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A Class of Cylindrically-Symmetric Models in String Cosmology

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A new class of physically relevant explicit solutions for string cosmological models endowed with cylindrical symmetry on the background of singularity-free cosmological space times has been obtained and their physical and kinematical features are discussed. The matter-free limits of this class of solutions are observed to be the singularity-free vacuum solutions of Patel and Dadhich.

1. INTRODUCTION

The standard Friedman–Robertson–Walker (FRW) cosmological model, which prescribes a homogeneous and isotropic distribution for its matter content, has been quite successful in describing the present state of the universe. It has been realized, however, that the homogeneous and isotropic character of the space time cannot be sustained at all scales, especially for the early times. One of the main features of relativistic cosmology is the prediction of the big bang singularity in the finite past. This conviction arose out of the general result that under physically reasonable conditions of positivity of energy, causality and regularity etc., the initial singularity is inescapable in cosmology so long as we adhere to Einstein's equations

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and it can only be avoided by invoking quantum effects and/or modifying Einstein's theory.

Recently Senovilla [1] obtained a new class of exact solutions of Einstein's equations without big bang singularity. It represents a cylindrically symmetric universe filled with perfect fluid ($\rho = 3p$) which is smooth and regular everywhere, and satisfies the energy and causality conditions. All the physical and the geometrical invariants for it are finite and regular for the whole of space-time. Later Ruiz and Senovilla [2] separated out a fairly large class of singularity-free models through a comprehensive study of a general cylindrically-symmetric metric with separable functions of r and t as metric coefficients. Dadhich, Tikekar and Patel [3] have established a link between the FRW models and the singularity-free family by deducing the latter through a natural and simple inhomogenisation and anisotropisation of the former.

It would be interesting to study cosmic strings, which have received considerable attention in cosmology, in this framework. Strings are important in the early stages of evolution of the universe before the particle creation. The present day observations do not rule out the possible existence of large scale networks of strings in the early universe. The gravitational effects of cosmic strings, which are considered as objects endowed with stress energy and coupled to gravitational fields, have been extensively discussed by Vilenkin [4], Gott [5] and Garfinkle [6].

General relativistic treatment of strings was developed by Letelier [7] and Stachel [8] from which Letelier [9] obtained relativistic models of strings in Bianchi I and Kantowski–Sach space times by introducing the energy momentum tensor

$$T_{ik} = \rho u_i u_k - \lambda w_i w_k, \qquad u_i u^i = -w_i w^i = 1, \; w^i u_i = 0, \tag{1}$$

as the source term in the Einstein field equations

$$R_{ik} - \tfrac{1}{2} R g_{ik} = -8\pi T_{ik}. \tag{2}$$

T_{ik} in (1) represents the energy momentum tensor associated with a cloud of strings with particles attached to them. ρ and λ respectively denote the energy density and the string tension density of the string cloud which are related by [9]

$$\rho = \rho_{\mathrm{p}} + \lambda, \tag{3}$$

where ρ_{p} is the particle density in the string cloud. The energy conditions imply $\rho > 0$, leaving the sign of the string tension density λ unrestricted. The unit time like vector u^i is the flow vector of the matter and the space like vector w^i, specifies the string direction in the cloud. Relativistic string models in the context of space times of Bianchi types of other kinds have subsequently been obtained by Krori et al. [10], Banerjee et al. [11], Tikekar and Patel [12]. The solutions of the Einstein field equations in the context of spherically-symmetric space-times with Kantowski–Sach symmetry and classical strings as the source for the gravitational field have been obtained by Matravers [13]. Nevin [14] has obtained solutions for static spherically-symmetric and cylindrically-symmetric space-times assuming string dust source and has further matched them at finite boundary $r = a$ with appropriate extended vacuum solutions with respective symmetry.

We report here a new class of specific inhomogeneous solutions of string cosmology in the context of space-times endowed with cylindrical symmetry, obtained by adopting the metric form that can represent singularity free fluid models for the background space-time. The matter-free limit of this class is the singularity-free cylindrically symmetric vacuum metrics obtained by Patel and Dadhich [15].

