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Cosmology of General Relativity without Energy-Momentum Conservation

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A modified version of the field equations of general relativity is obtained on relaxing the covariant energy-momentum conservation condition. This introduces a single arbitrary constant and does not appear to upset the successes of general relativity in or outside cosmology. The matter-dominated cosmological model, based on the generalized field equations, is discussed. It is shown to provide more room for consistency with the observational data.

1. INTRODUCTION

In this paper we derive a modified form of Einstein's field equations, by relaxing the condition of covariant conservation of the energy-momentum tensor

$$T^{\mu\nu}{}_{;\mu} = 0, \tag{1}$$

and investigate the changes that are introduced in the standard matter-dominated cosmological model by this modification. Our original motivation in considering this modification has been the desire to seek a solution to the cosmological entropy problem within standard Friedmann–Robertson–Walker cosmology. This problem, which is not the subject of the present paper, follows from the constancy of entropy during cosmic evolution. It is a puzzle that the universe initially possesses the large entropy that we presently observe in the thermal background radiation. In a

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homogeneous Robertson–Walker (RW) model with a perfect-fluid energy-momentum tensor, the constancy of entropy is an immediate consequence of condition (1). A solution to this puzzle that does not assume a specified time-dependent cosmological constant [1–4] or introduce an extraneous inflaton field [5] of arbitrary potential, may therefore be sought within a modified form of general relativity (MGR) in which the condition (1) is relaxed.

We find that it is indeed possible to relax the restriction (1) without upsetting the successes of general relativity (GR)* either in or outside cosmology. It does, therefore, appear that the covariant conservation condition (1) has not been specifically tested by observation. The construction of a theory in which (1) does not necessarily hold would thus provide an opportunity for testing this condition. In fact the relaxation of this condition introduces a single arbitrary constant, η. The restriction (1) requires $\eta = 1$. Our investigation of the matter-dominated cosmological model, based on MGR and the homogeneous Robertson–Walker metric indicates that $\eta \neq 1$.

It should be stressed that our approach to the derivation of the modified field equations is the conventional heuristic and intuitive approach [6,7]. We simply drop the requirement (1) for $T_{\mu\nu}{}^{(\mathrm{matter})}$, and impose the other criteria for the field equations. No attempt is made to construct the invariant action that would yield the modified field equations from a variational principle. This is obviously desirable, but it is not our concern in the present paper, which is mainly concerned with the cosmological consequences of the modified field equations.

The suggestion that cosmological considerations may require the covariant conservation condition (1) to be relaxed for $T_{\mu\nu}{}^{(\mathrm{matter})}$ has previously been advanced in various forms; notably in the work on variable-Λ cosmology [1–4] to solve the entropy problem, and also [8] to resolve the horizon problem of the standard model. One usually adopts the interpretation that there is a covariantly conserved $T_{\mu\nu}{}^{(\mathrm{universe})} \equiv T_{\mu\nu}{}^{(\mathrm{matter})} +$ (extra piece), which is to be used in the standard field equations. Such an interpretation is, of course, also possible in the present work. It obviously does not affect the mathematical structure and is of no significant consequence. We find it more appealing to adopt the simple attitude that MGR is a classical theory of gravitation that involves two independent constants, which appear to be equally fundamental. One of these constants does not survive in the Newtonian limit and seems to be operative in non-Newtonian

* Editor's note: It should be mentioned that the field equations discussed here do *not* follow from a variational principle as GR does.

contexts. It would, therefore, be appropriate to seek its determination in the cosmological domain.

The results of our work are summarized in Section 4. The main cosmological consequence is a new relationship between the age of the universe $t_{\rm p}$ and the present value $H_{\rm p}$ of Hubble's constant,

$$t_{\rm p}H_{\rm p} = \frac{2}{2+\eta}\left(\frac{\eta}{\Omega_{\rm p}}\right)^{1/2} F\left(\frac{1}{2}, \frac{1}{2}+\frac{1}{\eta}; \frac{3}{2}+\frac{1}{\eta}; 1-\frac{\eta}{\Omega_{\rm p}}\right), \tag{2}$$

where F is the hypergeometric function, $\Omega_{\rm p}$ the density parameter and η the new constant. This relation widens the margin for agreement with observational data and provides, in principle, a basis for the determination of the constant η.

The arrangement of the paper is as follows. In Section 2 we derive the modified field equations. In Section 3 we discuss some of the immediate consequences of the modified field equations. In Section 4 the cosmological model based on these equations is discussed. A summary of the results and concluding remarks are presented in Section 5.

We show elsewhere [9] that the proposed cosmological model is consistent with the recent data on Hubble's constant [10], does not lead to an increase in the probability of gravitational lensing and yields $\Omega_{\rm B} > 0.03$, where $\Omega_{\rm B}$ is the baryon contribution to $\Omega_{\rm p}$, in agreement with recent analysis [11] of light element abundances.

After the completion and limited preprint distribution of this work, it was brought to our attention that our modified field equations, eqs. (8), had already been obtained by Rastall [12], on the assumption that condition (1) be replaced by

$$T^{\mu}{}_{\nu;\mu} = \lambda R_{,\nu}\,, \tag{3}$$

where λ is a constant. The motivation of Rastall is that the assumptions that lead to (1) are all questionable, including the principle of equivalence.

