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TWO-DIMENSIONAL MODAL LOGIC*

I. INTRODUCTION

This paper arose in response to a problem posed, in conversation, by Lennart Åqvist. Working with ordinary Kripke-type semantics for propositional logic, Åqvist wanted to add to his language two new operators **O** and **O**′ of an unusual kind. What made them unusual was that formulas in which they occurred could not be evaluated in a model at one point, as is usually done: such formulas would have to be evaluated at two points simultaneously. Suppose we are dealing with a certain model, and let us write, with reference to that model, $W(\mathbf{A}, x, y)$ for the tentative notion "The formula **A** is true at the point x with respect to the point y". The semantical conditions Åqvist wanted to impose on **O** and **O**′ were these:

$$W(\mathbf{OA}, x, y) \quad \text{iff} \quad W(\mathbf{A}, y, y);$$
$$W(\mathbf{O'A}, x, y) \quad \text{iff} \quad W(\mathbf{A}, y, x).$$

On the other hand, for a formula **B** not containing **O** or **O**′ we would have

$$W(\mathbf{B}, x, y) \quad \text{iff} \quad W(\mathbf{B}, x, z).$$

O and **O**′ are special cases of an operator **C** that satisfies, for some functions f and g, the condition

$$W(\mathbf{CA}, x, y) \quad \text{iff} \quad W(\mathbf{A}, f(x, y), g(x, y)).$$

Such operators provide simple instances of what may be termed 'two-dimensional' operators. More complicated examples are easily found, for instance,

$$W(\mathbf{C'A}, x, y) \quad \text{iff} \quad x = y;$$
$$W(\mathbf{C''}(\mathbf{A}, \mathbf{B}), x, y) \quad \text{iff} \quad \text{for all } z \text{, if } W(\mathbf{A}, x, z) \quad \text{then}$$
$$W(\mathbf{B}, z, y).$$

Are such operators of any philosophical interest? It has been argued that

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at least some new operators of that vaguely delineated category are needed if one wants to give an account, within a post-Kripkean framework, of certain aspects of English. That thesis will not be defended here, but we shall offer one example to illustrate the alleged need.

Consider the sentence

> *It has always been the case that there will be a sea battle tomorrow.*

Here we shall take it for granted that to assert this sentence is not to assert that it was always true in the past that there would be a sea battle on the following day (and hence, since time has no beginning, that every day a sea battle has taken place). According to the reading we recognize, the word *tomorrow* refers to the day after the day the sentence is uttered; so in order to decide whether the sentence is uttered truthfully it is important to know when it is uttered. In fact, unless the sentence is regarded as uttered at a certain time, or is associated in some other way with a particular time, its truth-value at a particular time cannot be determined. To bring this point out more clearly, let us think of time as the set of integers under the natural ordering. Letting x, y, z range over the set of integers, we now understand $W(\mathbf{A}, x, y)$ to mean "The sentence **A**, regarded as being uttered at y, is true at x".

Consider the operators defined as follows:

$$W(\mathbf{HA}, x, y) \quad \text{iff} \quad \text{for all } z < x,\ W(\mathbf{A}, z, y).$$
$$W(\mathbf{SA}, x, y) \quad \text{iff} \quad W(\mathbf{A}, x+1, y).$$
$$W(\mathbf{TA}, x, y) \quad \text{iff} \quad W(\mathbf{A}, y+1, y).$$

If **A** is a particular sentence, then **HA**, **SA**, and **TA** may perhaps be said to formalize, respectively,

> *Every day in the past*, **A**;
> *On the following day*, **A**;
> *Tomorrow*, **A**.

Essentially, **H** is the well known operator introduced by A. N. Prior, and S, discussed for example in [5], is closely related to G. H. von Wright's binary "and-next" operator. **T**, however, belongs to the new category. Now, it is easily seen that

> **HS** (*a sea battle takes place*)

cannot be regarded as a formalization of the sentence under analysis – it yields the reading ruled out above as inadmissible. But

HT (*a sea battle takes place*)

is a rather good approximation to the given sentence.

Crude as it is, this analysis shows why in 'two-dimensional' modal logic one wants to evaluate formulas at two points: at a point x, with respect to a point y. Of course, he who insists that formulas be evaluated at one point only will achieve the same result by evaluating at points of type $\langle x, y\rangle$, thus endowing points with a structure.

The history of the ideas expressed in the preceding paragraphs is not clear to the author. First to study operators of type **O** were probably Hans Kamp and A. N. Prior; our discussion of *tomorrow* was inspired by their treatment of *now* in [2] and [4]. Also of interest in this connection is David Lewis's discussion of *actual* in [3]. As far as the author knows, the more complicated operator **O′** is new with Åqvist. However, it is worth quoting the anonymous referee who remarked on the penultimate version of this paper that "insofar as there is any one inventor of two-dimensional modal logic, it is Frank Vlach". When Vlach's U.C.L.A. thesis becomes available it will be possible to evaluate this claim.

The topic of this paper is Åqvist's problem, by which is meant the problem of finding a way to handle operators of type **O′** within the framework of Kripke semantics. Before tackling this problem it is well to note one peculiarity of Åqvist's new logic which is shared by few other systems: it is not closed under substitutivity. For example, whereas the formula

$$\mathbf{Op} \leftrightarrow \mathbf{O'p}$$

is valid, for each propositional letter **p**, the formula

$$\mathbf{OOp} \leftrightarrow \mathbf{O'Op}$$

is not. This fact already may be an indication that Åqvist's problem has not yet been given a sufficiently general formulation. A moment's reflection will bear out the correctness of this observation. Naïvely we may think of an "Åqvist model" in the following way: its frame is made up of identical copies of an ordinary frame which are piled up on top of each other, with one particular element singled out on each level; and its valuation function treats each point the same regardless of its level.

Clearly the valuation condition is not an essential ingredient of the formal problem, and we shall dispense with it in the general treatment below.

We shall now first formulate in exact terms a semantics of the sort outlined. Then – and this is the major effort of the paper – we shall axiomatize the resulting logic. At the end of the paper we shall return to Åqvist's logic to give a brief examination of it in the light of the general situation.

