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Classical Relevant Logics II

The purpose of this note is to extend the simplifications of [1] to the system *R* of relevent implication analyzed semantically in [2]. In [1], it was established that the system R^+, which is the negation-free fragment of *R*, could be furnished with a negation $\neg$ more classical in most respects than the preferred negation $-$ of [3]. This was rather astonishing, since an important motivating condition on relevant logics had been absence of the classical paradoxes of implication[1]. It turned out, however, that one could have the most objectionable of the paradoxes anyway, namely $A\&-A\rightarrow B$ and $A\rightarrow B\vee\neg B$, without the least interference with the positive ideas.

It now turns out, however, that *not only* can classical negation be added to R^+, producing the system *CR* of [2], *but also* that the original relevant negation $-$ of *R* can be explicated *directly* as a certain kind of classical negation. That is to say, from a certain viewpoint classical negation is a *more general* kind of negation than is relevant negation, the latter being subsumed under the former. (Meyer in particular wishes to note that, in view of his interest in systems that are not trivialized by the presence of contradiction, he again finds these results disappointing; his job here, however, is to report them, not to like them.)

The subsumption of relevant $-$ under classical $\neg$ is effected by the direct introduction of a new, 1–place connective * into the language of [1], producing a system CR^*. The formal semantics of CR^* is then given by simplifying [2] on the plan of [1], in the following way.

A *CR^* model structure* (henceforth, *CR^*ms*) is a quadruple $\langle 0, K, R, ^*\rangle$, where *K* is a set, $0\in K$, * is a 1-place operation on *K*, and *R* is a ternary relation on *K*, satisfying the following definitions and postulates:

d1. $R^0ab =_{df} a=b$; and, recursively, for $0\leqslant k<w$,

d2. $R^{k+1}\, c_0\ldots c_k ab =_{df} \exists x(R^k c_0\ldots c_k x\&Rxab)$

d3. $a_0\ldots a_k Rb =_{df} R^k a_0\ldots a_k b$

Under these definitions, $R=R^1$, and R^2 has the meaning $\exists x\,(Rabx\&Rxcd)$ assigned it in previous publications. But the present notation is more perspicous, and makes the essential point that *R* is profitably thought of alternatively, as in d3, as a binary relation between a finite sequence in *K* and a point in *K*; this facilitates algebraic

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[1] Other scholars, notably Urguhart and Gabbay, have thought independently about classical negation in relevant logics. But [1], so far as we know, contained the first demonstration that the system R^+ does not collapse under the admission of such negation.

connections and connections with Gentzen-style consecution calculi, application to R of these alternative modes of logical analysis having been initiated in [4] and [5] respectively. The postulates are

p1. $0aRb$ iff $a=b$ (iff, by d1, d3, aRb)

p2. $abcRd$ iff $acbRd$

p3. $aaRa$

p4. $a^{**}=a$

p5. $abRc$ iff ac^*Rb^*

p2—p5 are taken from [2]; p1 is the simplification of [1].

Intuitive motivation for the postulates has been given in [1] and [2]. To visualize algebraic connections, think of R as a partial order relation $\geqslant$, and of K as a *subset* of a partially ordered commutative monoid $\langle K;\cdot, 0\rangle$, partially ordered by $\geqslant$, in the sense that if $a \geqslant b$ then $a\cdot c \geqslant b\cdot c$, that $\cdot$ is associative and commutative, and that 0 is a *multiplicative* (!) identity for $\cdot$[2]. We cannot, however, deal profitably with the algebra K', which is intuitively a calculus of theories, except at the price of formal ugliness[3]. So we do not define the *operation* $\cdot$ on the subset $K \subseteq K'$, where K corresponds intuitively to a collection of *prime* theories[4]. We save what we need of it, however, by saving the *relation* $ab \geqslant c$; even when $a\cdot b \notin K$, the relation continues to make sense. The most interesting *special* property of K' is the square-increasing property $a^2 \geqslant a$, signalled by p3; this corresponds intuitively to the fact that theories are closed under *modus ponens*, and is on that ground well-motivated; p1—p2 express the underlying commutative monoid properties, p1 making the extra point that the elements of K itself shall constitute a *totally unordered* set with respect to $\geqslant$.