Our derivation of the modified field equations in Section 2 may be considered an alternative approach to these equations. It yields (3) and does not assume it. The rest of our paper is different. Reference 12 does not discuss the cosmological consequences of the modified field equations.

2. GENERAL RELATIVITY WITHOUT THE CONSERVATION CONDITION

We derive the modified field equations using the conventional approach [6,7] except that we do not impose the covariant conservation condition. The field equations are

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}\,, \tag{4}$$

where $G_{\mu\nu}$ is the gravitational tensor to be determined. The requirement that $G_{\mu\nu}$ contains only terms that are linear in the second, or quadratic in the first, derivatives of the metric tensor restricts $G_{\mu\nu}$ to the form

$$G_{\mu\nu} = \alpha R_{\mu\nu} + \beta R g_{\mu\nu}\,, \tag{5}$$

where α and β are constants. Then one requires that the non-relativistic Newtonian equation

$$\nabla^2 g_{00} = -8\pi G T_{00} \tag{6}$$

be obtained in the limit of the weak stationary field. This imposes the condition

$$\beta = \frac{\alpha(\alpha-2)}{2(3-2\alpha)}\,, \qquad \alpha \neq 0, \frac{3}{2}\,. \tag{7}$$

When this is substituted into (5) and (4), one obtains the modified field equations

$$R_{\mu\nu} - \tfrac{1}{2}\gamma R g_{\mu\nu} = -k T_{\mu\nu}\,, \tag{8}$$

where

$$\gamma = \frac{2-\alpha}{3-2\alpha}\,, \qquad k = \frac{8\pi G}{\alpha}\,. \tag{9}$$

In these equations the constant α is an arbitrary parameter. Standard GR has $\alpha = 1$.

Since the conservation condition (1) has not been explicitly imposed, it will obviously not be automatically satisfied. In fact one finds

$$T^{\mu}{}_{\nu;\mu} = \frac{\gamma-1}{2k} R_{,\nu}\,, \tag{10}$$

or, using $R = k(2\gamma-1)^{-1}T$,

$$T^{\mu}{}_{\nu;\mu} = \frac{-1}{2}(1-\alpha)T_{,\nu} \tag{11}$$

where $T = g_{\mu\nu}T^{\mu\nu}$. We shall assume that α is a universal gravitational parameter, i.e. the same for all physical systems, and that $\alpha \neq 1$. Nevertheless it would still be possible for some systems to satisfy the covariant conservation condition (1). Equation (11) then shows that such systems are characterized by constant T. The constant value must then be zero, if one requires that $T_{\mu\nu} = 0$ at spatial infinity, i.e.,

$$T^{\mu\nu}{}_{;\mu} = 0 \implies T = 0\,. \tag{12}$$

Thus, a viable covariantly conserved system would necessarily possess a traceless energy-momentum tensor. This conclusion, which requires $\alpha \neq 1$, is a remarkable consequence of MGR. There is in flat space no connection between the conservation and tracelessness conditions.

The important case of the pure electromagnetic field is an example of this result. In this case

$$T^{\mu\nu} = \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - g^{\mu\alpha} F^{\nu\beta} F_{\alpha\beta} \,, \tag{13}$$

and $T^{\mu\nu}{}_{,\mu} = 0$. The generalization of this to $T^{\mu\nu}{}_{;\mu} = 0$ requires $g_{\mu\nu} T^{\mu\nu} = 0$ in curved space-time. Since this is already satisfied in flat space, both conditions may consistently be transformed from flat to curved space-time without imposing any additional constraint on $T^{\mu\nu}$.

Conversely, one deduces from eq. (11) that MGR coincides with standard GR for all systems with a traceless energy-momentum tensor. For such systems $R = 0$ so that

$$R_{\mu\nu} = -k T_{\mu\nu} \,. \tag{14}$$

The only difference between this and the corresponding equation of GR is the replacement of G by G/α on the right-hand side. Any observable consequences of this change should lead to a determination of α. For the change to be possible, $\alpha > 0$ is required.

As previously stated we consider MGR to be a theory with two constants: k and α, or equivalently G and α. The Newtonian limit shows that αk, or G, is the universal gravitational constant. Equation (11) indicates a universal rate of energy dissipation, or energy generation, for matter systems in interaction with the gravitational field. The universal constant may thus be determined by studying (11) in simple systems for which this equation is non-trivial.

It is also possible to write the field equations (8) in the conventional form

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = -k \theta_{\mu\nu} \,, \tag{15}$$

where

$$\theta_{\mu\nu} = T_{\mu\nu} + \tfrac{1}{2}(1 - \alpha) T g_{\mu\nu} \,. \tag{16}$$

One notes that $\theta_{\mu\nu}$ is completely determined by $T_{\mu\nu}$ and vanishes in the absence of matter. This form of the field equations shows that one could equivalently regard MGR as a modification of $T_{\mu\nu}$, within standard GR, by the addition of a piece representing a matter-gravitation interaction term that involves the new universal constant α. The total energy-momentum tensor, $\theta_{\mu\nu}$, is covariantly conserved. This point of view is always tenable and does not affect any of the formal or physical consequences.