II. SEMANTICS AND SYNTAX

Suppose that U is a set and that X and Y are functions defined on U. We shall say that $\langle U, X, Y\rangle$ is a *frame* if the following conditions are satisfied:

(i) For all $u, v \in U$, if $Xu = Xv$ and $Yu = Yv$ then $u = v$.

(ii) For all $u, v \in U$ there is some $w \in U$ such that $Xu = Xw$ and $Yv = Yw$.

(iii) For each $u \in U$ there is some $v \in U$ such that $Xv = Yv = Xu$.

(iv) For each $u \in U$ there is some $v \in U$ such that $Xv = Yv = Yu$.

(v) For each $u \in U$ there is some $v \in U$ such that $Xu = Yv$ and $Yu = Xv$.

For u an element of U, Xu is the *X-coordinate* of u and Yu is the *Y-coordinate* of u, and we shall say that (Xu, Yu) are the coordinates of u. The import of (i) is that every element of U is uniquely determined by its coordinates; in particular, the elements the existence of which is claimed in (ii)–(v) are unique.

V is defined to be a *valuation* in $\langle U, X, Y\rangle$ if V is a function defined on *Nat* (the set of natural numbers including 0), taking values in $\mathfrak{P}U$ (the power set of U). We call $\langle U, V, X, Y\rangle$ a *model*.

Obviously, the present concept of frame is not the only possible one of its kind. Condition (i) is what makes the whole enterprise interesting, and the completeness proof below will show that there would be no point relaxing it. Conditions (ii)–(v) have a different status: one might want to drop any combination of them, and if this is done a slightly weaker logic may result. They are adopted here both because they seem natural and because they are motivated by Åqvist's ideas. Moreover, the tiny bit of generality gained by omitting (ii)–(v) from the definition of frame has to be paid by an increase of tedious detail, and at the present time the price does not appear worth paying.

There are various possibilities of constructing a language suitable for our new concept of frame. The one we adopt here has infinitely many propositional letters $\mathbf{P}_0$, $\mathbf{P}_1, \ldots, \mathbf{P}_k, \ldots$, some functionally complete set of Boolean connectives, and the modal operators $\Box$, $\boxbar$, $\boxminus$, $\varobar$, $\ominus$, and $\otimes$. The following readings of the latter are tentatively suggested:

$\Box$ **A** – everywhere, **A**.
$\boxbar$ **A** – everywhere on this longitude, **A**.
$\boxminus$ **A** – everywhere on this latitude, **A**.
$\varobar$ **A** – at the diagonal point on this longitude, **A**.
$\ominus$ **A** – at the diagonal point on this latitude, **A**.
$\otimes$ **A** – at the mirror point, **A**.

These readings are suggested as mnemonic devices only. Naturally, if one studies the above operators with some specific objective in mind, then very different readings may be chosen.

Suppose $\langle U, V, X, Y\rangle$ is a given model. We now define the concept of *truth at a point* in this model. Suppose $u \in U$. Whenever $\mathbf{P}_k$ is a propositional letter, then

$$\models_u \mathbf{P}_k \quad \text{iff} \quad u \in V(k).$$

If **A** is a Boolean compound the definition goes as usual. If **A** is of the form **oB** with **o** a modal operator, then we stipulate:

$\models_u \Box \mathbf{B}$ iff for all $v \in U$, $\models_v \mathbf{B}$.
$\models_u \boxbar \mathbf{B}$ iff for all $v \in U$ such that $Xu = Xv$, $\models_v \mathbf{B}$.
$\models_u \boxminus \mathbf{B}$ iff for all $v \in U$ such that $Yu = Yv$, $\models_v \mathbf{B}$.
$\models_u \varobar \mathbf{B}$ iff for that $v \in U$ such that $Xv = Yv = Xu$, $\models_v \mathbf{B}$.
$\models_u \ominus \mathbf{B}$ iff for that $v \in U$ such that $Xv = Yv = Yu$, $\models_v \mathbf{B}$.
$\models_u \otimes \mathbf{B}$ iff for that $v \in U$ such that $Xv = Yu$ and $Yv = Xu$, $\models_v \mathbf{B}$.

We read '$\models_u \mathbf{A}$' as '**A** is true at u' or '**A** holds at u'. Similarly, '$\not\models_u \mathbf{A}$' (the denial of $\models_u \mathbf{A}$) is read '**A** is false at u' or '**A** fails at u'. Note that the notion of truth at a point is relative to a model, even though we have not brought the reference to the model explicitly into the symbolism.

A formula is said to be true in a model if it is true at every point of the model. A formula is valid in a frame if it is true in every model definable on the frame. A valid formula is a formula valid in every frame.

We shall prove in the next section that the set of valid formulas is axiomatized by the following axiom system. There are two rules:

Modus Ponens. *From* **A** *and* **A → B**, *infer* **B**.
Universal Necessitation. *From* **A**, *infer* **□A**.

The axioms are divided into three groups. The first group is, simply, the set of all truth-functional tautologies. The second group is the set of all instances of the following schemata, where **o** is a parameter running over the six-element set of modal operators:

#1. **o(A → B) → (oA → oB)**.
#2. **oA → A**, *provided* **o** *is* □, ◫, *or* ⊟.
#3. **A → o¬o¬A**, *provided* **o** *is* □, ◫, *or* ⊟.
#4. **oA ↔ ¬o¬A**, *provided* **o** *is* ⦶, ⊖, *or* ⊗.
#5. **oA ↔ ooA**, *provided* **o** *is not* ⊗.
#6. **A ↔ ooA**, *provided* **o** *is* ⊗.

The third group is the set of instances of the following schemata, in which the modal operators are interrelated:

#7.	□A ↔ ◫⊟A;	□A ↔ ⊟◫A.	
#8.	◫A → ⦶A;	⊟A → ⊖A;	□A → ⊗A.
#9.	◫A ↔ ⦶◫A;	⊟A ↔ ⊖⊟A.	
#10.	⦶A ↔ ◫⦶A;	⊖A ↔ ⊟⊖A.	
#11.	⦶A ↔ ⦶⊖A;	⊖A ↔ ⊖⦶A.	
#12.	⦶A ↔ ⦶⊗A;	⊖A ↔ ⊖⊗A.	
#13.	⦶A ↔ ⊗⊖A;	⊖A ↔ ⊗⦶A.	
#14.	□(◫A ∨ ⊟B) → □A ∨ □B.		

