (f5), (f6) and (fr.) Conditions (f7) and (f8) are true by (5) and (6), and since $\subseteq$ is the set-theoretical inclusion we have the conditions (f9), (f11) and (f12).

THEOREM 4.5. (Open sets characterization of D*-N*-spaces). *Let* $\mathbf{S}$ *be a system similar to D***-space. Then the following two conditions are equivalent*:

(i) $\mathbf{S}$ *is a D***-N***-space.*

(ii) *There exists an algebra* $\langle P(S), \bigvee, \bigwedge, \cup, \cap, \Rightarrow, \neg, D_1, \ldots, D_{m-1}, e_0, \ldots, e_{m-1} \rangle$ *similar to D-algebra and satisfying the following conditions*:

(ii1) $P(S)$ *is a set of subsets of* S,

(ii2) *each condition of the set* N *(corresponding to* N**)*

is satisfied,

(ii3) *for any* $x, x_1, x_2 \in S$ *and* $A, B \in P(S)$ *the following hold*:

(φ1) $x \in \bigvee$,

(φ2) $x \notin \bigwedge$,

(φ3) $x \in A \cap B$ iff $x \in A$ and $x \in B$,

(φ4) $x \in A \cup B$ iff $x \in A$ or $x \in B$,

($\varphi 5$) $x \in A \Rightarrow B$ iff $(\forall y \in S)((x \subset y \ \& \ y \in A) \to y \in B)$,
($\varphi 6$) $x \in \neg A$ iff $(\forall y \in S)(x \subset y \to y \notin A)$,
($\varphi 7$) $x \in D_i(A)$ iff $d_i x \in A$ $i = 1, \ldots, m-1$,
($\varphi 8$) $x \in e_i$ iff $x \in E_i$ $i = 0, \ldots, m-1$,
($\varphi 9$) $(x_1 \subset x_2)\,(\forall A \in P(S))(x_1 \in A \to x_2 \in A)$,
($\varphi 11$) $(\forall A \in P(S))(x_1 \in A \to x_2 \in A) \to (x_1 \subset x_2)$,
($\varphi 12$) $(\forall A \in P(S))(x_1 \in A \equiv x_2 \in A) \to (x_1 = x_2)$.

PROOF. (ii)$\to$(i). Suppose (ii). Then the following is also true:

(φr) $(\forall x \in S)(x \in A \equiv x \in B) \to (A = B)$

Define for any $x \in S$ and $A \in P(S)$: $x \Vdash A$ iff $x \in A$. Then by ($\varphi 1$), ..., ($\varphi 9$), ($\varphi 11$), ($\varphi 12$) and (φr) we obtain that $\Vdash$ is a strong forcing from S to $P(S)$. By (ii2) and theorem 4.1. (ii) $\mathfrak{P}(\boldsymbol{S})$ is a D-N-algebra and by theorem 4.1. (iii) $\boldsymbol{S}$ is a D*-N*-space.

(i)$\to$(ii). Suppose $\boldsymbol{S}$ is a D*-N*-space. Take the algebra $\mathfrak{P}(\boldsymbol{S})$ to be the D-N-algebra of sets over S, as in theorem 3.1 where $P(S)$ was the set of all open subsets of S in the sense of (9), and operations, defined by (10), ..., (16). Then (ii2) is fulfilled. Conditions ($\varphi 1$), ..., ($\varphi 8$) are other notations of (10), ..., (16). ($\varphi 9$) is true because A is an open subset of S. To prove ($\varphi 11$) suppose x_1 *non* $\subset x_2$ and let $A = \{y \in S / x_1 \subset y\}$. Obviously A is an open set and $x_1 \in A$ but $x_2 \notin A$. Then by contraposition we get ($\varphi 11$). Since $\subset$ is antisimmetric relation then ($\varphi 12$) follows from ($\varphi 11$). This completes the proof of the theorem.