3. CONSEQUENCES OF THE FIELD EQUATIONS

3.1. Empty space

In empty space the field equations reduce to those of standard GR

$$R_{\mu\nu} = 0 . \tag{17}$$

This is extremely significant on two accounts. The first is that one is allowed to keep, in MGR, the extensive literature [13] that exists on the classes of exact solutions of Einstein's vacuum field equations of various types and symmetries. The second is that some of these solutions, such as the Schwarzschild and Kerr solutions, have been widely used to explore the most important consequences of GR. In particular, one notes that the *crucial tests* of the perihelion of Mercury, the deflection of light, the gravitational red shift and the delay in radar echoes, as well as the demonstration of the existence of black holes, are all based on the Schwarzschild solution. The physics of rotating black holes is described by the Kerr solution. Thus all these features, which are characteristic of standard GR, are maintained in MGR.

3.2. Radiation

As previously remarked, for systems with traceless energy-momentum tensor, the form of the field equations of GR and MGR are the same except for the replacement of G by G/α. These include Einstein–Maxwell fields for which many exact solutions exist [13]. They also include general radiation and highly relativistic thermodynamic systems. Thus the modified and the standard field equations are essentially equivalent in their description of the pure radiation era in cosmology. It follows that, even with the proposed modification, entropy could not have been generated during the pure radiation era. One must therefore associate the generation of entropy with the advent of massive particles , i.e., with the phase transitions that create mass. It thus seems that the successes of the standard model in early cosmology are maintained while deviations are provided for, where the standard model is expected to be inadequate.

3.3. Perfect fluid solutions

In a homogeneous RW space-time, an energy-momentum tensor of the form

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} , \tag{18}$$

where $\rho = \rho(t)$, $p = p(t)$ and $u^2 = -1$, will be designated "of perfect fluid type" (PFT). To represent a physical perfect fluid, $T_{\mu\nu}$ must also satisfy certain other conditions, such as positive energy density and a physically acceptable equation of state, which are not necessary for our definition of PFT.

We now observe that eq. (16) shows that if $T_{\mu\nu}$ is PFT then so is $\theta_{\mu\nu}$ and vice versa. But $\theta_{\mu\nu}$ is covariantly conserved and satisfies the field equations (15) of standard GR. There exist many exact solutions of eqs. (15) when $\theta_{\mu\nu}$ is PFT [13]. These may therefore be utilized to yield exact solutions to the modified field equations (8) when $T_{\mu\nu}$ is given by eq. (18) for certain equations of state $p = p(\rho)$.

The procedure is to transform eq. (18) into

$$\begin{aligned} \theta_{\mu\nu} &= (\rho_\theta + p_\theta)u_\mu u_\nu + p_\theta g_{\mu\nu}\,, \\ \rho_\theta &= \tfrac{1}{2}(3-\alpha)\rho - \tfrac{3}{2}(1-\alpha)P, \\ P_\theta &= \tfrac{1}{2}(5-3\alpha)P - \tfrac{1}{2}(1-\alpha)\rho, \end{aligned} \tag{19}$$

and the equation of state $p = p(\rho)$ into the constraint $P_\theta = P_\theta(\rho_\theta)$. If, under this constraint, an exact solution to eqs. (16) and (19) exists, one would have a corresponding exact solution to eqs. (8) and (18) under the equation of state $p = p(\rho)$. One should note that the constraint $P_\theta = P_\theta(\rho_\theta)$ need not be a physically acceptable equation of state since many exact PFT solutions of GR extend beyond physically admissible regions and may still be formally used to obtain exact PFT solutions of MGR.

As an example, suppose we seek a solution to the field equations (8) for a dust universe with flat RW metric and

$$T_{\mu\nu} = \rho u_\mu u_\nu\,, \tag{20}$$

where $\rho = \rho(t) > 0$. From eqs. (16) and (19) this corresponds to $\rho_\theta = \frac{1}{2}(3-\alpha)\rho$ and

$$p_\theta = -\frac{1-\alpha}{3-\alpha}\,\rho_\theta\,. \tag{21}$$

Although $p_\theta < 0$, for $0 < \alpha < 1$, there exists an exact solution to eqs. (15), (19) and (21) in a flat RW metric with scale factor $a(t)$, namely

$$a(t) = \left(\frac{2}{3-\alpha}\,\frac{t}{t_0}\right)^{1-\alpha/3}, \tag{22}$$

and

$$\rho_\theta = \frac{\alpha(3-\alpha)^2}{24\pi G}\,\frac{1}{t^2}\,, \tag{23}$$

where t_0 is an arbitrary constant. This yields a solution to eqs. (8) and (20) with

$$\rho = \frac{\alpha(3-\alpha)}{12\pi G}\,\frac{1}{t^2} \tag{24}$$

and the same scale factor (22), since the RW metric is the same. Setting $\alpha = 1$ in (22) and (24) one obtains the well-known solution for the dust or matter-dominated universe of standard GR.

