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Stochastic Gravity

D. K. Ross[1] **and William Moreau**[2]

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We investigate stochastic gravity as a potentially fruitful avenue for studying quantum effects in gravity. Following the approach of stochastic electrodynamics (SED), as a representation of the quantum gravity vacuum we construct a classical state of isotropic random gravitational radiation, expressed as a spin-2 field, $h_{\mu\nu}(x)$, composed of plane waves of random phase on a flat spacetime manifold. Requiring Lorentz invariance leads to the result that the spectral composition function of the gravitational radiation, $h(\omega)$, must be proportional to $1/\omega^2$. The proportionality constant is determined by the Planck condition that the energy density consist of $\hbar\omega/2$ per normal mode, and this condition sets the amplitude scale of the random gravitational radiation at the order of the Planck length, giving a spectral composition function $h(\omega) = \sqrt{16\pi c^2} L_P/\omega^2$. As an application of stochastic gravity, we investigate the Davies–Unruh effect. We calculate the two-point correlation function $\langle R_{i0j0}(0, \tau - \delta\tau/2) R_{k0l0}(0, \tau + \delta\tau/2)\rangle$ of the measureable geodesic deviation tensor field, R_{i0j0}, for two situations: (i) at a point detector uniformly accelerating through the random gravitational radiation, and (ii) at an inertial detector in a heat bath of the random radiation at a finite temperature. We find that the two correlation functions agree to first order in $a\delta\tau/c$ provided that the temperature and acceleration satisfy the relation $kT = \hbar a/2\pi c$.

1. INTRODUCTION

In stochastic electrodynamics (SED), the electromagnetic vacuum is re-

[1] Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011, USA

[2] Department of Physics and Astronomy, University of Canterbury, Christchurch 1, New Zealand

garded as a classical state of *real*, isotropic, random electromagnetic radiation. This picture of zero-point fluctuations in a real field has its origins in early investigations of the quantum aspects of the electromagnetic field by Planck [1], Nernst [2], and Einstein and Stern [3] before the advent of quantum mechanics. After the introduction of quantum mechanics the picture came to be regarded as naive, and the vacuum was considered to consist only of virtual particles without any measurable consequence. There are, however, measureable vacuum effects that are realized when the vacuum interacts with charged particles as in, for example, the Lamb shift [4] and quantum noise in electronic devices (Ref. 5, Ch.7), and when physical boundary conditions are altered as in the Casimir effect [6], the Casimir–Polder force [7], and spontaneous emission in cavities [8]. A further vacuum effect that is, in principle, measureable is the Davies–Unruh effect [9,10] which is due to an altered physical vacuum as perceived from an accelerated frame of reference. In an attempt to obtain more physically meaningful and less abstract explanations for vacuum phenomena, Marshall [11] and Boyer [12] in some far-reaching work in the 1960s and 1970s resurrected the earlier classical ideas of the electromagnetic vacuum and applied them, respectively, to the problem of the stability of the hydrogen atom ground state and to the derivation of the Planck spectrum. Since this pioneering work considerable progress has been made, and SED has been applied successfully to many of the standard problems of QED [13]. In fact, Milonni [14] has shown that, with few exceptions, SED and QED treatments give identical results. Some good arguments supporting the reality of the vacuum have been given by Haisch et al. [15]. A consensus may be emerging that the vacuum is an all pervasive state of a field, not fundamentally different from the excited (particle) states except that, because it is the ground state, it cannot give up any energy-momentum to a particle detector. A comprehensive review of SED, including many references, has been given by de la Peña [16].

At the present time there is no satisfactory theory of quantum gravity. With the possible exception of superstrings [17], which have their own problems, theories of quantum gravity [18] are essentially nonrenormalizable. But even more serious conceptual problems exist. The metric $g_{\mu\nu}(x)$ is not only a tensor field on the spacetime manifold, but through the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ it also defines the proper distances between the points x on the manifold which label the degrees of freedom of the metric field itself. Thus, when the metric field becomes quantized, the degrees of freedom of the metric field also become quantized, and such a situation is exceedingly difficult, if not impossible, to interpret both physically and mathematically.