We now give the interpretative machinery. Let $\mathscr{K} = \langle 0, K, R, {}^*\rangle$ be a CR^*ms, S the set of sentential variables of CR, and F the set of formulas built up from members of S and the connectives $\rightarrow$, &, $\neg$, $*$. We also use T, F, the latter unambiguous in context, for classical truth-values. Then

(1) Any function v: $S \times K \rightarrow \{T, F\}$ is a *valuation* of CR^* in $\mathscr{K}$.

(2) A function I: $F \times K \rightarrow \{T, F\}$ is a *possible interpretation* of CR^* in $\mathscr{K}$.

[2] Perhaps the time has come to start writing '0' as '1'. Since '1' enters reasonably in section IV in another sense, however, we stick here to the old convention.

[3] This is our sole complaint about [6], which otherwise emphasizes exactly the technical ideas we think most important, with a couple of nice twists. Unfortunately, neither we nor Fine knew of the work of the other until our respective projects were far advanced. (Prof. Wolf suggests that those of us interested in relevant logics should have periodic conferences, to avoid this sort of unnecessary duplication. Right on, Bob!) On the main points, we add, we anticipated Fine by about a year, and were ourselves partially anticipated by Urquhart; we greatly admire the contributions of both Fine and Urquhart.

[4] This is necessary to get the natural truth-condition TV below, a fact also pointed out in [6].

Let I be a possible interpretation of CR^* in $\mathscr{K}$. We shall write

(3) $A_I a$ for $I(A, a) = T$.

Where I is clear in context, we write simple

(4) Aa for $I(A, a) = T$.

We also use $\sim$, &, $\vee$, $\Rightarrow$, (x), $\exists x$ as intuitive quantifiers and connectives in the metalanguage. Given I, we may also write

(5) Aab for $(x)(ab\, R\, x \Rightarrow Ax)$.

(5) may be generalized to an arbitrary number of arguments by

(6) $Aa_0 \ldots a_n$ for $(x)(a_0 \ldots a_n\; R\, x \Rightarrow Ax)$.

The advantage of (5) and its generalization (6) is drawn from the algebraic remarks, and it brings our semantics closer in *appearance* to that of [6], where [6] is prettier; in contrast to [6], however, the move is a notational one only, and so the economies of our basic approach form [2] on are retained[5].

We come now to the key semantic notion of *interpretation*. A possible interpretation I in $\mathscr{K}$ is an *interpretation* of I in $\mathscr{K}$ provided that the following *truth-conditions* are satisfied, for all B, A in F and a in K:

T&. $[A\&B]a$ iff $Aa\&Ba$

T$\urcorner$. $[\urcorner A]a$ iff $\sim Aa$

T*. $[A^*]a$ iff Aa^*

T$\rightarrow$. $[A \rightarrow B]a$ iff $(y)(Ay \Rightarrow Bay)$

Given (5), only T^* is new. Note that every valuation v uniquely determines an interpretation I, so that we may speak interchangeably of valuations and interpretations. We say also, for given I, $\mathscr{K}$,

(7) A *is verified on* I iff $A_I 0$

(8) A *is valid in* $\mathscr{K}$ iff, for all interpretations I in $\mathscr{K}$, A is verified on I.

(9) A is CR^**-valid* iff A is valid in all CR^**ms.*

I

The connective * builds directly into the language of CR* the notion of *weak assertion* underlying the treatment of relevant negation in [2]. The idea was that $\bar{A}$ is to be true in a set-up a just in case A is false in a counterpart set-up a^*; for

[5] Thus the truth-condition T$\rightarrow$ coincides, in appearance, with those of [15] and [6]. The appearance is more than *just* appearance, since interpretation of the semantics in the calculus of theories—as e.g. in [2] and [6] — is intended. But T$\rightarrow$ below is shorthand nonetheless.

further motivation, consult [2]. Underlying all was the intuitive notion that what has not been refuted on certain evidence may in some sense be asserted on that evidence. Previous formal treatments of negation have tended to pass over this point lightly, in that they take it for granted that, were all the evidence in, the propositions denied would be precisely those that had failed of being asserted.