THEOREM 4.6. *In any pseudo-Post (quasi-Post, Post) algebra $\mathfrak{P}$ the axiom* (J) *holds.*

PROOF. By theorem 4.2 there exists a pseudo-Post (quasi-Post, Post) space $\boldsymbol{S}$ and a strong forcing from S to P. By theorem 2.5 the condition (J*) is fulfilled in $\boldsymbol{S}$ and by teorem 4.1. (iii) (J) holds in $\mathfrak{P}$.

THEOREM 4.7. *Let $\mathfrak{P}, \mathfrak{P}'$ be two algebras similar to D-algebra,* $\boldsymbol{S}$ *be a relational system similar to D*-space, h be a homomorhism from $\mathfrak{P}$ into $\mathfrak{P}'$ and $\Vdash_1$ be a forcing from S to P'. Define for any $x \in S$ and $a \in P$: $x \Vdash a$ iff $x \Vdash_1 h(a)$. Then*

(i) *$\Vdash$ is a forcing from S to P,*
(ii) *for any $a \in P$: $h(a) = \bigvee$ iff $x \Vdash a$ for any $x \in S$.*

The easy proof is left to the reader.

5. Set-theoretical representations for D-N-algebras and D*-N*-spaces

THEOREM 5.1 *Let $\mathfrak{P}$ be an algebra similar to D-algebra,* $\boldsymbol{S}$ *be a D*-N*-space and $\Vdash$ a forcing from S to P. Define h and h^* as follows:*

(20) $h(a) = \{x \in S / x \Vdash a\}$ *for any* $a \in P$,
(21) $h^*(x) = \{a \in P / x \Vdash a\}$ *for any* $x \in S$.

Then the following hold:

(i) *h is a homomorphism from $\mathfrak{P}$ into the D-N-algebra of sets $\mathfrak{P}(\boldsymbol{S})$ over $\boldsymbol{S}$.*

If $\Vdash$ is a strong forcing then

(ii) *$\mathfrak{P}$ is a D-N-algebra and h is an isomorphism from $\mathfrak{P}$ into the D-N-algebra of sets $\mathfrak{P}(\boldsymbol{S})$ over $\boldsymbol{S}$.*

(iii) *h^* is an isomorphism from $\boldsymbol{S}$ into the D^*-N^*space of sets $\boldsymbol{S}(\mathfrak{P})$ over $\mathfrak{P}$.*

PROOF. (i) Let $x \in h(a)$ and $x \subset y$. Then $x \Vdash a$ and by (F9) $y \Vdash a$. Thus $y \in h(a)$ and $h(a)$ is an open subset of S. We shall prove that h preserves the operations of P. As an example we shall take only the case of $\Rightarrow$. We have: $h(a \Rightarrow b) = \{x \in S / x \Vdash a \Rightarrow b\} = \{x \in S / (\forall y \in S)\big((x \subset y \;\&\; y \Vdash a) \rightarrow y \Vdash b\big) = \big\{x \in S / (\forall y \in S)\big((x \subset y \;\&\; y \in h(a)\big) \rightarrow y \in h(b)\big)\big\} = h(a) \Rightarrow h(b)$. We have used (20), (F5), (20) and (13).

(ii). Suppose now that $\Vdash$ is a r-strong forcing. Then by theorem 4.1 $\mathfrak{P}$ is a D-N-algebra. If $h(a) = h(b)$ then for any $z \in S$: $z \in h(a)$ if and only if $z \in h(b)$, and by (20) for any $z \in S$: $z \Vdash a$ if and only if $z \Vdash b$. Consequently, by (Fr) $a = b$ and h is an isomorphism from $\mathfrak{P}$ into $\mathfrak{P}(\boldsymbol{S})$.