Given the many successes of SED in treating quantum aspects of the electromagnetic field and the lack of success of attempts to quantize the gravity field, it seems entirely reasonable and appropriate to apply the classical-statistical ideas of SED to gravity as a potentially powerful analytical tool. At the outset, however, we must recognize that where Maxwell's theory is linear, general relativity is nonlinear. While a linear superposition of plane waves satisfies Maxwell's equations, it does not satisfy Einstein's field equation. So we must immediately retreat to a weak field satisfying a linear field equation. But this restriction is compatible with our goal in this paper which is limited to the vacuum state for distance scales well above the Planck scale. For distance scales on the order of the Planck scale, we probably have a topological foam, and Hawking [19] has shown that Planck-scale black holes contribute strongly. Our formalism is totally inadequate to handle these non-linear configurations. We are considering a random classical field as a representation of the quantum gravity vacuum. Our results will be compatible with a weak-field assumption.

Thus we are led to consider a tensor field $h_{\mu\nu}(x)$ on a flat manifold, with $|h_{\mu\nu}| \ll 1$, composed of an isotropic superposition of plane waves with random phases as a classical representation of the quantum gravitational vacuum. Weinberg [20] and Deser [21] have shown that a self-coupled spin 2 quantum field (without random phases) satisfies the Einstein field equations and has all of the properties of GR when it is finally interpreted as a metric field on a corresponding curved manifold. We interpret the zero-point fluctuations as occurring primarily in the tensor field on the flat manifold where it is unrelated to the metric which is Minkowskian. Then the full metric on the associated curved manifold is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and the fluctuations in $h_{\mu\nu}$ cause the manifold *itself* to fluctuate.

In this paper we do two things with the random classical gravitational radiation that we are taking to represent the gravitational vacuum: (i) through the relativity principle and the Planck condition we determine the form and magnitude of the spectral composition function $h(\omega)$, and (ii) we investigate the Davies–Unruh effect for a particle detector accelerating through this gravitational vacuum. In applying the relativity principle, we insist that the stochastic average, $\langle h_{\mu\nu}h^{\mu\nu}\rangle$, be manifestly Lorentz invariant when expressed as an integral over the wave vector and sum over polarizations. Applying the Planck condition, that $\langle T_{00}\rangle$ should have an energy $\hbar\omega/2$ per normal mode, sets a scale for the amplitude of the gravitational radiation at the order of the Planck length. For the Davies–Unruh effect we calculate the two-point correlation function, $\langle R_{i0j0}(\mathbf{0}, \tau - \delta\tau/2)R_{k0l0}(\mathbf{0}, \tau + \delta\tau/2)\rangle$, of the measureable geodesic deviation tensor field, R_{i0j0}, at a point detector accelerating through random gravitational radiation and at an inertial detector in a thermal bath of random radiation. The two cases agree to first order in $a\delta\tau/c$ if the temperature and acceleration are related by $kT = \hbar a/2\pi c$.

2. THE STOCHASTIC GRAVITATIONAL VACUUM

Following the procedure of Boyer in the electromagnetic case [22], we construct a rank two tensor field $h_{\mu\nu}(x)$ on a flat spacetime manifold as an isotropic linear superposition of classical plane waves propagating at the speed of light with random phases, integrated over all wave vectors $\mathbf{k}$ and summed over two independent polarizations $\lambda = 1, 2$. We work in the transverse, traceless (TT) gauge (Ref. 23, p.946) where $h_{\mu 0} = 0$, $\sum_i h_{ii} = 0$, and $\sum_j h_{ij,j} = 0$. The nonzero elements of $h_{\mu\nu}(x)$ are given by

$$h_{ij}(\mathbf{r}, t) = \sum_{\lambda=1}^{2} \int d^3k \epsilon_{ij}(\mathbf{k}, \lambda) h(\omega) \cos[\mathbf{k} \cdot \mathbf{r} - \omega t - \theta(\mathbf{k}, \lambda)], \tag{1}$$

where the $\epsilon_{ij}(\mathbf{k}, \lambda)$ for $\lambda = 1, 2$ are two independent polarization tensor fields on $\mathbf{k}$ space, $h(\omega) = h(ck)$ is a spectral composition function to be determined by considerations of Lorentz invariance and quantum mechanics, and the $\theta(\mathbf{k}, \lambda)$ are a set of random phase angles.