Our theory of relevant negation —, which it is still our principal business to explicate, remains a more dialectical affair, even at the limit. I.e., even when the evidence is in a certain sense complete (producing a *prime* theory), we do not identify ordinary (*strong*) assertion and *weak* assertion. So it makes sense to introduce the proposition A^*, the weak assertion of A, directly into the vocabulary via a primitive connective *. Then, in a certain sense, we can subsume relevant under classical negation, via the definitional scheme

d4. $\overline{A} = \text{df} \neg A^*$

It is interesting, incidentally, to see what now becomes of such classical principles as the law

(10) $A \vee \overline{A}$

of excluded middle. It now takes the form

(11) $A \vee \neg A^*$

From this viewpoint, that (10) holds *truth-functionally* for R can no longer be strictly maintained, for (11) is now a *substantive* assumption[6]. Accordingly, the opening confession of motivational disappointment need not be taken too seriously; although we show in a strong sense that relevant negation is classically explicable, there is no particular reason to *prefer* the classical point of view on which it is explicable; for statements involving relevant negation, even when they have classical counterparts — e.g., the counterpart

(12) $A \vee \neg A$

of (10) — are rarely simply derivable from those counterparts in CR^*—e.g., (11) is by no means an *instance* of (12), but results rather from substantive assumptions on *. And these, in the end, amount to just the original assumptions on —, viewed from a different but essentially equivalent perspective[7].

Motivational gain and loss aside, however, the detour through CR^* illumines the system R technically in interesting and important ways. In certain fragments

[6] Involved, among other things, is the principle v discussed in [3]. This is the question whether the inference from A and $A \supseteq B$ to B is admissible in, say, R, where $\supseteq$ is the material implication defined using — in d7 below. The present semantics builds in an affirmative answer, again, substantively. Cf. p. 7, top.

[7] The reason for making such assumptions, according to [11], does not lie in semantical analysis at all but in the theory of deduction.

of R, for example, it has been known for a long time that, algebraically, the intensional complement $-$ amounted to the composition of *Boolean complementation* $\neg$ and a *permutation* $*$; these results rest mathematically on the investigation of so-called DeMorgan lattices (quasi-Boolean algebras) conducted in the late 1950's by Białynicki-Birula, Rasiowa, and Kalman (e.g., in [7] and [8].) The import of these results for relevant logics, both algebraically and motivationally, was first grasped by Dunn in [9], on suggestions of and in collaboration with Belnap; a partial account appears in [10]; a fuller one, in [3]; cf. also van Fraassen [12]. $*$ is introduced semantically in [13], and is used in [6], [2], and succeeding papers.

Nevertheless, there was a very important failure with respect to the application to the algebraization of R via DeMorgan monoids in [9] and [3] of the algebraic ideas of [7] and [8]. The problem is that if one is to view $-$ as the composition $\neg *$, one must be able to choose, or at least to embed, the DeMorgan monoids that algebraize R among such DeMorgan monoids as are Boolean algebras. Here an interesting situation developed. For it was known, on the one hand, that a DeMorgan monoid could always be embedded *as a lattice* in a Boolean algebra. But that was to lose their most interesting feature, preserving the operations that correspond to &, $\vee$, $-$ at the loss of the operation one chooses to explicate the distinctive connective $\to$ of R. So, on the researches of [9] and [10], Boolean algebras played an interesting role in the algebra of R only in that fragment of R which contains no occurrences of $\to$, properly speaking, as an operation $-$ i.e., the first-degree fragment of R investigated in [14]. Thus there is a tension in [9] between the formal explication of $-$, which doesn't explain $\to$, and the formal explication of $\to$, which leaves $-$ unanalyzed save as one requires postulates on DeMorgan monoids that make all the axioms of R true.

It is this situation which was set right for R^+ in [1], and which will be set right for all of R here, with the added benefit here that, while it was merely of interest that the Dunn monoids of [1] could always be embedded in Boolean monoids, our work here rests upon and in a certain sense completes the theory of propositional negation proposed in [9].

III

In this section we show that the system CR^*, characterized so that its set of theorems is exactly the CR^* valid formulas, exactly contains the system R of relevant implication on the definition of $-$ by d4. We note that, by definition, $-$ has the truth-condition,

T$-$. $\quad \bar{A}a$ iff $\sim A^*$.