(iii). First we shall prove that for any $x \in Sh^*(x)$ is a prime filter in $\mathfrak{P}$. By (F1) and (F2) $h^*(x)$ is a non-empty proper subset of P. Further by (21), (F3) and (21) we have:

$$a \cap b \in h^*(x) \equiv x \Vdash a \cap b \equiv x \Vdash a \quad \textit{and} \quad x \Vdash b \equiv a \in h^*(x) \quad \textit{and} \quad b \in h^*(x).$$

Thus $h^*(x)$ is a filter. In the same way it can be proved, using (F4), that $h^*(x)$ is a prime filter. So $h^*(x) \in S(P)$. It remains to prove the following:

(iii1) $x \subset y$ *iff* $h^*(x) \subseteq h^*(y)$,
(iii2) $h^*(d_i x) = d_i h^*(x)$ $i = 1, \ldots, m-1$,
(iii3) $x \in E_i$ *iff* $h^*(x) \in E_i(P)$ $i = 0, \ldots, m-1$,
(iii4) if $h^*(x) = h^*(y)$ *then* $x = y$.

Condition (iii1) follows from (F9) and (F11). For (iii2) and (iii3) it have to be used (F7) and (5), and (F8) and (6) respectively. Condition (iii4) follows from (F12).

THEOREM. 5.2. *For any D-N-algebra $\mathfrak{P}$ there exists a D^*-N^*-space $\boldsymbol{S}$ and an isomorphism h from $\mathfrak{P}$ into the D-N-algebra of sets $\mathfrak{P}(\boldsymbol{S})$ over $\boldsymbol{S}$.*

PROOF. By theorem 4.2 there exist a D^*-N^*-space $\boldsymbol{S}$ and a strong forcing $\Vdash$ from S to P. Define h by (20). Then by theorem 5.1 (ii) h is the required isomorphism.

THEOREM. 5.3. *For any D^*-N^*-space $\boldsymbol{S}$ there exists a D-N-algebra $\mathfrak{P}$ and an isomorphism h^* from $\boldsymbol{S}$ into the D^*-N^*-space of sets $\boldsymbol{S}(\mathfrak{P})$ over $\mathfrak{P}$.*

PROOF. By theorem 4.3 there exist a D-N-algebra $\mathfrak{P}$ and a strong forcing $\Vdash$ from S to P. Define h^* by (21). Then by theorem 5.1. (iii) h^* is the required isomorphism.

THEOREM 5.4. *In any quasi-pseudo-Post (pseudo-Post, quasi-Post, Post) algebra the following equations hold:*

$$(22)\qquad D_i(a \Rightarrow b) = \big(D_1(a) \Rightarrow D_1(b)\big) \cap \ldots \cap \big(D_i(a) \Rightarrow D_i(b)\big) \quad i = 1, \ldots, m\text{-}1,$$
$$(23)\qquad D_i(\neg a) = \neg D_1(a) \quad i = 1, \ldots, m-1.$$

PROOF. By theorem 3.2 and theorem 5.2.

6. McKinsey-type embedding theorem for D-N-algebras

THEOREM 6.1. *Let* $\mathfrak{P} = \langle P, \bigvee, \bigwedge, \cup, \cap, \Rightarrow, \neg, D_1, \ldots, D_{m-1}, e_0, \ldots$ $\ldots, e_{m-1}\rangle$ $m \geqslant 2$ *be a D-N-algebra for which axiom* R $\in$ N *but* J $\notin$ N, *and let* $\{a_1, \ldots, a_n\}$ *be a finite subset of* P. *Then there exists a finite D-N-algebra* $\mathfrak{P}' = \langle P', \bigvee', \bigwedge', \cup', \cap', \Rightarrow', \neg', D_1', \ldots, D_{m-1}', e_0', \ldots, e_{m-1}'\rangle$ *with the following properties:*

(i) P' *contains at most* $2^{(m+n).m}$ *elements,*
(ii) $\{a_1, \ldots, a_n\} \subseteq P' \subseteq P$,
(iii) $\bigvee' = \bigvee$, $\bigwedge' = \bigwedge$, $e_i' = e_i$ $\quad i = 0, \ldots, m-1$,
(iv) *if* $a_i \cap a_j = a_k$ *then* $a_i \cap' a_j = a_k$
(v) *if* $a_i \cup a_j = a_k$ *then* $a_i \cup' a_j = a_k$
(vi) *if* $a_i \Rightarrow a_j = a_k$ *then* $a_i \Rightarrow' a_j = a_k$
(vii) *if* $\neg a_i = a_k$ *then* $\neg' a_i = a_k$
(viii) *if* $D_i(a_j) = a_k$ *then* $D_i'(a_j) = a_k$ $\quad i = 1, \ldots, m-1$.