We hasten to remark that this axiom system is not independent. Note that □, ◫, and ⊟ are *S5*-modalities, while ⦶, ⊖, and ⊗ are very strong *K*-modalities of rather unusual kinds (where *K* is the fundamental normal modal logic named in honor of Kripke).

By a logic we understand, in this paper, any set of formulas closed under modus ponens and universal necessitation. The set of formulas provable in the preceding axiom system is a logic which we shall call *B* (for 'basic'). As it happens, *B* is closed under substitution as well. However, it is not required that a logic is thus closed; this point becomes important in Section 8.

III. CANONICAL STRUCTURES

Throughout this section let L be any fixed logic at least as strong as B. Let U_L be the set of all maximal L-consistent sets of formulas. (A set u of formulas is maximal L-consistent if all of these conditions are met: (i) $L \subseteq u$; (ii) u is closed under modus ponens; (iii) for every formula $\mathbf{A}$, $\mathbf{A} \in u$ if and only if $\neg \mathbf{A} \notin u$.) For each $k \in Nat$ we define

$$V_L(k) = \{u \in U_L \colon \mathbf{P}_k \in u\}.$$

Furthermore we define, for $u \in U_L$,

$$X_L u = \{\mathbf{A} \colon \oplus \mathbf{A} \in u\};$$
$$Y_L u = \{\mathbf{A} \colon \ominus \mathbf{A} \in u\}.$$

We shall call $\mathfrak{S}_L = \langle U_L, V_L, X_L, Y_L \rangle$ the *canonical structure for L*. Throughout this section, let t be any fixed element of U_L. By the *canonical substructure for L generated by t* we shall understand the structure $\mathfrak{S}_L(t) = \langle U, V, X, Y \rangle$ where U is the set of elements $u \in U_L$ such that, for all formulas $\mathbf{A}$, if $\Box \mathbf{A} \in t$ then $\mathbf{A} \in u$; and where V, X, and Y are the restrictions to U, in the appropriate senses, of V_L, X_L, and Y_L, respectively. It is worth observing that, for all u, $v \in U$ and every formula $\mathbf{A}$, $\Box \mathbf{A} \in u$ iff $\Box \mathbf{A} \in v$. Hence if $\mathbf{B} \in u$, for some $u \in U$, then $\neg \Box \neg \mathbf{B} \in v$, for every $v \in U$.

We introduce the following auxiliary definition: if $u \in U_L$ then

$$\tilde{u} = \{\mathbf{A} \colon \otimes \mathbf{A} \in u\}.$$

One readily verifies – here #4 is essential – that $X_L u$, $Y_L u$, $\tilde{u} \in U_L$, for all $u \in U_L$. Hence, since $\Box$ is an $S5$-modality, ##7-9 imply that Xu, Yu, $\tilde{u} \in U$, for all $u \in U$. Using ##5, 6, 11, 13 one can show that, for all $u \in U$,

$$Xu = XXu = YXu = Y\tilde{u};$$
$$Yu = YYu = XYu = X\tilde{u};$$
$$u = \tilde{\tilde{u}}.$$

LEMMA 1. *Let u be any element of U. Then there exist elements of U having coordinates* (Xu, Xu), (Yu, Yu), *and* (Yu, Xu). *If v is also an element of U, then there exists an element of U having coordinates* (Xu, Yv).

Proof. The former part of the lemma is immediate from the remarks preceding it. To prove the latter part it will be enough to establish the consistency of the set

$$\Sigma = \{\mathbf{A}: \Box\mathbf{A} \in t\} \cup \{①\mathbf{B}: ①\mathbf{B} \in u\} \cup \{\ominus\mathbf{C}: \ominus\mathbf{C} \in v\}.$$

To do this, assume that Σ is inconsistent. Then there must be formulas **A**, **B**, **C** such that $\Box\mathbf{A}\in t$, $①\mathbf{B}\in u$, and $\ominus\mathbf{C}\in v$, and the formula

$$\mathbf{A} \wedge ①\mathbf{B} \wedge \ominus\mathbf{C} \rightarrow \bot$$

is derivable in L. By universal necessity, $①\mathbf{B} \wedge \ominus\mathbf{C} \notin w$, for every $w \in U$. Using # #10, 14 among the third group of axioms one may show that

$$\neg\Box\neg\mathbf{B} \wedge \neg\Box\neg\mathbf{C} \rightarrow \neg\Box\neg(①\mathbf{B} \wedge \ominus\mathbf{C})$$

is a theorem of L. Since u and v are elements of U, $\neg\Box\neg\mathbf{B}, \neg\Box\neg\mathbf{C} \in t$. Consequently,

$$\neg\Box\neg(①\mathbf{B} \wedge \ominus\mathbf{C}) \in t,$$

and consequently there must be some $w \in U$ such that $①\mathbf{B} \wedge \ominus\mathbf{C} \in w$. This is a contradiction, so Σ must be consistent.

IV. A DIGRESSION

According to Lemma 1, $\mathfrak{S}_L(t)$ satisfies four of the five conditions on modelhood. Some reflection will probably make it plausible to the reader that the remaining condition is not in general satisfied. This impression is correct, and it may be instructive to exhibit a case in which it is violated.

Consider the frame $\mathfrak{F} = \langle U, X, Y \rangle$ where

$$U = \{00, 01, 10, 11\};$$
$$X00 = X01 = Y00 = Y10 = 0;$$
$$X10 = X11 = Y01 = Y11 = 1.$$

Let V be the valuation such that, for every $k \in Nat$,

$$V(k) = \{00, 11\}.$$

Let $\mathfrak{M} = \langle U, V, X, Y \rangle$. Notice that for all **A**

$$\models_{00}\mathbf{A} \text{ iff } \models_{11}\mathbf{A}; \models_{01}\mathbf{A} \text{ iff } \models_{10}\mathbf{A}.$$

Define

$$\Sigma = \{\mathbf{A}: \mathbf{A} \text{ is true at } 00 \text{ in } \mathfrak{M}\};$$
$$\Sigma' = \{\mathbf{A}: \mathbf{A} \text{ is true at } 01 \text{ in } \mathfrak{M}\}.$$

Both Σ and Σ' are maximal B-consistent sets. Thus $\mathfrak{S}_B(\Sigma)$, in particular, is well defined. Which are its elements?