Other interesting connectives can be defined in CR^* as follows:

d5. $\quad A \vee B =_{\text{df}} \neg(\neg A \& \neg B)$

d6. $\quad A \supset B =_{\text{df}} \neg A \vee B$

d7. $A \supset B =_{df} \bar{A} \vee B$

d8. $A \equiv B =_{df} (A \supset B) \& (B \supset A)$

d9. $A \leftrightarrow B =_{df} (A \to B) \& (B \to A)$

d10. $A \circ B =_{df} \overline{A \to \bar{B}}$

d11. $A + B =_{df} \bar{A} \to B$

Truth-conditions corresponding to these connectives are

T$\vee$. $[A \vee B]a$ iff $Aa \vee Ba$

T$\supset$. $[A \supset B]a$ iff $Aa \Rightarrow Ba$

T$\supseteq$. $[A \supseteq B]a$ iff $Aa^* \Rightarrow Ba$

T$\equiv$. $[A \equiv B]a$ iff Aa iff Ba

T$\leftrightarrow$. $[A \leftrightarrow B]a$ iff $(x)([Ax \Rightarrow Bax] \& [Bx \Rightarrow Aax])$

T$\circ$. $[A \circ B]a$ iff $\exists x \exists y (Ax \& By \& xyRa)$

T$+$. $[A+B]a$ iff $(x)(y)(x^*y^*Ra^* \Rightarrow Ax \vee By)$

Of these connectives, $\vee$, $\Leftrightarrow$, $\circ$, $+$ are familar connectives of R; their truth-conditions, together with T$-$, are as in [2] (and as in [1] also, for $\vee$, $\Leftrightarrow$, $\circ$). $\supset$, $\equiv$ are *extensional* connectives, with straightforward meanings. $\supseteq$ is included for contrast; it is *not* straightforwardly extensional, which made it for a while an interesting open problem whether B is an R-theorem whenever A, $A \supseteq B$ are R-theorems. (Yes — automatically for CR^*, as the reader is invited to verify.) (Another interesting point is that had $-$ appeared *everywhere* in d5 for $\neg$, we'd have an equivalent definition of $\vee$.)

To get our desired result, we must show that, for any formula A of R, A is a theorem of R iff its translation via d4 into CR^* is a theorem of CR^*. (To handle $\vee$, either take it as primitive for R or use the suggested alternative definition just above; in any case, its truth-condition will be T$\vee$.)

Our proof procedure will be wholly semantic. The best result from [2] that we have to go on is the following:

Normality Theorem. *A is a theorem of R iff A is normally valid.* (Proof in [2].)

What the normality theorem means is explained as follows: a *normal Rms* is characterized like a *CR*ms*, as a quadruple $\langle 0, K, R, * \rangle$, except that for p1 are substituted the weaker postulates, in the context p2—p5,

p6. $R0aa$.

p7. $R^2 0abc \Rightarrow Rabc$.

p8. $0^* = 0$.

It's quickly verified that every *CR*ms* satisfies p2—p8, whence every *CR*ms* is a normal *Rms*, though not conversely. Let an interpretation be defined *mutatis* *mutandis* as above, in a normal *Rms*, satisfying T&, T$\vee$, T$\rightarrow$, T$-$. An interpretation is *hereditary* provided that it satisfies also

H. $R0ab \& Aa \Rightarrow Ab$

Note that an interpretation in a CR^*ms is automatically hereditary, as remarked above, but without p1 this isn't always true. Then the normality theorem states that A is a thorem of R iff A is true at 0 (i.e., verified) on all hereditary interpretations in all normal *Rms*. Evidently,

LEMMA 1. *If A is a theorem of R, then A is CR^* valid.*

PROOF immediate from the normality theorem, and the observations that a CR^*ms is an *Rms* and that interpretations therein are hereditary, since the truth--conditions are the same for formulas of R in *Rms* and in CR^*ms.

The converse of lemma 2 is the main point.

LEMMA 2. *If a formula A of R is CR^* valid, then A is a theorem of R.*

PROOF. Let A be a formula of R, and assume that A is not a theorem of R. We show that A is invalid in some CR^*ms, whence the lemma follows by contraposition.

At any rate, by the normality theorem, A is not verified on some hereditary interpretation I in some normal *Rms* $\mathscr{K} = \langle 0, K, R, * \rangle$. Following the recipe of [1], we shall make $\mathscr{K}$ into a CR^*ms by adding a new 0. So let $0'$ be a new element, and let $K' = K \cup \{0'\}$. Let $*'$ be like $*$ on elements of K, and define

(13) $0'^{*'} = 0'$.