Equation (1) is a classical, stochastic representation of the quantum gravitational vacuum. With the random phases $\theta(\mathbf{k}, \lambda)$ held constant, $h_{\mu\nu}(x)$ is a solution of the Einstein field equations in the weak field (linearized) limit (see Ref. 23, p.435) for any composition function $h(\omega)$. The classical ground state is a zero-field configuration. So eq. (1) represents zero-point fluctuations (ZPF) about the classical vacuum.

Because the classical equations of motion are linear in the weak-field limit, the spectral composition function $h(\omega)$ is so far unrestricted. But we require that average (or expectation) values be in accord with the relativity principle. In particular we require $\langle h_{\mu\nu} h^{\mu\nu} \rangle$ to be a Lorentz scalar. From eq. (1) with $h_{\mu\nu}$ in the TT gauge, we have

$$\langle h_{\mu\nu} h^{\mu\nu} \rangle = \langle h_{ij} h^{ij} \rangle = \sum_{\lambda=1}^{2} \int d^3k \frac{1}{2} k^3 h^2(\omega) \epsilon_{ij}(\mathbf{k}, \lambda) \epsilon^{ij}(\mathbf{k}, \lambda) \tag{2}$$

where $k \equiv |\mathbf{k}|$ and we have used

$$\begin{aligned} \langle \cos[\theta(\mathbf{k}, \lambda)] \cos[\theta(\mathbf{k}', \lambda')] \rangle &= \langle \sin[\theta(\mathbf{k}, \lambda)] \sin[\theta(\mathbf{k}', \lambda')] \rangle \\ &= \tfrac{1}{2} k^3 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'}, \qquad (3) \\ \langle \cos[\theta(\mathbf{k}, \lambda)] \sin[\theta(\mathbf{k}', \lambda')] \rangle &= 0 \,. \qquad (4) \end{aligned}$$

The factor of k^3 in eq. (3) arises from dimensional considerations. The angled brackets in eqs. (2), (3), and (4) denote time averages over random phase angles, $\theta(\mathbf{k}, \lambda)$, associated with each normal mode (wave vector $\mathbf{k}$ and polarization λ). The fluctuations in phase between different modes are uncorrelated. Therefore the averages of eq. (3) should give a Dirac delta function in the continuous mode label $\mathbf{k}$ and a Kronecker delta in the discrete label λ. But these averages are pure numbers without units. Consequently the Dirac delta function must be dimensionless and of the form

$$\delta^3[(\mathbf{k}-\mathbf{k}')/k] \equiv \frac{1}{(2\pi)^{3/2}} \int d^3\xi e^{i\xi\cdot[(\mathbf{k}-\mathbf{k}')/k]} = k^3\delta^3(\mathbf{k}-\mathbf{k}'), \tag{5}$$

where ξ is nondimensional. The argument of this delta function uniquely satisfies all of the requirements: (i) it is dimensionless; (ii) it is a function exclusively of the components of the two wave vectors; and (iii) it is zero only when the two wave vectors are equal. The factor of $\frac{1}{2}$ is due to the fact that when $\mathbf{k} = \mathbf{k}'$ and $\lambda = \lambda'$ the two averages in eq. (3) are equal and their sum is equal to one. Boyer's disregard of the units of delta functions has resulted in his spectral function $h^2(\omega)$ for both the scalar field and the electromagnetic field being incorrect by a factor of $(c/\omega)^3$ [22]. However, if one consistently ignores the units of delta functions, as Boyer has done, a compensating error is made in doing the integrations over wave vectors for the correlation functions. Consequently Boyer's correlation functions are correct.

In order for eq. (2) to be Lorentz invariant, the right-hand side must have the invariant form (Ref. 24, p.112)

$$\int \frac{d^3k}{2\omega}\,(\text{const.}). \tag{6}$$

Since $\sum_{\lambda=1}^{2} \epsilon_{ij}(\mathbf{k}, \lambda)\epsilon^{ij}(\mathbf{k}, \lambda) = 4$, independent of $\mathbf{k}$,[3] and $\omega = ck$, inspection of eq. (2) shows that Lorentz invariance requires the spectral composition function to be of the form

$$h(\omega) = \frac{A}{\omega^2}, \tag{7}$$

where A is a constant.