PROOF. Let $P_0 = \{e_0, \ldots, e_{m-1}, a_1, \ldots, a_n, D_1(e_0), \ldots, D_1(e_{m-1}),$ $D_1(a_1), \ldots, D_1(a_n), \ldots, D_{m-1}(e_0), \ldots, D_{m-1}(e_{m-1}), D_{m-1}(a_1), \ldots, D_{m-1}(a_n)\}$. This set has at most $(m+n)\cdot m$ elements. Let $\mathfrak{P}'$ be the sublattice of $\mathfrak{P}$ generated by P_0 and containing $\bigvee$ and $\bigwedge$. Then P' contains no more than $2^{2^{(m+n).m}}$ elements and conditions (i), ..., (v) are satisfied. It is easy to see by induction and using axiom (R) that P' is closed under the operations D_i. Thus (viii) is fulfilled and the axioms (D$'$), (D$''$), (D$_0'$), (D$_0''$) and all those of the set N are satisfied. Since $\mathfrak{P}$ is a distributive lattice so is $\mathfrak{P}'$. It is well known that every finite distributive lattice is a pseudo-Boolean algebra with relatively-pseudo-complement $\Rightarrow'$ defined as follows:

$$(24)\qquad a \Rightarrow' b = \max\{c\colon c \cap a \leqslant b \ \&\ c \in P'\}.$$

It follows from (24) that $a \Rightarrow' b \leqslant a \Rightarrow b$ for any $a, b \in P$. Suppose now that $a, b, a \Rightarrow b \in P'$. Since $(a \Rightarrow b) \cap a \leqslant b$ then by (24) $a \Rightarrow b \leqslant a \Rightarrow' b$. Thus for $a, b, a \Rightarrow b \in P'$ $a \Rightarrow' b = a \Rightarrow b$ and (vi)is fulfilled. Putting $\neg a = a \Rightarrow \bigwedge$ we infer that (vii) follows from (vi). This completes the proof of the theorem.

We shall use later this theorem to prove that some logical calculi have the finite model property and are decidable. Since axiom (J) $\notin$ N this theorem does not cover the case of pseudo-Post and quasi-pseudo-Post algebras. In the next section we prove some theorems for these algebras, which theorems also give some decidability results for the corresponding logical systems.

7. Four special algebras

In this section we shall give another tipical example of quasi-pseudo-Post (quasi-Post, pseudo-Post, Post) algebra. In case of Post algebras the construction is known (see [4], [5]).

THEOREM 7.1. *Let* $\mathfrak{A}$ *be a pseudo-Boolean (Boolean) algebra and* $P[A] = \{(a_1, \ldots, a_{m-1})/a_1, \ldots, a_{m-1} \in A,\ a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{m-1}\}$. *Define for any* $(a_1, \ldots, a_{m-1}),\ (b_1, \ldots, b_{m-1}) \in A^{m-1}$.

(25) $\quad \bigvee = (\bigvee, \bigvee, \ldots, \bigvee)\ \bigwedge = (\bigwedge, \bigwedge, \ldots, \bigwedge)$

(26) $\quad (a_1, \ldots, a_{m-1}) \cup (b_1, \ldots, b_{m-1}) = (a_1 \cup b_1, \ldots, a_{m-1} \cup b_{m-1})$

(27) $\quad (a_1, \ldots, a_{m-1}) \cap (b_1, \ldots, b_{m-1}) = (a_1 \cap b_1, \ldots, a_{m-1} \cap b_{m-1})$