Likewise, let R' be like R on elements of K. And define, for all a, b in K',

(14) $R' 0' ab$ iff $a = b$

(15) $R' a0' b$ iff $a = b$

(16) $R' ab0'$ iff $a = b^{*'}$

(14)—(15) are from [1]; (16) is new. Then, first,

(17) $\mathscr{K}' = \langle 0', K', R', *' \rangle$ *is a* CR^*ms.

Verification of (17) is pretty trivial. p1 holds directly by (14), and the other postulates p2—p5 are verified quickly by the fact that the corresponding postulates hold in $\mathscr{K}$ and definitions (13)—(16). The only interest lies in the verification of p2 for $\mathscr{K}'$, which we leave to the reader for his amusement.

Let now v be the valuation got by restricting the given interpretation I to sentential variables. Let v' be like v at all points of K, and set, for each p in S,

(18) $v'(p, 0') = v(p, 0)$.

Let now I' be the interpretation in $\mathscr{K}'$ determined by v' and the truth-conditions. We show, for all formulas B of R, and all a in K,

(19) $I'(B, a) = I(B, a)$, and

(20) $I'(B, 0') = I(B, 0)$

Given the similar argument of [1], verification of (19)—(20) can also be safely left to the reader. (Prove the conjunction of (19)—(20) by induction on length of formula.) The normality postulate p8 enters into the argument ofr (20) in view of the case, on inductive hypothesis, where B is $\bar{C}$,

(21) $\bar{C}_I 0'$ iff $\sim C0^{*\prime}$, iff $\sim C0'$, iff $\sim C0$, iff $\sim C0^*$, iff $\bar{C}_I 0$.

But then in particular $I'(A, 0') = I(A, 0) = F$, for our chosen non-theorem A of R, ending the proof of its CR^* invalidity and so the proof of lemma 2.

Our desired objective has been obtained.

TRANSLATION THEOREM. *The translation by* d4 *of* R *into* CR^* *is exact; i.e.,* A *is a theorem of* R *iff, on translation* $'$, A' *is a theorem of* CR^*.

PROOF. By lemmas 1 and 2.

So the decomposition of relevant negation $-$ into $\neg$ and $*$ is accomplished.

IV

In this section, we extend to all of R the algebraic conclusions drawn in [1] for R^+. As in [1], a *Boolean monoid* shall be a structure $\mathscr{B} = \langle B, \circ, \rightarrow, \vee, ', 1 \rangle$ such that $\langle B, \vee, ' \rangle$ is a Boolean algebra, and $\langle B, \circ, 1, \wedge, \vee, \rightarrow \rangle$ is a Dunn monoid, $a \wedge b$ being defined as $(a' \vee' b')'$. A Boolean monoid is *complete*, it is recalled, if the following infinite distribution laws hold for $a \in B$, arbitrary $\{b_i\}_{i \in I} \subseteq B$, and if B is complete as a Boolean algebra:

(22) $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$

(23) $a \circ \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \circ b_i)$

A Boolean monoid is *regular*, again, provided that

(24) 1 is an *atom* under the Boolean ordering of $\langle B, \vee, ' \rangle$.

Finally,

(25) $\mathscr{B}$ is a *Boolean set monoid* if $\mathscr{B}$ is complete, regular, and $\langle B, \vee, ' \rangle$ is isomorphic to the set of all subsets of some set C (under set union and complementation), where $\mathscr{B}$ is a Boolean monoid.

An algebraic interpretation I of R^+ (or of CR) in a Boolean monoid $\mathscr{B}$ is a function which assigns a member a of B to each formula A, and which respects the connectives in the obvious sense (i.e., is a homomorphism from the algebra of formulas to $\mathscr{B}$). A is verified on I iff $1 \leqslant I(A)$ in $\mathscr{B}$. Then the best result of [1] was, algebraically, Boolean Algebra Theorem for R^+. *A is a theorem of R^+ iff A is verified on all interpretations in all Boolean set monoids.*

The theorem was welcome for many reasons. First, the regularity condiction (24) places 1 under rather firm control, which is important because 1 is the special element which figures in verification, as well as an identity for $\circ$. Second, the residual $\rightarrow$becomes superfluos, being definable by

(26) $a \rightarrow b = \bigvee\{c\colon c \circ a \leqslant b\}$,

as is well-known given commutativity of $\circ$, completeness under infinite joins, and (23). Third, the condition (25) means that every element of $\mathscr{B}$ is a join of atoms (except 0, of course).