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[3] The identity, $\sum_{\lambda=1}^{2} \epsilon_{ij}(\mathbf{k}, \lambda)\epsilon^{ij}(\mathbf{k}, \lambda) = 4$, follows from eqs. (22) and (23) in Section 3. It is obviously true for the polarization tensors given by eq. (22) and the sum is invariant under the orthogonal transformations of eq. (23).

We determine the proportionality constant, A, by considering the energy density of the gravitational radiation, T_{00}, which, according to Planck, should be composed of $\hbar\omega/2$ for each normal mode of oscillation. Thus from a quantum mechanical point of view, the energy density should be

$$T_{00} = \sum_{\lambda=1}^{2} \int d^3k \frac{1}{2}\,\hbar\omega. \tag{8}$$

On the classical side of the picture, the energy density for gravitational waves in the TT gauge is given by (see Ref. 23, p.955)

$$T_{00} = \frac{1}{32\pi}\,\frac{c^4}{G}\,\langle h_{ij,0} h^{ij}_{\ ,0} \rangle, \tag{9}$$

where $h_{ij,0}$ denotes a derivative with respect to $x^0 \equiv ct$, and the average is over a representative number of periods of the wave. As the stochastic average is also over such an interval, we make no distinction in our notation.

Substituting derivatives of eq. (1) with respect to x^0 and eq. (7) for $h(\omega)$ into eq. (9) and applying eqs. (3) and (4), we have the result

$$T_{00} = \frac{1}{64\pi}\,\frac{c^4}{G} \sum_{\lambda=1}^{2} \int d^3k \frac{A^2}{\omega^4} \left(\frac{\omega}{c}\right)^2 \epsilon_{ij}\epsilon^{ij} k^3, \tag{10}$$

where the factor of $(\omega/c)^2$ comes from the x_0 derivatives of eq. (1).

Comparing eqs. (8) and (10) and noting that $\sum_{\lambda=1}^{2} \epsilon_{ij}\epsilon^{ij} = 4$ in eq. (10) (see footnote 3 above), we see that the proportionality constant must be

$$A = \sqrt{16\pi}\,\sqrt{\hbar G/c^3}\,c^2 = \sqrt{16\pi}\,L_{\rm P} c^2, \tag{11}$$

where $L_{\rm P}$ is the Planck length, and the spectral composition function is given by

$$h(\omega) = L_{\rm P}\sqrt{16\pi}\,\frac{c^2}{\omega^2} = L_{\rm P}\sqrt{16\pi}\,/k^2. \tag{12}$$

Substituting eq. (12) into eq. (1), our final result for the random classical gravitational radiation in the TT gauge can now be written as

$$h_{ij}(\mathbf{r}, t) = L_{\rm P}\sqrt{16\pi} \sum_{\lambda=1}^{2} \int \frac{d^3k}{k^2}\, \epsilon_{ij}(\mathbf{k}, \lambda) \cos[\mathbf{k}\cdot\mathbf{r} - \omega t - \theta(\mathbf{k}, \lambda)\,], \tag{13}$$

where $k = |\mathbf{k}| = \omega/c$. Average values of scalars formed from $h_{\mu\nu}$ are Lorentz invariant, and the associated energy density consists of a collection of harmonic oscillators, with each oscillator in its ground state of energy $\hbar kc/2$ and each one corresponding to a normal mode $(\mathbf{k}, \lambda)$. The contact with quantum mechanics has set the scale for the amplitude of the gravitational radiation at the magnitude of the Planck length. The large k contribution to h_{ij} goes as $L_P \int dk$ and approaches unity only for $k \sim 1/L_P$.

We pointed out earlier that our linearized formalism with $|h_{ij}| \ll 1$ cannot be used for length scales on the order of the Planck length, so this result is nicely consistent. Clearly if eq. (13) is to be used as a tool for approximate quantum mechanical calculations, it must be cut off at $k < 1/L_P$.

3. THE CORRELATION FUNCTION AT AN ACCELERATING DETECTOR

The random classical gravitational radiation given by eq. (13) can be used in general relativity calculations to estimate quantum effects. As a first application of stochastic gravity we investigate the Davies–Unruh effect for an observer accelerating through the zero-point fluctuations of the gravitational vacuum. This application extends the work of Boyer on the scalar and electromagnetic fields [22] to the spin-2 gravitational case. In so doing we also clarify some of Boyer's earlier results.