Suppose $u \in \mathfrak{S}_B(\Sigma)$ and $u \neq \Sigma$. Take any $\mathbf{A} \in u - \Sigma$ and any $\mathbf{B} \in \Sigma'$. It is easy to see that $\Box(\mathbf{A} \to \mathbf{B}) \in \Sigma$, so since $u \in \mathfrak{S}_B(\Sigma)$, $\mathbf{A} \to \mathbf{B} \in u$. It follows that $\mathbf{B} \in u$. This argument shows that $\Sigma' \subseteq u$. But Σ' is maximal, so $\Sigma' = u$. In other words, $\mathfrak{S}_B(\Sigma)$ contains exactly two elements, Σ and Σ'. Moreover, $\mathfrak{S}_B(\Sigma)$ is not a model, for

$$X\Sigma = X\Sigma',$$
$$Y\Sigma = Y\Sigma',$$

as one easily verifies.

If the reader is not familiar with the kind of Henkin proof we are building up to, he is advised at this point to go ahead and take a look at the proof of Theorem I. He will then realize that we should like to discover a way of transforming canonical substructures into models without changing them in any 'essential' way. For example, the model $\mathfrak{M}$ above is 'essentially' the same structure as $\mathfrak{S}_B(\Sigma)$; and given $\mathfrak{S}_B(\Sigma)$ and the problem of finding a model "essentially like it", $\mathfrak{M}$ is probably the first model that comes to mind. This is, of course, a very loose way of speaking, but it will give the reader a reasonably good idea of what we are trying to do in the pages that lie ahead.

What we shall do, then, is to define a procedure that can be applied to any canonical substructure and which eventually yields a model. Since finite structures are more easily handled than infinite ones, we shall begin by showing how any canonical substructure can be reduced to finite size by the filtration technique. The reduced structure will then be transformed into a model.

V. FILTRATIONS

Suppose, throughout this section, that $\mathfrak{S} = \langle U, V, X, Y \rangle$ is some fixed generated canonical substructure for L, where L is some logic at least as strong as B, also fixed throughout this section. Let Ψ be a set of formulas closed under subformulas as well as under $\oplus$, $\ominus$, and $\otimes$. (That a formula set Ξ is closed under an operator $\mathbf{o}$ means that, for every $\mathbf{A} \in \Xi$, $\mathbf{oA} \in \Xi$.) Then the following defines an equivalence relation in U:

$$u \equiv v \text{ if and only if } u \cap \Psi = v \cap \Psi.$$

In fact, $\equiv$ is a congruence relation with respect to X, Y, $\sim$, and $V(k)$, if $\mathbf{P}_k \in \Psi$. That is to say, if $u \equiv v$, then $Xu \equiv Xv$, $Yu \equiv Yv$, $\tilde{u} \equiv \tilde{v}$, and, if $\mathbf{P}_k \in \Psi$, $u \in V(k)$ if and only if $v \in V(k)$. Let $[u]$ denote the equivalence class under $\equiv$ of u. In view of what was just said the following definitions are meaningful:

$$U^\circ = \{[u] : u \in U\}.$$
$$V^\circ(k) = \{[u] \in U^\circ : u \in V(k) \text{ and } \mathbf{P}_k \in \Psi\}.$$
$$X^\circ[u] = [Xu].$$
$$Y^\circ[u] = [Yu].$$
$$\widetilde{[u]} = [\tilde{u}].$$

The structure $\mathfrak{S}^\circ = \langle U^\circ, V^\circ, X^\circ, Y^\circ \rangle$ is called the *filtration of* $\mathfrak{S}$ *through* Ψ. Note that the ranges of X°, Y°, and $\sim$ are included in U°, and that, for all $[u] \in U^\circ$,

$$X^\circ[u] = X^\circ X^\circ[u] = X^\circ Y^\circ[u] = Y^\circ\widetilde{[u]};$$
$$Y^\circ[u] = Y^\circ Y^\circ[u] = Y^\circ X^\circ[u] = X^\circ\widetilde{[u]};$$
$$[u] = \widetilde{\widetilde{[u]}}.$$

We shall say that a subset $K \subseteq U^\circ$ is a *column* in $\mathfrak{S}^\circ$ if there is some $[u] \in U^\circ$ such that

$$K = \{[w] : Xw \equiv Xu\}.$$

Similarly, $R \subseteq U^\circ$ is a *row* in $\mathfrak{S}^\circ$ if there is some $[v] \in U^\circ$ such that

$$R = \{[w] : Yw \equiv Yv\}.$$

If K is a column and R a row in $\mathfrak{S}^\circ$, then $C = K \cap R$ is a *cluster* in $\mathfrak{S}^\circ$. The notation

$$\mathit{col}\ C = K,$$
$$\mathit{row}\ C = R,$$

should be self-explanatory.

By Lemma 1, a cluster is always nonempty. Evidently, the set of clusters forms a partition of U°. It may be noted that $\mathfrak{S}^\circ$ would be a model if every cluster were a singleton.

If $S \subseteq U^\circ$, then $\tilde{S}$ will denote the set $\{\widetilde{[u]} : [u] \in S\}$. It is straightforward to verify that if S is a column (row) then $\tilde{S}$ is a row (column). Observe that for any $S, T \subseteq U^\circ$,

$$\breve{\overline{S \cap T}} = \breve{S} \cap \breve{T}.$$

Hence $\breve{C}$ is a cluster if C is, and

$$\begin{aligned} col\ \breve{C} &= \breve{\overline{row\ C}}; \\ row\ \breve{C} &= \breve{\overline{col\ C}}. \end{aligned}$$

We say that a cluster C is *on the diagonal* if $C = \breve{C}$, *off the diagonal* if $C \neq \breve{C}$.

LEMMA 2. *If C is on the diagonal, then there is exactly one element* $[u] \in C$ *such that* $[u] = X^\circ[u] = Y^\circ[u]$. *Furthermore,* $[\tilde{u}] = [u]$.