(28) $\quad (a_1, \ldots, a_{m-1}) \Rightarrow (b_1, \ldots, b_{m-1}) = \big(a_1 \Rightarrow b_1, (a_1 \Rightarrow b_1) \cap (a_2 \Rightarrow b_2), \ldots$
$\quad \ldots, (a_1 \Rightarrow b_1) \cap (a_2 \Rightarrow b_2) \cap \ldots \cap (a_{m-1} \Rightarrow b_{m-1})\big)$

(29) $\quad \neg(a_1, \ldots, a_{m-1}) = (\neg a_1, \neg a_1, \ldots, \neg a_1)$

(30) $\quad D_i(a_1, \ldots, a_{m-1}) = (a_i, a_i, \ldots, a_i) \quad i = 1, \ldots, m-1$

(31) $\quad e_i = (\underbrace{\bigvee, \ldots, \bigvee}_{i-times}, \bigwedge, \ldots, \bigwedge) \quad i = 0, \ldots, m-1$

Then the system

(32) $\quad \mathfrak{P}[A] = \langle P[A], \bigvee, \bigwedge, \cup, \cap, \Rightarrow, \neg, D_1, \ldots D_{m-1}\rangle$

with operations defined by (25), ..., (30) *is a quasi-speudo-Post (quasi-Post) algebra and if we add the constants* e_i *defined by* (31) *then it turns to a pseudo-Post (Post) algebra.*

PROOF. By an easy verification.

THEOREM 7.2. *For any quasi-pseudo-Post (quasi-Post) algebra* $\mathfrak{P}$ *there exists a pseudo-Boolean (Boolean) algebra* $\mathfrak{A}$ *and an isomorphism* h *from* $\mathfrak{P}$ *into the quasi-pseudo-Post (quasi-Post) algebra* $\mathfrak{P}[A]$ *of the type* (32). *Any quasi-pseudo-Post (quasi-Post) algebra can be embedded into a pseudo-Post (Post)algebra.*

PROOF. Put $A = \{D_i(a)/a \in P,\ i = 1, \ldots, m-1\}$, and $h(a) = \big(D_1(a), \ldots$ $\ldots, D_{m-1}(a)\big)$. First it has to be proved that $\mathfrak{A}$ is a pseudo-Boolean (Boolean) subalgebra of $\mathfrak{P}$. The non-trivial case is the case of $\Rightarrow$. By theorem 5.4 in both cases the following equation holds

(33) $\quad D_i(a \Rightarrow b) = \big(D_1(a) \Rightarrow D_1(b)\big) \cap \ldots \cap \big(D_i(a) \Rightarrow D_i(b)\big), \quad i = 1, \ldots, m-1$

By (33) and (R) we have $D_i(D_j(a) \Rightarrow D_k(b)) = (D_1(Dj(a)) \Rightarrow D_1(D_k(b))) \cap \ldots \cap (D_i(D_j(a) \Rightarrow D_i(D_k(b)))) = (D_j(a) \Rightarrow D_k(b)) \cap \ldots \cap (D_j(a) \Rightarrow D_k(b)) = D_j(a) \Rightarrow D_k(b)$. So $D_j(a) \Rightarrow D_k(b) \in A$.

The proof that h is a homomorphism is easy. Let us prove that h is an isomorphism. First we shall prove the following:

(34) *If* $D_i(a) = D_i(b)$ *for* $i = 1, \ldots, m-1$ then $a = b$.

Suppose $D_i(a) = D_i(b)$ for $i = 1, \ldots, m-1$. Then $D_i(a) \Rightarrow D_i(b) = \bigvee$ for $i = 1, \ldots, m-1$ and by (33) $D_{m-1}(a \Rightarrow b) = \bigvee$. By (L_a) $D_{m-1}(a \Rightarrow b) \leqslant a \Rightarrow b$. So $a \Rightarrow b = \bigvee$ and $a \leqslant b$. From this it is easy to conclude (34).