We consider a particle detector that is uniformly accelerating in the x direction with respect to an inertial frame of reference containing the gravitational vacuum given by eq. (13). Let the detector be located at the origin of an accelerating reference frame which we will call the detector's rest frame. The world line of the detector is given by

$$t(\tau) = \frac{c}{a} \sinh\left(\frac{a\tau}{c}\right), \tag{14}$$

$$x(\tau) = \frac{c^2}{a} \cosh\left(\frac{a\tau}{c}\right), \tag{15}$$

where a is the magnitude of the acceleration and τ is the proper time. As an indication of how the vacuum appears to the particle detector, we calculate the two point correlation function at the origin of the detector's rest frame at proper time τ for two points separated by a small time $\delta\tau$. Correlation functions in stochastic field theories correspond to Wightman functions in quantum field theories [25].

But before we proceed, we recognize that the potential field $h_{ij}(x)$ is not directly observable. The equation of geodesic deviation is

$$\frac{d^2x^j}{d\tau^2} = -R^j_{\ 0k0}x^k, \tag{16}$$

where $R^j_{\ 0k0}$ is the Riemann tensor, which is directly observable. Therefore, we turn our attention to the related tensor field $R_{i0j0}(x)$ of geodesic deviation which, for weak fields in the TT gauge, is related to the potential field by (see Ref. 23, p.948)

$$\begin{aligned} R_{i0j0}(\mathbf{r},t) &= -\frac{1}{2}\,h_{ij,00}(\mathbf{r},t) \\ &= \frac{1}{2}\sum_{\lambda=1}^{2}\int d^3k\epsilon_{ij}(\mathbf{k},\lambda)h(\omega)\frac{\omega^2}{c^2}\cos[\mathbf{k}\cdot\mathbf{r}-\omega t-\theta(\mathbf{k},\lambda)\,], \end{aligned} \tag{17}$$

and corresponds to the $\mathbf{E}$ and $\mathbf{B}$ fields in electrodynamics.

Then in the detector's rest frame the two-point correlation function at proper time τ in which we are interested is given by

$$\Delta_{ijkl}(\delta\tau) = \langle R_{i0j0}(\mathbf{0},\tau-\delta\tau/2)R_{k0l0}(\mathbf{0},\tau+\delta\tau/2)\,\rangle. \tag{18}$$

In order to calculate $\Delta(\delta\tau)$ we must transform the $R_{i0j0}(x,t)$ given by eq. (17) to the instantaneously comoving inertial frames at $\tau\pm\delta\tau/2$ with t and x given by eqs. (14) and (15). When we do the Lorentz transformations, we get a factor of $\gamma(\tau-\delta\tau/2)$ for each 0 index and each 1 index in the set $\{i0j0\}$ and similarily a factor $\gamma(\tau+\delta/2)$ for the set $\{k0l0\}$, where from eqs. (14) and (15) the function $\gamma(\tau)$ is given by

$$\gamma(\tau) \equiv \left(\sqrt{1-\frac{1}{c^2}\left(\frac{dx}{dt}\right)^2}\right)^{-1} = \cosh\left(\frac{a\tau}{c}\right). \tag{19}$$

Substituting the Lorentz transformations of eq. (17) into eq. (18) and using eqs. (3) and (4) to do one integration over $\mathbf{k}$ and sum on λ, we have

$$\begin{aligned} \Delta_{ijkl}(\delta\tau) = \sum_{\lambda=1}^{2}\int & d^3k\frac{1}{2}\,k^3\epsilon_{ij}(\mathbf{k},\lambda)\epsilon_{kl}(\mathbf{k},\lambda)h^2(\omega)\left(\frac{\omega^2}{2c^2}\right)^2\times \\ & \times\gamma^{n_1+2}\left(\tau-\frac{\delta\tau}{2}\right)\gamma^{n_2+2}\left(\tau+\frac{\delta\tau}{2}\right)\times \\ & \times\cos\left[k_xx\left(\tau-\frac{\delta\tau}{2}\right)-\omega t\left(\tau-\frac{\delta\tau}{2}\right)\right. \\ & \qquad \left. -\,k_xx\left(\tau+\frac{\delta\tau}{2}\right)+\omega t\left(\tau+\frac{\delta\tau}{2}\right)\right], \end{aligned} \tag{20}$$

where the functional dependences of t, x, and γ are given respectively by eqs. (14), (15), and (19), and n_1 and n_2 are the total number of 1 indices respectively in the sets $\{i0j0\}$ and $\{k0l0\}$.