Proof. Take any element v such that $[v] \in C$. Since C is on the diagonal $[\tilde{v}] \in C$, so $Xv \equiv X\tilde{v}$. As $X\tilde{v} = Yv$ it follows that $Xv \equiv Yv$. Let

$$\Sigma = \{\mathbf{A}: \textcircled{\scriptsize |}\mathbf{A} \in v \cap \Psi\} \cup \{\mathbf{B}: \ominus\mathbf{B} \in v \cap \Psi\}.$$

Assume that Σ is inconsistent. Then there are formulas $\mathbf{A}$ and $\mathbf{B}$ such that $\textcircled{\scriptsize |}\mathbf{A} \in v \cap \Psi$ and $\ominus\mathbf{B} \in v \cap \Psi$ and $\mathbf{A} \wedge \mathbf{B} \rightarrow \perp$ is a theorem of L. Hence $\textcircled{\scriptsize |}\mathbf{A} \wedge \textcircled{\scriptsize |}\mathbf{B} \rightarrow \perp$ is also a theorem of L. However, $Xv \equiv Yv$, so $\textcircled{\scriptsize |}\mathbf{B} \in v$. This contradicts the consistency of v. Therefore Σ must be consistent. Let u be any maximal L-consistent extension of Σ. Then $[u] \in C$ and $u \equiv Xu \equiv Yu$. That $[u]$ is unique among the elements of C with this property is obvious.

The proof that $u \equiv \tilde{u}$ goes as follows. Assume that $\mathbf{A} \in \Psi$. Then

$$\begin{aligned} \mathbf{A} \in \tilde{u} \quad &\text{iff} \quad \otimes\mathbf{A} \in u \\ &\text{iff} \quad \otimes\mathbf{A} \in Xu \quad \text{(since } \Psi \text{ is closed under } \otimes) \\ &\text{iff} \quad \textcircled{\scriptsize |}\otimes\mathbf{A} \in u \\ &\text{iff} \quad \textcircled{\scriptsize |}\mathbf{A} \in u \quad (\text{by } \#\ 12) \\ &\text{iff} \quad \mathbf{A} \in Xu \\ &\text{iff} \quad \mathbf{A} \in u. \end{aligned}$$

VI. RECTIFICATIONS

We begin this section by defining, for every even positive integer n, an $(n+1) \times (n+1)$ matrix

$$\Gamma_n = (\Gamma_n(p, q))_{0 \leq p, q \leq n}$$

of natural numbers. The definition is recursive. The base step is trivial:

$$\Gamma_0 = (0).$$

For the recursive step, assume that n is an even integer and that Γ_{n-2} has already been defined. Then $\Gamma_n(p, q)$ is defined by cases.

(a) If $p + q \leqq n - 2$, then $\Gamma_n(p, q) = \Gamma_{n-2}(p, q)$.

(b) If $p + q \geqq n + 2$, then $\Gamma_n(p, q) = \Gamma_{n-2}(p - 2, q - 2)$.

(c) If $p + q = n - 1$, then

$$\Gamma_n(p, q) = \begin{cases} n - 1, \text{ if } p \text{ is even;} \\ n, \text{ if } p \text{ is odd.} \end{cases}$$

(d) If $p + q = n + 1$, then

$$\Gamma_n(p, q) = \begin{cases} n - 1, \text{ if } q \text{ is even;} \\ n, \text{ if } q \text{ is odd.} \end{cases}$$

(e) If $p + q = n$, then

$$\Gamma_n(p, q) = \begin{cases} n - 2p, \text{ if } p < q \text{ and } p \text{ is even;} \\ n - 2p - 1, \text{ if } p < q \text{ and } p \text{ is odd;} \\ 0, \text{ if } p = q; \\ n - 2q, \text{ if } p > q \text{ and } q \text{ is odd;} \\ n - 2q - 1, \text{ if } p > q \text{ and } q \text{ is even.} \end{cases}$$

The definition is forbidding, but this is due to the difficulty of describing matrices rather than to any complexity inherent in the matrices defined. As a short-cut to understanding how the Γ_n: s are constructed it may suffice to scrutinize the first few members of the family:

$$\Gamma_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}; \quad \Gamma_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 0 & 4 & 1 & 3 \\ 1 & 3 & 0 & 4 & 2 \\ 4 & 2 & 3 & 0 & 1 \\ 3 & 4 & 1 & 2 & 0 \end{pmatrix};$$

$$\Gamma_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 0 & 4 & 1 & 6 & 3 & 5 \\ 1 & 3 & 0 & 5 & 2 & 6 & 4 \\ 4 & 2 & 6 & 0 & 5 & 1 & 3 \\ 3 & 5 & 1 & 6 & 0 & 4 & 2 \\ 6 & 4 & 5 & 2 & 3 & 0 & 1 \\ 5 & 6 & 3 & 4 & 1 & 2 & 0 \end{pmatrix}.$$

As the reader may verify, each Γ_n satisfies the following conditions:

(1) For all $i, p \leqq n$ there is some q such that $i = \Gamma_n(p, q)$.

(2) For all $i, q \leqq n$ there is some p such that $i = \Gamma_n(p, q)$.

(3) Suppose $p, q, r \leqq n$, with $r \neq 0$ and $\Gamma_n(p, q) = r$. Then

$$\Gamma_n(q, p) = \begin{cases} r + 1, \text{ if } r \text{ is odd;} \\ r - 1, \text{ if } r \text{ is even.} \end{cases}$$

We now return to the subject of structure transformations. It will be assumed, throughout the section, that L, Ψ, $\mathfrak{S}$, and $\mathfrak{S}^\circ$ are as in Section 5, with the exception that from now on we shall assume that Ψ is logically finite. It is important to observe that the latter assumption implies that U° is finite. Our next enterprise is to define a structure $\mathfrak{S}^*$, which will be a model closely related to $\mathfrak{S}^\circ$. The greatest difficulty is to define the domain of $\mathfrak{S}^*$.

Let m be the largest number of elements of any one cluster in $\mathfrak{S}^\circ$. For each cluster C we intend to define an enumeration

$$C(0), C(1), \ldots, C(2m-2)$$

of elements of C, in such a way that the following conditions are satisfied:

(1) Every element of C occurs somewhere in the enumeration (perhaps more than once).