Suppose now $h(a) = h(b)$. Then $(D_1(a), \ldots, D_{m-1}(a)) = (D_1(b), \ldots, D_{m-1}(b))$ and by (34) $a = b$. Hence h is an isomorphism from $\mathfrak{P}$ into the quasipseudo-Post algebra $\mathfrak{P}[A]$. If $\mathfrak{P}$ is a quasi-Post algebra then by axiom (B) we obtain that $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{P}[A]$ a quasi-Post algebra. The remaining part of the theorem follows from the fact that $\mathfrak{P}[A]$ is itself a pseudo-Post (Post) algebra.

§ 8. Some propositional calculi based on D-N-algebras and D*-N*-spaces

In this section we introduce a class of propositional calculi named here DNPC. As a special case we obtain the propositional calculi of the classical and intuitionistic many-valued logics, introduced by Rousseau [7, 8]. The main purpose is to give the notions of algebraic and relational validity and to prove its equivalence.

Any of the DNPC is based on a language containing the following symbols:

(i) an infinite set $V = \{p, q, r, \ldots\}$ of propositional variables,

(ii) $\{\bigwedge, \bigvee, e_0, \ldots, e_{m-1}\}$ the set of propositional constants. $\bigvee$ and $\bigwedge$ are called truth and false respectively,

(iii) $\neg, D_1, \ldots, D_{m-1}$ – the set of one-argument connectives; $\neg$ is the sign of the negation,

(iv) $\cap, \cup, \Rightarrow$ – the set of two-argument connectives – conjunction, disjunction, and implication,

(v) ( ) – parentheses.

Note that we shall assume also languages without the symbols $e_0, \ldots, e_{m-1}$.

The notion of a formula is the usual one. The set of all formulas is denoted by F and the algebra of formulas $\langle F, \vee, \wedge, \cup, \cap, \Rightarrow, \neg, D_1, \ldots, D_{m-1}, e_0, \ldots, e_{m-1}\rangle$ will be denoted by $\mathfrak{F}$. It is clear that $\mathfrak{F}$ is similar to D-algebra of order m.

A pair $\mathfrak{A} = \langle \mathfrak{P}, h\rangle$ is said to be an *algebraic D-N-model structure* (briefly D-N-a.m.s.) if $\mathfrak{P}$ is a D-N-algebra and h is a homomorphism from the algebra of formulas $\mathfrak{F}$ into $\mathfrak{P}$. $\mathfrak{A}$ is said to be an *algebraic D-N-model for a formula* a, or a is algebraically valid in $\mathfrak{A}$ if $h(a) = \bigvee$. $\mathfrak{A}$ is a *model for a set of formulas* A if it is a model for any formula a of A. A formula a is said to be an *algebraic D-N-tautology* if it is valid in any D-N-a.m.s.

A pair $\mathfrak{G} = \langle \boldsymbol{S}, \Vdash \rangle$ is said to be a *relational D-N-model structure* (briefly D-N-r.m.s.) if $\boldsymbol{S}$ is a D*-N*-space and $\Vdash$ a forcing from S to the algebra of formulas F. $\mathfrak{G}$ is said to be a *relational D-N-model for a formula* a, or a is *relationally valid in* $\mathfrak{G}$ if for any $x \in S$ $x \Vdash a$. $\mathfrak{G}$ is a *model for a set of formulas* A if it is a model for any formula a of A. A formula a is said to be a *relational D-N-tautology* if it is valid in any D-N-r.m.s.

THEOREM 8.1. *For any formula a and set of formulas A the following two conditions are equivalent:*

(i) *any algebraic D-N-model for A is a model for a;*

(ii) *any relational D-N-model for A is a model for a.*

PROOF. (i)→(ii). Suppose not (ii). Then there exist a D-N-r.m.s. $\mathfrak{G} = \langle \boldsymbol{S}, \Vdash \rangle$ such that $\mathfrak{G}$ is a model for A but not for a. Let $\mathfrak{P}(\boldsymbol{S})$ be the D-N-algebra of sets over $\boldsymbol{S}$ and let $h(\beta) = \{x \in S / x \Vdash \beta\}$ for any $\beta \in F$. By theorem 5.1. (i) h is a homomorphism from $\mathfrak{F}$ into $\mathfrak{P}(\boldsymbol{S})$, so the pair $\mathfrak{A} = \langle \mathfrak{P}(\boldsymbol{S}), h \rangle$ is a D-N-a.m.s. It is clear that $\mathfrak{A}$ is an algebraic D-N-model for A but not for a. Thus, by contraposition (i)→(ii).