2. FIELD EQUATIONS

Assuming that the metric coefficients are separable functions of the radial coordinate r and the time t, the metric of a cylindrically-symmetric space-time is taken in the form

$$\begin{aligned} ds^2 &= \cosh^{2\alpha}(kt)\,\cosh^{2a}(mr)(dt^2 - dr^2) - \cosh^{2\beta}(kt)\cosh^{2b}(mr)dz^2 \\ &\quad - \frac{1}{m^2}\cosh^{2\gamma}(kt)\cosh^{2c}(mr)\sinh^2(mr)d\phi^2, \end{aligned} \tag{4}$$

where $\alpha, \beta, \gamma, a, b, c, m$ and k are constants.

Introducing the orthonormal tetrad

$$\begin{aligned} \theta^1 &= \cosh^{\alpha}(kt)\cosh^{a}(mr)dr, \\ \theta^2 &= \cosh^{\beta}(kt)\cosh^{b}(mr)dz, \\ \theta^3 &= \frac{1}{m}\cosh^{\gamma}(kt)\cosh^{c}(mr)\sinh(mr)d\theta, \\ \theta^4 &= \cosh^{\alpha}(kt)\cosh^{a}(mr)dt, \end{aligned} \tag{5}$$

and using Cartan's equations of structure, the surviving components of the Ricci tensor in this tetrad basis are found to have the following expressions:

$$
\begin{aligned}
A^2 R_{(11)} &= (b+3c+1)m^2 - \alpha k^2 \\
&\quad + m^2[b^2 - b(1+a) + c^2 - c(1+a) - a]\tanh^2(mr) \\
&\quad - k^2[\alpha(\beta+\gamma-1)]\tanh^2(kt), \qquad (6)\\
A^2 R_{(22)} &= 2bm^2 - \beta k^2 + bm^2(b+c-1)\tanh^2(mr) \\
&\quad - \beta k^2(\beta+\gamma-1)\tanh^2(kt), \qquad (7)\\
A^2 R_{(33)} &= m^2(1+b+3c) - \gamma k^2 + cm^2(b+c-1)\tanh^2(mr) \\
&\quad - \gamma k^2(\beta+\gamma-1)\tanh^2(kt) \qquad (8)\\
A^2 R_{(44)} &= -2am^2 + k^2(\alpha+\beta+\gamma) - am^2(b+c-1)\tanh^2(mr) \\
&\quad - k^2[\alpha+\beta+\gamma+\alpha(\beta+\gamma) - \beta^2 - \gamma^2]\tanh^2(kt) \qquad (9)\\
A^2 R_{(14)} &= mk(\gamma-\alpha)\tanh(kt)\operatorname{cotanh}(mr) \\
&\quad + mk[b(\beta-\alpha) + c(\gamma-\alpha) - a(\beta+\gamma)]\tanh(mr)\tanh(kt). \qquad (10)
\end{aligned}
$$

Here and in what follows

$$A^2 = \cosh^{2\alpha}(kt)\cosh^{2a}(mr) \tag{11}$$

and the bracketed indices are used to denote the tetrad components. The Einstein field equations (2) with T_{ik} as in (1) in the tetrad form are equivalent to

$$R_{(ij)} = -8\pi[\rho u_{(i)}u_{(j)} - \lambda w_{(i)}w_{(j)}] + 4\pi[\rho+\lambda]g_{(ij)}. \tag{12}$$

We adopt co-moving coordinates in which

$$u^{(i)} = (0,0,0,1). \tag{13}$$

We identify the string direction with z-axis which implies that

$$w^{(i)} = (0,1,0,0). \tag{14}$$

Subsequently eqs. (12) imply the following relations:

$$
\begin{aligned}
R_{(14)} &= 0, &(15)\\
R_{(11)} &= R_{(33)}, &(16)\\
R_{(22)} &= R_{(44)}, &(17)\\
8\pi\rho &= -R_{(11)} - R_{(22)}, &(18)\\
8\pi\lambda &= R_{(22)} - R_{(11)}, &(19)
\end{aligned}
$$

for the space-time metric (4).

The expansion scalar θ, the shear σ and the acceleration vector a_i for the velocity field u_i given by (13) have the explicit expressions

$$\begin{aligned}\theta &= k(\alpha+\beta+\gamma)\tanh(kt)/A,\\ a_i &= (-am\tanh(mr),0,0,0),\\ 9A^2\sigma^2 &= k^2[(\alpha+\beta-2\gamma)^2+(\beta+\gamma-2\alpha)\\ &\quad+(\gamma+\alpha-2\beta)^2]\tanh^2(kt).\end{aligned}\tag{20}$$

Here the coordinates are labelled as $x^1 = r, x^2 = z, x^3 = \theta$, and $x^4 = t$.