(2) If C is on the diagonal, then

$$\overline{C(0)} = C(0),$$

and for every odd number $i < 2m - 1$,

$$\overline{C(i)} = C(i + 1).$$

(3) If C is off the diagonal, then, for every even number $i < 2m - 1$,

$$C(i) = C(i + 1),$$

and for all $i < 2m - 1$,

$$\overline{C(i)} = \breve{C}(i).$$

The definition goes as follows. First for each cluster in $\mathfrak{S}^\circ$ we fix upon an enumeration of its elements which is exhaustive and without repetitions, in such a way that the following conditions hold:

(i) If C is a cluster on the diagonal and $\xi_0, \ldots, \xi_{c-1}$ is the enumeration of it, then ξ_0 is the element whose existence is guaranteed by Lemma 2.

(ii) If C is a cluster off the diagonal and if $\eta_0, \ldots, \eta_{c-1}$ and $\zeta_0, \ldots \zeta_{c-1}$ are the enumerations of C and $\tilde{C}$, respectively, then, for every $i<c$, $\tilde{\eta}_i = \zeta_i$.

It is clear that such enumerations can be found.

Suppose now that C is any cluster, and let $c = |C|$ ($|C|$ is the cardinality of C). Suppose that $\xi_0, \ldots, \xi_{c-1}$ is the enumeration of it. If C is on the diagonal, then define, for all $i < 2m-1$:

$$C(i) = \begin{cases} \xi_0, \text{ if } i = 0; \\ \xi_t, \text{ if } i = 2t-1 \text{ and } 0 < t < c; \\ \tilde{\xi}_t, \text{ if } i = 2t \text{ and } 0 < t < c; \\ \xi_0, \text{ otherwise.} \end{cases}$$

If C is off the diagonal, then define, for all $i < 2m-1$:

$$C(i) = \begin{cases} \xi_t, \text{ if } i = 2t \text{ or } i = 2t+1, \text{ and } t < c; \\ \xi_0, \text{ if } i = 2t \text{ or } i = 2t+1, \text{ and } t \geqq c. \end{cases}$$

It is clear that this definition satisfies conditions (1)–(3) above.

We are now in a position to define $\mathfrak{S}^*$. To make notation easier we shall write Γ_Ψ for Γ_{2m-2} (recall that, given $\mathfrak{S}$, Ψ uniquely determines m). Define

$$U^* = \{\langle C, p, q\rangle : C \text{ is a cluster in } \mathfrak{S}^\circ, \text{ and } 0 \leqq p, q < 2m-1\}.$$

For each $k \in Nat$, define

$$V^*(k) = \{\langle C, p, q\rangle \in U^* : C(\Gamma_\Psi(p, q)) \in V^\circ(k)\}.$$

Finally, for each $\langle C, p, q\rangle \in U^*$ define

$X^*\langle C, p, q\rangle = \langle D, p, p\rangle$, where D is the cluster on the diagonal such that $col\ C = col\ D$;

$Y^*\langle C, p, q\rangle = \langle E, q, q\rangle$, where E is the cluster on the diagonal such that $row\ C = row\ E$.

To have a term for it, we shall call the structure $\mathfrak{S}^* = \langle U^*, V^*, X^*, Y^*\rangle$ the *rectification of* $\mathfrak{S}^\circ$. Note that X^* and Y^* are operations on U^* and that, for all $\langle C, p, q\rangle \in U^*$,

$$\begin{aligned} X^*\langle C, p, q\rangle &= X^*X^*\langle C, p, q\rangle = \\ &= Y^*X^*\langle C, p, q\rangle = Y^*\langle \tilde{C}, q, p\rangle; \\ Y^*\langle C, p, q\rangle &= Y^*Y^*\langle C, p, q\rangle = \\ &= X^*Y^*\langle C, p, q\rangle = X^*\langle \tilde{C}, q, p\rangle. \end{aligned}$$

It is a noteworthy feature of our construction that, for every $u \in U$, there are some non-negative $p, q < 2m-1$ such that $\langle C, p, q\rangle \in U^*$, where C is the cluster in $\mathfrak{S}^\circ$ of which $[u]$ is an element.

LEMMA 3. *$\mathfrak{S}^*$ is a model.*

Proof. We have to verify that U^*, X^*, and Y^* satisfy the five conditions (i)–(v) in the definition of frame. Conditions (iii)–(v) are automatic, given previous remarks, but we shall prove that (i) and (ii) hold.

Condition (i). Suppose $\langle C, p, q\rangle$ and $\langle C', p', q'\rangle$ are elements of U^* such that

(1) $\quad X^*\langle C, p, q\rangle = X^*\langle C', p', q'\rangle$;

(2) $\quad Y^*\langle C, p, q\rangle = Y^*\langle C', p', q'\rangle$.

(1) implies that there are clusters D and D' on the diagonal such that

(3) $\quad \langle D, p, p\rangle = \langle D', p', p'\rangle$;

(4) $\quad col\ C = col\ D$;

(5) $\quad col\ C' = col\ D'$.

From (3) it follows that

(6) $\quad D = D'$;

(7) $\quad p = p'$.

(4), (5), and (6) imply that

(8) $\quad col\ C = col\ C'$.

By a similar argument we may infer from (2) that

(9) $\quad q = q'$;

(10) $\quad row\ C = row\ C'$.

A cluster is completely determined by its column and row, so (8) and (10) yield

(11) $\quad C = C'$.

Hence, by (7), (9), and (11),

(12) $\langle C,p,q\rangle = \langle C', p', q'\rangle$,

which is the desired result.

Condition (ii). Assume that $\langle C, p, q\rangle$ and $\langle D, r, s\rangle$ are elements of U^*. Then C and D are clusters in $\mathfrak{S}^\circ$, so *col* $C \cap$ *row* D is also a cluster in $\mathfrak{S}^\circ$; call it E. Evidently $\langle E, p, s\rangle \in U^*$. We have *col* $C =$ *col* E and *row* $D =$ *row* E. Therefore, at once,

$$X^*\langle E, p, s\rangle = X^*\langle C, p, q\rangle;$$
$$Y^*\langle E, p, s\rangle = Y^*\langle D, r, s\rangle;$$

which is what we wanted.