We make two observations on eqs. (22) and (24), that pave the way to our discussion of the cosmological model in MGR in Section 4.

(a) Equation (22) changes the relation between the present age of the universe, $t_{\rm p}$, and the present value of Hubble's constant, $H_{\rm p}$, from $t_{\rm p} = \frac{2}{3}H_{\rm p}^{-1}$ in standard GR to $t_{\rm p} = (1 - \alpha/3)H_{\rm p}^{-1}$ in MGR. For a given observational value of $H_{\rm p}$, this increases the estimate of the age of the universe by a factor of $(3 - \alpha)/2$ where $0 < \alpha < 1$.

(b) From eqs. (22) and (24) one deduces the expected energy non-conservation. The energy of non-relativistic matter increases as the universe expands since $\rho a^3 \sim a^s$ where $s = 3(1 - \alpha)(3 - \alpha)^{-1}$. This indicates the continuous transformation of gravitational energy into dust, with associated generation of entropy which may also be estimated.

Finally we mention an important consequence of eqs. (15) and (16). The proofs of the classical singularity theorems [14] in GR will hold for MGR with the usual energy conditions imposed on $\theta_{\mu\nu}$ and then transformed to $T_{\mu\nu}$. The condition that $(\theta_{\mu\nu} - \frac{1}{2}\theta g_{\mu\nu})t^\mu t^\nu \geq 0$, for every time-like vector t^μ, yields $(T_{\mu\nu} - \frac{1}{2}(2 - \alpha)T g_{\mu\nu})t^\mu t^\nu \geq 0$. When $T_{\mu\nu}$ is PFT this condition requires

$$\rho + p \geq 0, \qquad \rho + \frac{3}{\alpha}(2 - \alpha)p \geq 0, \tag{25}$$

which reduce to the usual conditions in GR when $\alpha = 1$. For $\rho > 0$, violation of (25) is not likely to occur with any reasonable equation of state.

3.4. Built-in cosmological constant

It is well known that the addition of a term $\Lambda g_{\mu\nu}$, Λ constant, to the left hand side of Einstein's field equations is compatible with the conservation constraint (1). It does, however, violate the condition on the Newtonian limit and leads to residual vacuum curvature. In a homogeneous RW universe, generation of entropy requires that Λ be time-dependent [1] so that (1) is violated. A number of variable-Λ cosmological models exist in the literature [1–4]. The field equations of MGR may, in fact, be written in a form that exhibits a time-dependent cosmological term and presents the theory as a variable-Λ model:

$$R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = -kT_{\mu\nu}\,, \tag{26}$$

$$\Lambda = \frac{1}{2}\,\frac{1 - \alpha}{3 - 2\alpha}\,R. \tag{27}$$

In this respect one observes that

(a) $\Lambda(t) \to 0$ as $t \to \infty$ in a perfect-fluid RW universe, i.e. one has a decaying-Λ cosmology.

(b) $\Lambda \equiv 0$ in vacuum, so that no additional Λ-force occurs in the vacuum solutions.

(c) $\Lambda \equiv 0$ for traceless $T_{\mu\nu}$, so that exponential de Sitter or inflationary expansion cannot arise during the pure-radiation era. Thus, if desired, exponential inflation can only occur in a pre-radiation period. This is consistent with the assumption that the initial period of the cosmic expansion was controlled by the interaction of a scalar field with gravitation [5].

4. THE COSMOLOGICAL MODEL

At any time during the evolution of the universe one may write the total energy density in the form

$$\rho = \rho_{\mathrm{r}} + \rho_{\mathrm{rm}} + \rho_{\mathrm{m}} \tag{28}$$

where ρ_{r} is the contribution of pure radiation (i.e. of the different types of massless particles), ρ_{rm} the contribution of massive matter in equilibrium with (electromagnetic) radiation and ρ_{m} the contribution of decoupled massive matter.

In a classical cosmological model, one may distinguish three stages of evolution. Stage I is an era of *pure radiation* with

$$\rho = \rho_{\mathrm{r}}\,, \qquad p = \tfrac{1}{3}\rho_{\mathrm{r}}\,, \qquad \rho_{\mathrm{r}} = bT^4, \tag{29}$$

where b is a known constant. Stage II is a *transitional era* of radiation and relativistic (elementary-particle) matter in thermal equilibrium, with

$$\rho = \rho_{\mathrm{r}} + \rho_{\mathrm{rm}}\,, \qquad p_{\mathrm{r}} = \tfrac{1}{3}\rho_{\mathrm{r}} = b'T^4. \tag{30}$$

The expressions for $\rho_{\mathrm{rm}}(T)$ and $p_{\mathrm{rm}}(T)$ are model-dependent. One usually assumes ideal gas distributions and applies the formulae of nuclear statistical equilibrium to study nucleosynthesis during this period. It would be useful to reconsider this standard analysis in MGR.