(ii)→(i). Suppose now not (i). Then there exists a D-N-a.m.s. $\mathfrak{A} = \langle \mathfrak{P}, h \rangle$ such that $\mathfrak{A}$ is a model for A but not for a, i.e. $h(a) \neq \bigvee$. Then by theorem 4.2. there exist a D-N-space $\boldsymbol{S}$ and a forcing $\Vdash_1$ from S to P. Define for $x \in S$ and $\beta \in F$: $x \Vdash \beta$ iff $x \Vdash_1 h(\beta)$. By theorem 4.7. (i) $\Vdash$ is a forcing from S to F. So the pair $\mathfrak{G} = \langle \boldsymbol{S}, \Vdash \rangle$ is a D-N-r.m.s. It follows from theorem 4.7. (ii) that $\mathfrak{G}$ is a relational D-N-model for A but not for a. Hence by contraposition (ii)→(i).

THEOREM 8.2. *For any formula a the following conditions are equivalent:*

(i) *a is an algebraic D-N-tautology;*

(ii) *a is a relational D-N-tautology.*

PROOF. By theorem 8.1. with empty A.

Now we shall axiomatize the set of all D-N-tautologies. As axioms we shall use some of the following formulas and scheems:

(t) $\bigvee$

(f) $\neg \bigwedge$

(D′) $D_i(a \cup \beta) \Leftrightarrow D_i(a) \cup D_i(\beta)$ $\quad i = 1, \ldots, m-1$

(D″) $D_i(a \cap \beta) \Leftrightarrow D_i(a) \cap D_i(\beta)$ $\quad i = 1, \ldots, m-1$

(D_0') $\neg D_i(\bigwedge)$, $\quad$ (D_0'') $D_i(\bigvee)$ $\quad i = 1, \ldots, m-1$

(R) $D_i(D_j(a)) \Leftrightarrow D_j(a)$ $\quad i, j = 1, \ldots, m-1$

(M)	$D_{i+1}(a) \Rightarrow D_i(a)$	$i = 1, \ldots, m-1$	
(L_ω)	$a \Rightarrow D_1(a)$		
(L_α)	$D_{m-1}(a) \Rightarrow a$		
(L)	$(a \cap D_i(\beta)) \Rightarrow (\beta \cup D_{i+1}(a))$		$i = 1, \ldots, m-2$
(C_1)	$D_i(e_j) \quad i \leqslant j$	$i = 1, \ldots, m-1$	$j = 0, \ldots, m-1$
(C_2)	$\neg D_i(e_j) \quad i > j$	$i = 1, \ldots, m-1$	$j = 0, \ldots, m-1$
(J)	$D_i(D_i(a) \Rightarrow a)$	$i = 1, \ldots, m-1$	
(B)	$\neg D_i(a) \cup D_i(a)$	$i = 1, \ldots, m-1$	

(We use the same notations as in the D-N-algebras)

Axioms for DNCP:

I. The full set of axiom-scheems for the intuitionistic propositonal calculus (see for example [4])

II. (t), (f), (D′), (D″), (D_0') and (D_0'')

III. any of the (R), (M), (L_ω), (L_α), (L), (C_1), (C_2), (J) and (B) which algebraic analogous occur in the set N.

The rules of inference are the following:

$$(r_1) \quad \frac{a,\ a \Rightarrow \beta}{\beta} \quad \text{— modus ponens and} \quad (r_2) \quad \frac{a \Leftrightarrow \beta}{(D_i a) \Leftrightarrow D_i(\beta)}$$

$i = 1, \ldots, m-1$

The notion of a theory based on a DNCP is analogous to that of Post logics (see [4]). If $\mathscr{T}$ is a theory with the set of axioms A then we shall denote this by $\mathscr{T}(A)$.