We are now ready to prove the fundamental result.

LEMMA 4. *For every formula* $\mathbf{A} \in \Psi$ *and every element* $\langle C, p, q\rangle$ *of* U^*, *the following conditions are equivalent*:

(*i*) $\mathbf{A}$ *is true at* $\langle C, p, q\rangle$ *in* $\mathfrak{S}^*$.

(*ii*) *For each* $u \in C(\Gamma_\Psi(p, q))$, $\mathbf{A} \in u$.

Proof. We assume that $\mathbf{A}$ is a formula of Ψ and that $\langle C, p, q\rangle$ belongs to U^*. That the lemma holds if $\mathbf{A}$ is a propositional letter or a Boolean compound, is easily seen. Thus it will be enough to confine attention to the case when $\mathbf{A} = \mathbf{oB}$, for some modal operator $\mathbf{o}$. In fact we shall only exhibit the subcases when $\mathbf{o}$ is ◫, ⦶, or ⊗.

First suppose that $\mathbf{A} =$ ◫$\mathbf{B}$. Consider the following conditions:

(1) ◫$\mathbf{B}$ holds at $\langle C, p, q\rangle$.

(2) For every D such that *col* $C =$ *col* D and every $r < 2m - 1$, $\mathbf{B}$ holds at $\langle D, p, r\rangle$.

(3) For every D such that *col* $C =$ *col* D and every $r < 2m - 1$, if $v \in D(\Gamma_\Psi(p, r))$ then $\mathbf{B} \in v$.

(4) For every $u \in C(\Gamma_\Psi(p, q))$, ◫$\mathbf{B} \in u$.

We wish to show that (1) and (4) are equivalent. The equivalence of (1) and (2) is given by the truth definition, the equivalence of (2) and (3) by the induction hypothesis. Hence it is enough to prove that (3) and (4) are equivalent.

First assume that (4) holds. Take any D and r of the appropriate kind. Pick any $u \in C(\Gamma_\Psi(p, q))$ and $v \in D(\Gamma_\Psi(p, r))$. From the fact that *col* $C =$

$= col\ D$ it follows that $X^\circ[u] = X^\circ[v]$, so $Xu \equiv Xv$. Now,

$$\begin{array}{llll} ◫\mathbf{B} \in u & \text{implies} & ⦶◫\mathbf{B} \in u & (\text{by } \#9) \\ & \text{implies} & ◫\mathbf{B} \in Xu & \\ & \text{implies} & ◫\mathbf{B} \in Xv & (\text{since } ◫\mathbf{B} \in \Psi) \\ & \text{implies} & ⦶◫\mathbf{B} \in v & \\ & \text{implies} & ◫\mathbf{B} \in v & (\text{by } \#9) \\ & \text{implies} & \mathbf{B} \in v & (\text{by } \#2). \end{array}$$

Since $◫\mathbf{B} \in u$ is supposed to hold, $\mathbf{B} \in v$. Thus (3).

Conversely, assume that (4) does not hold. Take any $u \in C(\Gamma_\Psi(p, q))$. It will be enough to show that the set of formulas

$$\{\neg \mathbf{B}\} \cup \{⦶\mathbf{C}: ⦶\mathbf{C} \in u\}$$

is consistent. Suppose it is not. Then, for some $\mathbf{C}$ with $⦶\mathbf{C} \in u$, $⦶\mathbf{C} \rightarrow \mathbf{B}$ is derivable in L. Hence $◫⦶\mathbf{C} \rightarrow ◫\mathbf{B}$ is also derivable in L, whence by #10 $⦶\mathbf{C} \rightarrow ◫\mathbf{B}$ is derivable in L. Consequently, $◫\mathbf{B} \in u$. Since $◫\mathbf{B} \in \Psi$, condition (4) is satisfied, contrary to our assumption.

Next suppose that $\mathbf{A} = ⦶\mathbf{B}$. Consider the following conditions:

(1) $⦶\mathbf{B}$ holds at $\langle C, p, q \rangle$.

(2) $\mathbf{B}$ holds at $\langle D, p, p \rangle$, where $col\ C = col\ D$ and $D = \tilde{D}$.

(3) For each $v \in D(\Gamma_\Psi(p, p))$, $\mathbf{B} \in v$.

(4) For each $u \in C(\Gamma_\Psi(p, q))$, $⦶\mathbf{B} \in u$.

Again we want to show that (1) and (4) are equivalent, and again it suffices to show that (3) and (4) are equivalent. This time the proof is quite short. For let D be such as required. Take any $u \in C(\Gamma_\Psi(p, q))$ and $v \in D(\Gamma_\Psi(p, p))$. By the way U^* was constructed, $v \equiv Xv$. Furthermore, $col\ C = col\ D$, so $Xu \equiv Xv$. Hence $v \equiv Xu$. Consequently,

$$\begin{array}{lll} \mathbf{B} \in v & \text{iff} & \mathbf{B} \in Xu \\ & \text{iff} & ⦶\mathbf{B} \in u. \end{array}$$

Finally, suppose that $\mathbf{A} = \otimes\mathbf{B}$. Consider these conditions:

(1) $\otimes\mathbf{B}$ holds at $\langle C, p, q \rangle$.

(2) $\mathbf{B}$ holds at $\langle C, q, p \rangle$.

(3) For each $v \in C(\Gamma_\Psi(q, p))$, $\mathbf{B} \in v$.

(4) For each $u \in C(\Gamma_\Psi(p, q))$, $\otimes\mathbf{B} \in u$.

Again, and for the same reason, it will be enough to show that (3) and (4) are equivalent. Take any $u \in C(\Gamma_\Psi(p, q))$ and $v \in C(\Gamma_\Psi(q, p))$. By the way

U^* was constructed, $[u] = [\widetilde{v}]$. Hence $u \equiv \tilde{v}$. Since $\otimes \mathbf{B} \in \Psi$,

$$\begin{aligned} \mathbf{B} \in v \text{ iff } & \otimes \mathbf{B} \in \tilde{v} \\ \text{iff } & \otimes \mathbf{B} \in u. \end{aligned}$$

This ends our proof of Lemma 4.