State III is the *matter-dominated* era in which non-relativistic (atomic then galactic) matter is decoupled from radiation. In this era,

$$\rho = \rho_{\mathrm{r}} + \rho_{\mathrm{m}}\,, \qquad \rho_{\mathrm{r}} = cT_{\mathrm{r}}^4\,, \qquad p_{\mathrm{r}} = \tfrac{1}{3}\rho_{\mathrm{r}}\,, \qquad p_{\mathrm{m}} = 0\,. \tag{31}$$

The matter component may be thermodynamically modelled by an ideal gas with $p_{\mathrm{m}} \neq 0$ and a temperature T_{m} related to ρ_{m} and p_{m} by the equation of state of a classical non-relativistic ideal gas. It is actually a classical thermodynamic system that is gradually transformed from an ideal gas of hydrogen into an ideal gas of galaxies. To circumvent this, one simply studies the matter component as a mechanical system that evolves under the field equations.

For a homogeneous universe with RW metric and $T_{\mu\nu}$ given by (18), the field equations (8) give

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{k}{3}\left[\rho - \beta(3p - \rho)\right] - \frac{\epsilon}{a^2}, \tag{32}$$

$$\frac{d}{da}\left[a^3\{\rho - \beta(3p - \rho)\}\right] + 3a^2\left[p + \beta(3p - \rho)\right] = 0, \tag{33}$$

where $\beta = \frac{1}{2}(1 - \alpha)$ and ϵ is the curvature constant.

As remarked previously, these equations coincide with those of the standard cosmological model for the stage of pure radiation, except that one replaces G by G/α:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\,k\rho_{\mathrm{r}} - \frac{\epsilon}{a^2}, \qquad \frac{d}{da}\left(a^3\rho_{\mathrm{r}}\right) + a^2\rho_{\mathrm{r}} = 0\,. \tag{34}$$

Thus for small t, irrespective of the value of ϵ, $a \sim t^{1/2}$, $\rho_{\mathrm{r}} \sim a^{-4}$, $T \sim a^{-1}$, S is constant and the model is indistinguishable from the standard model during this stage. An earlier field-theoretic era may be constructed to remove the initial singularity. This may, for example, be an extension of the string-motivated model of [15].

During the transitional era of stage II, the field equations give

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{k}{3}\left[\rho_{\mathrm{r}} + (1+\beta)\rho_{\mathrm{m}} - 3\beta p_{\mathrm{m}}\right] - \frac{\epsilon}{a^2}, \\ \frac{d}{da}&\left[a^3\{\rho_{\mathrm{r}} + (1+\beta)\rho_{\mathrm{rm}} - 3\beta p_{\mathrm{rm}}\}\right] \\ &+ 3a^2\left[\frac{1}{3}\,\rho_{\mathrm{r}} - \beta\rho_{\mathrm{rm}} + (1+3\beta)p_{\mathrm{rm}}\right] = 0\,. \end{aligned} \tag{35}$$

One may apply the usual assumptions of thermal and chemical equilibrium for the various species of radiation, elementary particles or nuclei that populate the universe in several successive ranges of temperature.

The results will not be much affected by the presence of β, since for the most part the conditions are relativistic. For example, one would still find that

$$\frac{T_\gamma}{T_\nu} \approx \left(\frac{11}{4}\right)^{1/3}, \tag{36}$$

since this is based on $\rho_{e^\pm} \sim T^4$, so that entropy is approximately constant in MGR, which is required for (36). A notable exception is the generation of entropy during this stage in MGR whereas entropy is strictly constant in the standard model. Equation (35) gives

$$T\frac{ds}{dt} = \frac{d}{dt}(\rho_{\rm rm}V) + p_{\rm rm}\frac{dv}{dt} = \beta V\frac{d}{dt}(3p_{\rm rm} - \rho_{\rm rm}). \tag{37}$$

In the ideal gas model, for large T, $(d/dt)(3p_{\rm rm} - \rho_{\rm rm}) > 0$. Thus $\beta > 0$ i.e. $0 < \alpha < 1$.

The generation of entropy during this stage is an important physical feature, since this period includes the spontaneous symmetry breaking that generated quark and lepton mass, as well as the phase transition that created hadrons.

During the matter-dominated era, the field equations give

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{k}{3}(1+\beta)\rho_{\rm m} - \frac{\epsilon}{a^2}, \tag{38}$$

$$(1+\beta)\frac{d}{da}(a^3\rho_{\rm m}) - 3\beta a^2\rho_{\rm m} = 0, \tag{39}$$

assuming that $\rho_{\rm m}, 3\beta\rho_{\rm m} \gg \rho_{\rm r}$. For the radiation component, the entropy is (assuming massless neutrinos)

$$S_{\rm r} = S_\gamma + S_\nu \propto a^3 T_\gamma^3 \tag{40}$$

which is *not* constant, since eq. (34) is no longer valid. Equation (39) yields

$$\rho_{\rm m} = \rho_{\rm mp}\left(\frac{a_{\rm p}}{a}\right)^{3/(1+\beta)}, \tag{41}$$

where the constant of integration is evaluated at the present time $t = t_{\rm p}$, writing $\rho_{\rm mp}$ and $a_{\rm p}$ for the present values of $\rho_{\rm m}$ and a. Thus, in contrast to standard cosmology, the total energy content $E_{\rm m}$ of non-relativistic matter is not constant. It increases with the expansion according to