3. EXPLICIT SOLUTION

We will show here that the relations (15)–(17) between $R_{(ik)}$ —implied by the field equations (12) in the set-up under consideration—lead to a unique one-parameter class of cylindrically-symmetric models in string cosmology.

Equation (15), $R_{(14)} = 0$, leads to the relation

$$\alpha = \gamma, \tag{21a}$$

$$\beta(b-a) = \gamma(a+b). \tag{21b}$$

Subsequently eq. (16), $R_{(11)} = R_{(33)}$, is equivalent to

$$b^2 - ab - bc - ca - a - b = 0. \tag{22}$$

Equation (17), $R_{(22)} = R_{(44)}$, then implies the following relations:

$$\beta^2 - (\beta+\alpha) = 0, \tag{23a}$$

$$(a+b)(b+c-1) = 0, \tag{23b}$$

$$(a+b)m^2 = (\alpha+\beta)k^2. \tag{23c}$$

It follows from the above relations that a non-static class of solutions can arise only when $a+b \neq 0$ and the cylindrically-symmetric space-time of the metric (4) for this class of solution will have

$$c = 1-b, \qquad a = b(b-1), \tag{24a}$$

$$\alpha = \gamma = \frac{2(2-b)}{b^2}, \qquad \beta = \frac{2}{b}, \tag{24b}$$

$$\frac{k}{m} = \frac{b^2}{2}, \tag{24c}$$

where b is the class parameter. The physical variables associated with the cosmological string model of this class are

$$8\pi\rho = \frac{4k^2(b^2-4)}{b^4}\{\operatorname{sech}(kt)\}^{2(1-\alpha)}\{\operatorname{sech}(mr)\}^{-2a}, \tag{25}$$

$$8\pi\lambda = \frac{4k^2(b^2-4)(1-b)}{b^4}\{\operatorname{sech}(kt)\}^{2(1-\alpha)}\{\operatorname{sech}(mr)\}^{-2a}, \tag{26}$$

$$8\pi\rho_{\mathrm{P}} = \frac{4k^2(b^2-4)}{b^3}\{\operatorname{sech}(kt)\}^{2(1-\alpha)}\{\operatorname{sech}(mr)\}^{-2a}. \tag{27}$$

The kinematical variables associated with this class of solutions have the following explicit expressions:

$$\theta = \frac{2k(4-b)}{b^2}\tanh(kt)\{\cosh(kt)\}^{-\alpha}\{\cosh(mr)\}^{-a}, \tag{28}$$

$$\sigma^2 = \frac{16k^2(b-1)^2}{3b^4}\{\tanh(kt)\}^2\{\cosh(kt)\}^{-2\alpha}\{\cosh(mr)\}^{-2a}, \tag{29}$$

$$a_i = (b(1-b)\tanh(mr), 0, 0, 0). \tag{30}$$

4. DISCUSSION

The energy conditions imply that $\rho > 0$ and $\rho_{\mathrm{P}} > 0$, leaving the sign of the string tension density unrestricted. For the one-parameter class of cosmological string dust solutions obtained above ρ and ρ_{P} are non-negative for $b > 2$. If the space-time of the class of solutions is to represent an expanding universe for $t > 0$ the parameter b is further restricted to comply with $b < 4$ which ensures $\theta > 0$. The values of the other parameters are subsequently found to be in the respective ranges, so that

$$2 < a < 12, \qquad 2 < b < 4, \qquad -3 < c < -1$$
$$-\tfrac{1}{4} < \alpha = \gamma < 0 \qquad \text{and} \qquad \tfrac{1}{2} < \beta < 1.$$

Further, $2 < k/m < 8$.

The energy density ρ, string tension density λ and particle density ρ_{P} at all finite spatial locations tend to zero as time t increases indefinitely. These variables diverge as the radial variable r increases indefinitely. The kinematical variables θ and σ^2 on the other hand diverge as $t \longrightarrow \infty$ and vanish as $r \longrightarrow \infty$.

For $b = 2$ (as well as $b = -2$), $\rho = \rho_{\mathrm{P}} = \lambda = 0$ and this class of solutions degenerates into the singularity-free vacuum solutions of cylindrical symmetry of Patel and Dadhich [15] which are the matter-free limit of this class of solutions. The string direction w^i is aligned with the z-axis which is the axis of symmetry of the space-time.

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