Since these are the benefits that we wish to reap for *R* (and *CR**) also, we shall restrict our explicit generalizing to Boolean set monoids, our assertions automatically holding for wider natural classes of structures algebraizing *R*. So let $\langle B, \circ, \bigvee', 1\rangle = \mathscr{B}$ be a Boolean set monoid, $\bigvee$ being the infinite join and $\wedge$, $\rightarrow$ being defined as above. (Special properties of $\circ$ are given by requiring that $\langle B, \circ, 1\rangle$ be a commutative monoid, and that (23) and the square-increasing property $a \leqslant a \circ a$ hold.)

A Boolean set monoid is, as we have seen, a very natural structure, since it's just a power set with a continuous multiplication o defined on it, such that some atom (unit set) is the identity for this multiplication; $a \subseteq a^2$ is thus the only *really* special property. It is also natural, following the algebraic work cited earlier, to define a unary operation * as an *involution* on *B* provided that it has the following properties: Let *a*, *b* be arbitrary members of *B*.

(27) $1^* = 1$

(28) $a^{**} = a$

(29) $(a \vee b)^* = a^* \vee b^*$

(30) $a^{*\prime} = a^{\prime *}$

Thus an involution here is just a permutation in the ordinary sense, with the additional requirements that * be of period 2 (28), that 1 be a fixed point for * (27), and that * be a Boolean automorphism from *B* onto *B* (29), (30). Given $\mathscr{B}$, *, we may then follow work cited by defining the relevant complement $-$ by

(31) $-a =_{\mathrm{df}} a^{*\prime}$

Finally, letting $\langle B, *\rangle$ be a Boolean set monoid with a permutation *, we call it a *DeMorgan set monoid* provided that the residual $\rightarrow$ given by (26) is alternatively characterized by

(32) $a \rightarrow b = -(a \circ -b)$

The effect of (32), we note, is alternatively got by Dunn's antilogism postulate, namely

(33) $a \circ b \leqslant c$ iff $a \circ -c \leqslant -b$

We may now extend the notions of algebraic interpretation, verification, etc., to all of R and CR^*, finding as for R^+ in [1] a set algebra which will refute any non--theorem. I.e.,

BOOLEAN ALGEBRA THEOREM FOR *R*. *A is a theorem of R iff A is verified on all interpretations in all DeMorgan set monoids.*

PROOF. As of the analogous theorem of [1]. First, a DeMorgan set monoid is evidently a DeMorgan monoid in the sense of [3], whence by Dunn's results all theorems are verified on all interpretations therein.

Suppose, conversely, that A is a non-theorem of R. By the translation theorem, A' is a non-theorem of CR^*. So there is a semantic interpretation I in a CR^*ms $\langle 0, K, R, * \rangle$ such that $\sim A'_I 0$. Let $\beta(K)$ be the power set of K. We wish to define the De Morgan operations on $\beta(K)$. $\bigcap$, $\bigcup$ are of course set intersection and union. Moreover, for $S \subseteq K$, $T \subseteq K$, we define

(34) $S \circ T = \{c : \exists X \exists y(x \in S \& y \in T \& Rxyc\}$

(35) $S^* = \{c : c^* \in S\}$

(36) $1 = \{0\}$

$-$ and $\rightarrow$ we may take as automatically defined on $\beta(K)$ by (31)$-$(32), set-theoretic complementation $'$ being of course defined on $\beta(K)$ with respect to K; i.e., $a' = K\text{-}a$.

We wish to refute our chosen non-theorem A in the DeMorgan set monoid $\beta(\mathscr{K}) = \langle \beta(K), \circ, U, ', 1, * \rangle$; evidently it will do to refute A', since the syntactical and algebraic definitions d4, (31) of relevant negation mirror each other. In this, we have two tasks. First, we must show that $\beta(\mathscr{K})$ really is a DeMorgan set monoid— i.e., that it satisfies all our postulates on such structures. Second, we must use our semantical interpretation I that refutes A' in $\langle 0, K, R, * \rangle$ to construct an algebraic interpretation I° that refutes A' in $\beta(\mathscr{K})$. We turn to the first task.