$$E_{\rm m} \sim a^{3\beta/(1+\beta)}. \tag{42}$$

The time-dependence of the scale factor, $a(t)$, is obtained from eqs. (38) and (41). From eq. (38) one gets

$$\epsilon = a_{\rm p}^2 H_{\rm p}^2 \left[\frac{1+\beta}{1-2\beta}\,\Omega_{\rm p} - 1\right] \tag{43}$$

where

$$\Omega = \frac{8\pi G \rho_{\rm m}}{3H^2}\,. \tag{44}$$

Thus the universe is critical for

$$\Omega_{\rm p} = \eta, \qquad \eta = \frac{1-2\beta}{1+\beta}\,. \tag{45}$$

Since $\eta < 1$, this is a considerable departure from the standard model, which is critical for $\Omega_{\rm p} = 1$. If, for example, $\beta = \frac{1}{4}$, the universe is closed when $\Omega_{\rm p} > 0.4$.

The observational limits on $\Omega_{\rm p}$ are

$$0.1 < \Omega_{\rm p} < 4. \tag{46}$$

It thus appears that, with any appreciable value of β, a closed universe is more likely. We shall first consider the case $\epsilon = 1$, so that

$$\dot{a}^2 = \left(\frac{a_0}{a}\right)^{\eta}\left[1 - \left(\frac{a}{a_0}\right)^{\eta}\right], \tag{47}$$

where

$$a_0 = a_{\rm p}\left[\frac{\Omega_{\rm p}}{\Omega_{\rm p} - \eta}\right]^{1/\eta}. \tag{48}$$

Equation (47) should be integrated subject to the boundary condition $a = a_{\rm dec}$ at $t = t_{\rm dec}$, where 'dec' denotes decoupling of matter and radiation. We shall, however, replace this condition by $a = 0$ at $t = 0$. Then eq. (47) gives

$$t = \frac{a_0}{\eta}\int_0^{(a/a_0)^\eta} u^{1/\eta - 1/2}(1-u)^{-1/2}\,du,$$

i.e.,

$$t = \frac{2a_0}{2+\eta}\left(\frac{a}{a_0}\right)^{1+\eta/2} F\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{\eta}; \frac{3}{2} + \frac{1}{\eta}; \left(\frac{a}{a_0}\right)^{\eta}\right), \tag{49}$$

where F is the hypergeometric function. This equation determines $a = a(t)$ implicitly. The corresponding equation for the standard model ($\eta = 1$) is

$$t = a_0 \left[\sin^{-1}\left(\frac{a}{a_0}\right)^{1/2} - \left(\frac{a}{a_0}\right)^{1/2}\left(1 - \frac{a}{a_0}\right)^{1/2} \right]. \tag{50}$$

Equations (47), (48) and (49) may be used to express the age of the universe t_p in terms of H_p, Ω_p and η. For eqs. (47) and (48) give

$$a_\mathrm{p} = \frac{1}{H_\mathrm{p}} \left(\frac{\eta}{\Omega_\mathrm{p} - \eta}\right)^{1/2}, \tag{51}$$

so that eq. (49) yields

$$t_\mathrm{p} = N^{(+)}(\Omega_\mathrm{p}, \eta) H_\mathrm{p}^{-1}, \tag{52}$$

where

$$N^{(+)}(\Omega_\mathrm{p}, \eta) = \frac{2}{2+\eta}\left(\frac{\eta}{\Omega_\mathrm{p}}\right)^{1/2} F\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{\eta}; \frac{3}{2} + \frac{1}{\eta}; 1 - \frac{\eta}{\Omega_\mathrm{p}}\right). \tag{53}$$

For the standard model (with $\epsilon = 1$) one has

$$N^{(+)}(\Omega_\mathrm{p}, 1) = \frac{1}{\Omega_\mathrm{p} - 1}\left[\frac{\Omega_\mathrm{p}}{(\Omega_\mathrm{p} - 1)^{1/2}} \sin^{-1}\left(\frac{\Omega_\mathrm{p} - 1}{\Omega_\mathrm{p}}\right)^{1/2} - 1\right]. \tag{54}$$

We shall consider Ω_p, H_p and t_p as given observables. Equation (54) is then a rigid test of the (closed) standard model. From work on the age of the globular star clusters in the halo of the Milky Way [16] one deduces that

$$t_\mathrm{p} \geq (16 \pm 2) \times 10^9\,\mathrm{yr}. \tag{55}$$

Taking for the Hubble parameter, h_p, in the relation

$$H_\mathrm{p}^{-1} = 0.98 h_\mathrm{p}^{-1} \times 10^{10}\,\mathrm{yr}, \tag{56}$$

the observable range $0.5 \leq h_\mathrm{p} \leq 0.85$, we get

$$11.53 \times 10^9\,\mathrm{yr} \leq H_\mathrm{p}^{-1} \leq 19.6 \times 10^9\,\mathrm{yr}. \tag{57}$$

From these one secures the lower bound

$$t_\mathrm{p} H_\mathrm{p} \geq 0.71, \tag{58}$$

and, if equality is assumed in (55), also the upper bound $t_p H_p \leq 1.56$. The bound (58) appears to exclude the closed standard model, since eq. (54) gives $t_p H_p = (0.67, 0.57, 0.51, 0.47)$ for $\Omega_p = (1, 2, 3, 4)$, which covers the whole of the allowed range, (46), for the closed model.