THEOREM 8.3 (The strong completeness theorem). *For any formula a and D-N-theory $\mathscr{T}(A)$ the following conditions are equivalent:*

(i) *a is a theorem in the D-N-theory $\mathscr{T}(A)$*
(ii) *any algebraic D-N-model for A is a model for a*
(iii) *any relational D-N-model for A is a model for a*

PROOF. The equivalence of (i) and (ii) can be easily proved using the method of Lindenbaum algebras; (ii) is equivalent to (iii) on account of theorem 8.1.

THEOREM 8.4. (The weak completeness theorem). *For any formula a the following conditions are equivalent:*

(i) *a is a D-N-theorem*
(ii) *a is an algebraic D-N-tautology*
(iii) *a is a relational D-N-tautology*

PROOF. From theorem 8.3 with empty A.

THEOREM 8.5 *Let L be a DNPC for which* (R)$\in$N *and* (J)$\notin$N. *Then L has the finite model property and is decidable.*

PROOF. By theorem 6.1 the set of all algebraic D-N-models can be restricted to the set of all finite models, which proves the theorem.

The propositional calculi corresponding to the quasi-pseudo-Post (quasi-Post, pseudo-Post, Post) algebras will be called *quasi-pseudo-Post* (*quasi-Post*, *pseudo-Post*, *Post*) *propositional calculi* (PC). The language of quasi-pseudo-Post and quasi-Post PC does not contain the constants e_i, $i = 0, \ldots, m-1$

THEOREM 8.6. *Pseudo-Post PC* (*Post PC*) *is a conservative extention of quasi-pseudo-Post* (*quasi-Post*) *PC.*

PROOF. By theorem 7.2.

THEOREM 8.7. *Quasi-pseudo-Post* (*pseudo-Post*, *quasi-Post*, *Post*) *PC is decidable.*

For the pseudo-Post PC and Post PC the proof is given by Rousseau [8]. For the remaining calculi this is a consequence of theorem 8.6.

THEOREM 8.8. (A separation theorem for pseudo-Post and Post PC). *For any formula* a *not containing constants* e_i $(i = 0, \ldots, m-1)$ *the following conditions are equivalent*:

(i) a *is a theorem in the pseudo-Post* (*Post*) *PC*

(ii) a *is a theorem in the quasi-pseudo-Post* (*quasi-Post*) *PC*

PROOF. This is another form of theorem 8.6.

Let us note that, in view of theorem 2.4, the relational semantics for pseudo-Post PC can be much simplified.

References

[1] B. I. DAHN, *Kripke-style semantics for some many-valued propositional calculi*, **Bulletin de l'Académie Polonaise des Sciences**. *Série des sci. math., astr. et phys.* 21 (1973), p. 1073.

[2] G. C. MOISIL, *Essais sur les logiques non chrysyppiennes*, Bucarest 1972.

[3] M. C. FITTING, *Intuitionistic logic model theory and forcing*, North–Holland, Amsterdam–London 1969.

[4] H. RASIOWA, *An algebraic approach to non classical logics*, STUDIES IN LOGIC AND THE FOUNDATIONS OF MATHEMATICS, Vol. 78, Amsterdam 1974, xv + 403 pp.

[5] H. RASIOWA, *Post algebras as a semantic foundation of many-valued logics*, in: **Studies in algebraic logic**, **MAA** STUDIES IN MATHEMATICS, Vol. 9 (1974), pp. 92-142.

[6] P. C. ROSENBLOOM, *Post algebras I. Postulates and general theory*, **American Journal of Mathematics** 64 (1942), pp. 167-188.

[7] G. ROUSSEAU, *Logical systems with finitely many truth-tables*, **Bulletin de l'Académie Polonaise des Sciences**, *Série des sci. math., astr. et phys.*, 17 (1969), pp. 189-194.

[8] G. ROUSSEAU, *Post algebras and pseudo-Post algebras*, **Fundamenta Mathematicae**, 67 (1970), pp. 133-145.

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