As Boyer has shown [22], there is no preferred time for accelerated motion in a Lorentz invariant theory. The instantaneously comoving inertial frame is the equivalent of any other inertial frame. Therefore, we lose no generality if we set $\tau = 0$ in eq. (20), and so doing we obtain

$$\Delta_{ijkl}(\delta\tau) = \sum_{\lambda=1}^{2} \int d^3k \frac{1}{2}\, k^3 \epsilon_{ij}(\mathbf{k},\lambda)\epsilon_{kl}(\mathbf{k},\lambda) h^2(\omega) \left(\frac{\omega^2}{2c^2}\right)^2 \times$$
$$\times \cosh^n\left(\frac{a\delta\tau}{2c}\right) \cos\left[\frac{2\omega c}{a} \sinh\left(\frac{a\delta\tau}{2c}\right)\right], \qquad (21)$$

where we have substituted eq. (19) for the functional dependence of γ and $n \equiv 4 + n_1 + n_2$.

It is convenient to perform the integral of eq. (21) in spherical polar coordinates. The polarization tensors depend on the direction (θ, ϕ) of the $\mathbf{k}$ vector, but not its magnitude, i.e., $\epsilon(\mathbf{k}, \lambda) = \epsilon(\theta, \phi, \lambda)$. They can be obtained from $\epsilon(0, 0, \lambda)$ by a sequence of rotations. In the TT gauge, we have two independent polarization tensors for waves travelling in the $\theta = 0$ direction (see Ref. 23, p.952):

$$\epsilon(0,0,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon(0,0,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (22)$$

Then the general polarization tensors are given by

$$\epsilon(\theta, \phi, \lambda) = R_z(\phi) R_y(\theta) \epsilon(0, 0, \lambda) R_y^T(\theta) R_z^T(\phi), \qquad (23)$$

where the R's represent positive, active rotations.

Since only the polarization tensors in eq. (21) depend on the direction of $\mathbf{k}$ we define

$$Q_{ijkl} \equiv \sum_{\lambda=1}^{2} \int_0^{\pi} \int_0^{2\pi} \sin\theta d\theta d\phi\, \epsilon_{ij}(\theta, \phi, \lambda) \epsilon_{kl}(\theta, \phi, \lambda), \qquad (24)$$

with values given by

$$Q_{1111} = Q_{2222} = Q_{3333} = \frac{32\pi}{15},$$
$$Q_{1122} = Q_{1133} = Q_{2233} = -\frac{16\pi}{15}, \qquad (25)$$
$$Q_{1212} = Q_{1313} = Q_{2323} = \frac{8\pi}{15}.$$

The other components of Q_{ijkl} which cannot be derived from eqs. (25) by the obvious symmetries of eq. (24) are all zero.

Using eq. (24), we can write eq. (21) as

$$\Delta_{ijkl}(\delta\tau) = \frac{Q_{ijkl}}{8c^{10}} \cosh^n\left(\frac{a\delta\tau}{2c}\right) \int_0^\infty d\omega\omega^9 h^2(\omega)\times \times \cos\left[\frac{2\omega c}{a}\sinh\left(\frac{a\delta\tau}{2c}\right)\right]. \tag{26}$$

Finally, substituting eq. (12) for $h(\omega)$ and evaluating the integral over ω with the help of a temporary damping factor,

$$\int_0^\infty d\omega\omega^5 \cos(b\omega) = \text{Re} \lim_{\lambda\to 0} \int_0^\infty d\omega\omega^5 \exp[(ib-\lambda)\omega] = -\frac{120}{b^6}, \tag{27}$$

we have

$$\Delta_{ijkl}(\delta\tau) = -\frac{Q_{ijkl}}{8c^6}\left(\frac{a}{2c}\right)^6 \frac{32\pi L_P^2}{\sinh^6(a\delta\tau/2c)} \left[120\cosh^n\left(\frac{a\delta\tau}{2c}\right)\right]. \tag{28}$$

Equation (28) gives the correlation functions of two geodesic deviation tensor fields, $R_{i0j0}(\mathbf{0},\tau-\delta\tau/2)$ and $R_{k0l0}(\mathbf{0},\tau+\delta\tau/2)$, for an arbitrary proper time τ and small proper time separation $\delta\tau$ along the world line of an accelerating detector. The non-zero values are indicated by eq. (25). We will now show that these correlations are the same as would be detected by an observer who is stationary in a thermal bath of isotropic, random gravitational radiation at a temperature $T = \hbar a/2\pi ck$.