For $\Omega_p < 1$, the standard model is open and eq. (54) is replaced by

$$N^{(-)}(\Omega_p, 1) = \frac{1}{1-\Omega_p}\left[1 - \frac{\Omega_p}{(1-\Omega_p)^{1/2}} \sinh^{-1}\left(\frac{1-\Omega_p}{\Omega_p}\right)^{1/2}\right]. \tag{59}$$

Using this equation one finds that the lower bound (58) is satisfied only if $\Omega_p \leq 0.7$. It thus appears that the presently available observational data would exclude the matter dominated standard model for all $\Omega_p > 0.7$. It is only tenable in the range $0.1 \leq \Omega_p \leq 0.7$.

Equation (53) shows that the situation is quite different in the matter-dominated model of MGR, due to the presence of the parameter η.

We first note that eq. (53) also holds for the case $\epsilon = -1$, i.e., for the open model with $\Omega_p < \eta$, so that

$$N^{(-)}(\Omega_p, \eta) = N^{(+)}(\Omega_p, \eta). \tag{60}$$

This follows from the fact that, in this case, eq. (49) is replaced by

$$t = \frac{2a_1}{2+\eta}\left(\frac{a}{a_1}\right)^{1+\eta/2} F\left(\frac{1}{2}, \frac{1}{2}+\frac{1}{\eta}; \frac{3}{2}+\frac{1}{\eta}; -\left(\frac{a}{a_1}\right)^{\eta}\right), \tag{61}$$

where

$$a_1 = a_p\left(\frac{\Omega_p}{\eta-\Omega_p}\right)^{1/\eta} = H_p^{-1}\left(\frac{\eta}{\Omega_p}\right)^{1/2}\left(\frac{\Omega_p}{\eta-\Omega_p}\right)^{1/\eta+1/2}. \tag{62}$$

Thus eq. (53) may be used for all cases and we shall denote $t_p H_p$ by $N(\Omega_p, \eta)$. One notes the special cases:

$$N(\Omega_p, 1) = \frac{2}{3}\,\Omega_p^{-1/2} F\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; 1-\Omega_p^{-1}\right), \tag{63}$$

which may be written in the forms (54) or (59) for $\Omega_p \gtrless 1$;

$$N(\Omega_p, \Omega_p) = \frac{2}{2+\Omega_p}, \qquad \Omega_p \leq 1, \tag{64}$$

$$N(\Omega_p, 0) = \sqrt{\pi}\,\Omega_p^{-1/2} e^{1/\Omega_p}\left[1 - \Phi(\Omega_p^{-1/2})\right], \tag{65}$$

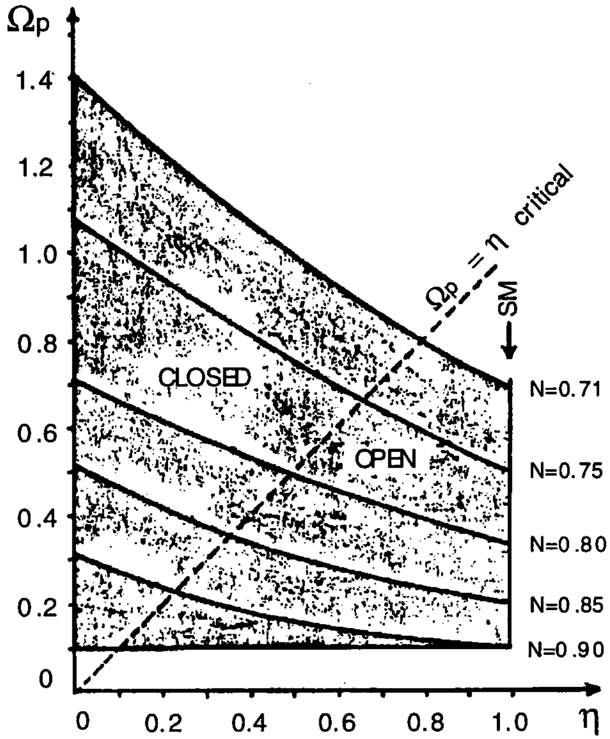

Figure 1. Region consistent with the modified matter-dominated model is shaded. The standard model is the edge $\eta = 1$ of this region.

where $\Phi(x)$ is the error function.

Using these equations one finds (see Figure 1) that the observational data restrict the values of N to the range

$$0.71 \leq N \leq 0.96, \tag{66}$$

where the upper limit corresponds to $\Omega_\mathrm{p} = 0.1$, $\eta = 0$. For $0.1 \leq \Omega_\mathrm{p} \leq 0.7$, it is possible to admit values of η in the whole range [0,1] , where the unique fixed value is determined by the values of Ω_p and N. For $0.7 < \Omega_\mathrm{p} \leq 1.4$ we must have $0 \leq \eta < 1$ and the standard model is excluded. Thus the admissible values of Ω_p lie in the range

$$0.1 \leq \Omega_\mathrm{p} \leq 1.4. \tag{67}$$

Should observations show N or $\Omega_{\rm p}$ to exceed 0.96 or 1.4, respectively, one must discard the matter-dominated, zero-pressure, cosmological model, even when energy conservation is relaxed.