But $\langle 0, K, R, * \rangle$ is of course an *Rms* in the sense of [2]. And, now that hereditary conditions have lapsed in the presence of p1, for CR^*ms, $\beta(\mathscr{K})$ with operations as defined is an *algebra of propositions* in the sense of [2]. So theorem 9 of [2] applies, and $\beta(K)$ according to that theorem is a complete De Morgan monoid with operations as defined. This automatically takes care of the underlying Dunn monoid conditions on $\circ$, $\rightarrow$, $\wedge$, $\vee$, and the completeness conditions, including (22)$-$(23); moreover, it assures that $-$ as characterized by (31) and the alternative characterization of $\rightarrow$ in (32), used definitionally here, turn out right. Since we chose $\beta(K)$ as a power set, (25) holds; since we chose 1 as a singleton in (36), the choice we had to make to follow the policy of [2], 1 is an atom in the ordering under $\subseteq$, satisfying (24). And (27)$-$(30) are immediate on our definition of $*$, given (for (27)) that $0^* = 0$ in all CR^*ms. So $\beta(\mathscr{K})$ is a DeMorgan set monoid, as required.

Next, we wish to define the refuting interpretation I°. We do this by setting, for all formulas B of CR^*, $I^\circ(B) = \{x : B_I x\}$, where I is the semantic interpretation that refutes A. Straightforward verification, on appeal to the semantic truth-conditions $T\&$, etc., then establishes that I° is indeed a homomorphism from the algebra of formulas of CR^*, and derivatively of R, into $\beta(\mathscr{K})$. Finally, our chosen non--theorem A is indeed refuted on I°; for $\sim A_I 0$, and so accordingly $1 = \{0\} \subseteq I^{\circ}(A) = \{x: A_I x\}$. This completes the proof of the Boolean algebra theorem for R.

We end this section with some incidental remarks. First, the technical benefits claimed for the Booleanization above of R^+ do in fact apply now to all of R; [11] is a first attempt to collect on some of these benefits. Second, these results pave the way for the resuscitation of some old pet schemes of Belnap and of Dunn. Belnap remarked some years ago, in conversation, that the lattices he had used to explicate first-degree relevant implication were just Boolean algebras that had got mixed up on complementation, and Dunn pointed out (cf. references above) just how they had got mixed up. So long, however, as that fact couldn't be squared technically with the overall analysis of relevant implication algebraically in [9], from the point of view of the enterprise entire it was an idle fact; it can now be sent back to work. Third, as pointed out in [1], one of the best ways to get concrete formal insights about a system is to have a look at its finite models, especially the small ones. Few subjects, of course, are under such firm control as the theory of finite Boolean algebras, and if we can only discover now the right way to multiply by 2, all will be revealed. Finally, the completion of the development of a relevantly acceptable theory of propositions, initiated in [14] and its predecessors, is now in sight. E.g., on the theory adumbrated in [14] by Belnap, things were said like, „We think the stronger assumption of complete distributivity to be as true for complete propositional lattices as it is for complete fields of sets". Well, yes, since on the theorem just proved the completest complete propositional lattices overall acceptable for R *are* complete fields of sets[8].

In conclusion, it will be noted that we have neglected to axiomatize CR^*. The reason isn't that it's unaxiomatizable or anything like that; indeed, we presume that just putting together the axiomatization of CR in [1] and of R in [2] or [3] one would have an axiomatization of CR^*, near enough, reversing d4 by then defining A^* as $\neg\bar{A}$. Frankly, however, we can't at this point stomach yet another completeness proof on ground that we have been over so often before; any readers that have stuck with us through the series of papers that began with [2] feel as we do, no doubt, letting the semantic characterization of CR^* above suffice. But the case is now pretty strong that $\neg$ was just left out of Anderson-Belnap formulations of their logics, and evidence is building that the entire project of relevant logic is unified and simplified when the *semantic* $\neg$, with a different function from the *deduction-theoretic* — that has been present from the start, is added. This paper is part of that evidence[9].

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[8] While agreeing with the quoted statement, we pass here on the question whether it means "equally true" or "equally false".

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