4. THE GRAVITATIONAL DAVIES–UNRUH EFFECT

We now want to look at the correlation function for a stationary observer in a radiation field with the same zero-point distribution as above, but now with a finite-temperature Planck distribution as well. Thus, instead of eq. (12) for the spectral function, we have

$$\tilde{h}^2(\omega) = h^2(\omega)2\left[\frac{1}{2} + \frac{1}{e^{\hbar\omega/kT}-1}\right]. \tag{29}$$

where the first term in the bracket corresponds to the usual zero-point fluctuations and the second term to the Planck distribution. Using $\tilde{h}$ in place of h in eq. (17) and averaging over the random phases as before, the new correlation function is given by

$$\begin{aligned}\tilde{\Delta}_{ijkl}(\delta t) &\equiv \langle R_{i0j0}(\mathbf{0}, t - \delta t/2) R_{k0l0}(\mathbf{0}, t + \delta t/2) \rangle \\ &= \sum_{\lambda=1}^{2} \int d^3k \frac{1}{2} k^3 \epsilon_{ij}(\mathbf{k}, \lambda) \epsilon_{kl}(\mathbf{k}, \lambda) \tilde{h}^2(\omega) \left(\frac{\omega^2}{2c^2}\right)^2 \cos\omega(\delta t), \quad (30)\end{aligned}$$

which does not depend upon the absolute time t.

Carrying out the angular integrations and summing over the polarizations then gives

$$\begin{aligned}\tilde{\Delta}_{ijkl}(\delta t) &= \frac{Q_{ijkl}}{8c^{10}} \int_0^\infty d\omega \omega^5 \times \\ &\times \cos[\omega(\delta t)] \left(1 + \frac{2}{\exp(\hbar\omega/kT) - 1}\right) 32\pi L_P^2 c^4. \quad (31)\end{aligned}$$

Evaluating the integral over ω with

$$\int_0^\infty (\cos bx) \frac{x^5 dx}{e^x - 1} = \frac{\partial^5}{\partial b^5} \left(\frac{\pi}{2} \operatorname{cotan} \pi b - \frac{1}{2b}\right), \quad (32)$$

we finally obtain

$$\begin{aligned}\tilde{\Delta}_{ijkl}(\delta t) &= -\frac{Q_{ijkl}}{8c^{10}} \left(\frac{kT\pi}{\hbar}\right)^6 \frac{32\pi L_P^2 c^4}{\sinh^6[(kT(\delta t)\pi)/\hbar]} \times \\ &\times \left[16\cosh^4\left(\frac{kT(\delta t)\pi}{\hbar}\right) + 88\cosh^2\left(\frac{kT(\delta t)\pi}{\hbar}\right) + 16\right]. \quad (33)\end{aligned}$$

Comparing eqs. (33) and (28) with $\delta t = \delta\tau$, we see that the correlation functions agree for a temperature of

$$T = \frac{\hbar a}{2\pi ck} \quad (34)$$

if the quantities in the square brackets in the two equations are the same. Now let us assume that $a\delta\tau/2c \ll 1$ for any reasonable acceleration and correlation interval. Then $\cosh(a\delta\tau/2c) = 1 + 1/2(a\delta\tau/2c)^2 + \ldots$ is 1 through first order in $a\delta\tau/2c$ and to this order the [...] in both eq. (33) and eq. (28) become equal to 120, independent of which components of Δ_{ijkl} and $\tilde{\Delta}_{ijkl}$ we are considering, that is, the dependence on n goes away.

We have shown to first order in $a\delta\tau/2c$ that an observer accelerating through the zero-point radiation of the gravitational vacuum will see the same spectral distribution as a stationary observer would see at a finite temperature with the magnitude of the acceleration and the temperature related by eq. (34). This is the Davies–Unruh effect for stochastic gravity.