VII. COMPLETENESS OF B

We are finally able to prove that *B* is complete with respect to our semantics.

THEOREM I. *Every valid formula is derivable in B.*

Proof. Suppose that A is a formula not derivable in *B*. We shall show that there is a model in which A is false somewhere. It follows from our assumption that $\{\neg \mathbf{A}\}$ is a consistent set, so there will be some $t \in U_B$ such that $\neg \mathbf{A} \in t$ (every consistent set can be extended to a maximal consistent set). Form the canonical substructure $\mathfrak{S}$ generated by t. Let Ψ be the closure under ⦶, $\ominus$, and $\otimes$ of the set of subformules of A. Let $\mathfrak{S}^\circ$ be the filtration of $\mathfrak{S}$through Ψ. Because of the reduction axioms # #5, 6, Ψ is logically finite, so $\mathfrak{S}^\circ$ is finite. The rectification $\mathfrak{S}^*$ of $\mathfrak{S}^\circ$ is therefore definable. $\mathfrak{S}^*$ is a model. Let C be the cluster of which $[t]$ is an element. It follows from the construction of $\mathfrak{S}^*$ that there exist p, $q \in Nat$ such that $[t] = C(\Gamma_\Psi(p, q))$. Since $\mathbf{A} \notin t$, Lemma 4 implies that A is false in $\mathfrak{S}^*$ at $\langle C, p, q \rangle$. Q.E.D.

As is usually the case with proofs of this sort, there is more information in the proof than spelled out by the theorem. Most important, the rejecting model $\mathfrak{S}^*$ (which of course varies with the formula to be rejected) is always finite. Actually it would not be difficult to compute an upper bound to the cardinality of U^*, given the number of subformulas of A. However, here we are content to state just this result:

COROLLARY. *B has the finite model property.*

Hence, since *B* is recursively enumerable:

THEOREM II. *B is decidable.*

VIII. OUR SOLUTION OF ÅQVIST'S PROBLEM

It is now possible to obtain a better picture of the logic contemplated by Åqvist. We see how his idea of evaluating formulas at a point with respect to a point is accommodated in our semantics by identifying each point with the ordered pair of its coordinates. If the operators **O** and **O′** are identified with our ⊖ and ⊗, respectively, then all we have to do in order to arrive at a definition of *Åqvist models* is to impose this one condition on our models $\langle U, V, X, Y\rangle$:

(vi) For all $u, v \in U$ and all $k \in Nat$, if $Xu = Xv$ then $u \in V(k)$ if and only if $v \in V(k)$.

Note that this is a condition on models, not on frames! This explains why Åqvist's logic is not closed under substitution.

In the completeness proof carried out above, the only condition on L was closure under modus ponens and universal generalization: nowhere was it assumed that L was closed under substitution. Hence the proof works for Åqvist's logic too. In fact, it is axiomatized by adding to B one new axiom schema:

#15. $\mathbf{P}_k \leftrightarrow ⊞\mathbf{P}_k$, *provided* $\mathbf{P}_k$ *is a propositional letter.*

We shall sketch a proof of this contention. Let us call the new axiomatization B'.

THEOREM III. *Every formula true in all Åqvist models is derivable in B'.*

Proof. Proceed as in the proof of Theorem I, with this exception: we add as a condition on Ψ that, for every propositional letter $\mathbf{P}_k$ if $\mathbf{P}_k \in \Psi$ then $⊞\mathbf{P}_k \in \Psi$. Even under the new condition Ψ is logically finite, so $\mathfrak{S}^*$ exists and is a model, and Lemma 4 holds. It only remains to prove that $\mathfrak{S}^*$ is an Åqvist model.

Suppose that

$$X^*\langle C, p, q\rangle = X^*\langle D, p, r\rangle,$$

and that for some $k \in Nat$,

$$\langle C, p, q\rangle \in V^*(k).$$

Take any elements $u \in C(\Gamma_\Psi(p, q))$ and $v \in D(\Gamma_\Psi(p, r))$. Since $V^*(k) \neq \emptyset$, it follows from the definition of V^* that $\mathbf{P}_k \in \Psi$. Therefore, as $\mathbf{P}_k$ holds at $\langle C, p, q\rangle$, Lemma 4 implies that $\mathbf{P}_k \in u$. By #15 – the new schema – $⊞\mathbf{P}_k \in u$. Hence, by #9, $⦶⊞\mathbf{P}_k \in u$, so $⊞\mathbf{P}_k \in Xu$. Since *col* $C =$ *col* D, $Xu \equiv Xv$. By the new condition on Ψ, since $\mathbf{P}_k \in \Psi$, also $⊞\mathbf{P}_k \in \Psi$. Therefore $⊞\mathbf{P}_k \in Xv$. Hence $⦶⊞\mathbf{P}_k \in v$, so, by #9 and #2, $\mathbf{P}_k \in v$. Applying Lemma 4 once more we conclude that $\mathbf{P}_k$ holds at $\langle D, p, r\rangle$. That is to say,

$$\langle D, p, r\rangle \in V^*(k).$$

COROLLARY. *B′ has the finite model property.*
THEOREM IV. *B′ is decidable.*

IX. POST SCRIPTUM

When this paper was almost completed, Lennart Åqvist published his own solution to what we have termed Åqvist's problem. His paper, an appendix to [1], is reprinted in the present issue of the *Journal of Philosophical Logic.*

A prima facie Åqvist's semantics may not appear very similar to ours. In effect the two are the same, however, even though in the present formulation Åqvist's semantics is less general than ours – not surprising as we consciously sought generality. Whereas no effort is made to prove Theorem IV, Åqvist gives a Henkin-type proof of Theorem III. The most interesting feature is that completeness of *B′* is proved with no appeal to filtrations, let alone rectifications. Åqvist exploits the fact that in the presence of #15 the canonical structure will satisfy the condition that each point be uniquely determined by its coordinates. Unfortunately there is no such fact to exploit in the general case. Thus, while Åqvist's strategy is simpler and more straightforward than ours, it does not achieve as much.

It would be interesting to know whether there are much shorter proofs of Theorems I and II than those we have presented here.

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NOTE

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