If, on the other hand, observations fix definite values for $\Omega_{\rm p}$ and N that lie within the admissible region of the MGR matter-dominated model, then a unique value of η is determined. For example, the values $\Omega_{\rm p} = 1$, $N = 0.73$ yield $\eta = 0.25$. One is then able to measure the energy non-conservation parameter at a value that definitely excludes the standard model. Figure 1 shows the standard model to be extremely restrictive since N is uniquely determined by $\Omega_{\rm p}$ when $\eta = 1$. For example $\Omega_{\rm p} = 0.5$ yields $N = 0.75$ (when $\eta = 1$), so that $t_{\rm p} = 15 \times 10^9$ yr implies $H_{\rm p}^{-1} = 2 \times 10^{10}$ yr, i.e. $h = 0.49$, which is at the limit of the observational range. In the MGR model, $\Omega_{\rm p} = 0.5$ is consistent with $0.75 \leq N \leq 0.85$, so that $t_{\rm p} = 15 \times 10^9$ yr yields $0.49 \leq h \leq 0.56$ which provides some margin for agreement with observation.

We finally note that eq. (27) enables one to obtain an expression of the effective decaying cosmological "constant" of the matter-dominated model based on MGR:

$$\Lambda = -\frac{1-\alpha}{\alpha} 4\pi G \rho_{\rm m} \sim a^{-[3/(1+\beta)]}. \tag{68}$$

If α is small, as seems to be the indication of Fig. 1, then $\beta \approx \frac{1}{2}$ and the rate of decay in (68) will be close to that of the critical-density model of [1] in which $\Lambda \sim a^{-2}$. From (68) one obtains for the present value

$$|\Lambda_{\rm p}| = \frac{3}{2}\,\frac{1-\alpha}{\alpha}\,\Omega_{\rm p} H_{\rm p}^2 , \tag{69}$$

of the order of 10^{20} yr^{-2}.

In Section 5 we give a summary of our results and some concluding remarks.

5. SUMMARY AND CONCLUSIONS

(i) Motivated by our desire to solve the entropy problem of the standard cosmological model, we have proposed that the energy-conservation condition be relaxed. The result is a modified version of classical general relativity with a single free parameter, embodied in the field equations (8).

(ii) A number of immediate consequences follow from the modified field equations:

(a) Covariantly conserved systems must possess a traceless energy momentum tensor.

(b) In empty space the modified field equations coincide exactly with those of standard general relativity; eqs. (17).
(c) It follows from (b) that the crucial tests of general relativity, the existence of singularities, black holes and all features connected with the exact vacuum solutions remain intact.
(d) For systems with a traceless energy-momentum tensor the modified field equations coincide with the original ones, eq. (14), except that G is replaced by G/α where $\alpha \neq 1$.
(e) It follows from (d) that the treatment of Einstein–Maxwell fields, general fields of pure radiation, and highly relativistic thermodynamic systems remains unchanged. In particular, the modified and original field equations are equivalent in their description of the radiation era in cosmology.
(f) The modified field equations may be written in a form that exhibits a cosmological term [eq. (26)] with a definite dependence on the scalar curvature [eq. (27)]. In a perfect-fluid RW universe, this provides an effective decaying cosmological "constant", that vanishes identically in vacuum and in pure-radiation systems.

(iii) In the matter-dominated era of the homogeneous RW universe based on the modified field equations one finds that the total energy of non-relativistic matter increases with the expansion, eq. (42). This might be interpreted as the continuous transformation of gravitational energy into massive matter, as the curvature of the universe unfolds and it evolves towards flatness.

(iv) The present value of the product $t_{\rm p}H_{\rm p}$ is given for the modified matter-dominated model by eq. (53), where $0 \leq \eta \leq 1$ and $\eta = 1$ is the un-modified model. The universe is closed for $\Omega_{\rm p} > \eta$. One finds that the observational lower bound in eq. (58), which seems to exclude the closed unmodified model, is consistent with open and closed possibilities in the modified version.

(v) The observational lower bound in eq. (58) restricts the original matter-dominated model to be open and admits values of $\Omega_{\rm p}$ that are limited to $\Omega_{\rm p} \leq 0.7$. For the modified version both open and closed possibilities are consistent with the data and the upper bound on $\Omega_{\rm p}$ is 1.4. Exact knowledge of $\Omega_{\rm p}$ and $t_{\rm p}H_{\rm p}$ would either uniquely determine the free parameter η in the range $0 \leq \eta \leq 1$ or show that the matter-dominated cosmological model is inconsistent with the data, even when energy conservation is relaxed.

(vi) Should the need arise for generalizing the matter-dominated zero-pressure cosmological model, the obvious extension is to include electromagnetic radiation and relativistic massive neutrinos, where the latter are the dominant component. Exploration of the wider margin for consistency with the observational data, that is provided by the modified model, should however precede such generalizations.

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