5. DISCUSSION AND CONCLUSION

In addition to fields of spin 0 and spin 1, the Davies–Unruh effect has now been demonstrated for a spin 2 tensor field. But while the effect is exact for the scalar field, it is only a first-order effect for the vector and tensor fields. We reproduce the correlation functions for the three cases here (in our notation) for the purpose of comparison and discussion. The correlation functions for the scalar field have been given by Boyer [22] as

$$\Delta(\delta\tau) = -\frac{\hbar}{\pi c}\left(\frac{a}{2c}\right)^2 \sinh^{-2}\left(\frac{a\delta\tau}{2c}\right), \tag{35}$$

$$\tilde{\Delta}(\delta t) = -\frac{\hbar}{\pi c}\left(\frac{\pi kT}{\hbar}\right)^2 \sinh^{-2}\left(\frac{\pi kT\delta t}{\hbar}\right), \tag{36}$$

and for the vector field as

$$\Delta_{ij}(\delta\tau) = \delta_{ij}\frac{4\hbar}{\pi c^3}\left(\frac{a}{2c}\right)^4 \sinh^{-4}\left(\frac{a\delta\tau}{2c}\right), \tag{37}$$

$$\tilde{\Delta}_{ij}(\delta t) = \delta_{ij}\frac{4\hbar}{\pi c^3}\left(\frac{\pi kT}{\hbar}\right)^4 \sinh^{-4}\left(\frac{\pi kT\delta t}{\hbar}\right)\left[\frac{1}{3}+\frac{2}{3}\cosh^{2}\left(\frac{\pi kT\delta t}{\hbar}\right)\right], \tag{38}$$

where the indices $i, j = 1, \ldots, 6$ run over the components of the electric and magnetic fields. For the tensor field we have from eqs. (28) and (33)

$$\Delta_{ijkl}(\delta\tau) = -Q_{ijkl}\frac{4\pi L_P^2}{c^6}\left(\frac{a}{2c}\right)^6 \sinh^{-6}\left(\frac{a\delta\tau}{2c}\right)\left[120\cosh^{n}\left(\frac{a\delta\tau}{2c}\right)\right], \tag{39}$$

$$\begin{aligned}\tilde{\Delta}_{ijkl}(\delta t) = {}& -Q_{ijkl}\frac{4\pi L_P^2}{c^6}\left(\frac{\pi kT}{\hbar}\right)^6 \sinh^{-6}\left(\frac{\pi kT\delta t}{\hbar}\right)\times \\ & \times\left[16\cosh^{4}\left(\frac{\pi kT\delta t}{\hbar}\right)+88\cosh^{2}\left(\frac{\pi kT\delta t}{\hbar}\right)+16\right].\end{aligned} \tag{40}$$

Upon examining eqs. (35)–(40) with $T = \hbar a/2\pi ck$ and $\delta t = \delta\tau$, it is clear that the Davies–Unruh effect is exact only for the scalar field, and it is valid to first order in $a\delta\tau/2c$ for the vector and tensor fields. This was the conclusion of Boyer [22] for the scalar and vector fields. Our results extend the first-order effect to the tensor field. Where the effect is not exact we have written the correlation functions as a product of an exact factor and a first-order factor in square brackets. It appears that the exact factors represent the scalar Davies–Unruh effect, and the first-order factors represent departures from the scalar effect due to a multicomponent field. The exponents in the exact factors are equal to $\pm 2\,(r+1)$ where r is the rank of the tensor field.

Our equivalence of the two correlation functions, eqs. (39) and (40), to first order in $a\delta\tau/2c$ is somewhat remarkable in that the coefficients of the leading terms in eq. (40) add up exactly to 120, and $120\cosh^n(a\delta\tau/2c) \rightarrow$ 120 in eq. (39) in leading order for any value of the integer n.

To the best of our knowledge, this paper is the first attempt to bring the ideas of SED to bear on gravitation. The representation of the quantum gravity vacuum as an isotropic superposition of random-phase plane waves is intuitive, and has lead to results that are in agreement with similar results for other fields with which we are more familiar in the quantum domain. We hope that this initial investigation will lead to further progress in understanding quantum effects in gravity.

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