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A THEOREM ON THE MARKOV PERIODIC APPROXIMATION IN ERGODIC THEORY

A. M. Vershik

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One presents a new variant of the theory of periodic approximations of dynamical systems and C^*-algebras, namely the construction for each automorphism of the Lebesgue space of a Markov tower (or adic model) of periodic automorphisms. One gives several examples.

1. Introduction

In [1] we have outlined the proof of the theorm on the simultaneous uniform approximation of the multiplication and shift operators in the space of L^2-functions on a dynamical system with an invariant measure; more exactly: let S be an automorphism of the segment [0, 1] with an invariant Lebesgue measure and let φ be a bounded measurable function; for each $\varepsilon > 0$ there exist operators in L^2, $U^{(\varepsilon)}$ and $V^{(\varepsilon)}$, such that $\|U_S - U^{(\varepsilon)}\| < \varepsilon$, $\|M_\varphi - V^{(\varepsilon)}\| < \varepsilon$ and the *-algebra generated by $U^{(\varepsilon)}$ and $V^{(\varepsilon)}$ is finite-dimensional; here U_S is a unitary shift operator of the dynamical system, M_φ is a multiplier, and $\|\cdot\|$ is the operator norm. If $(Sf)(x) = f((x+\alpha) \bmod 1)$, $\varphi = \exp i2\pi x$, then the problem consists in the simultaneous approximation of the operators, satisfying the relation $UMU^{-1}M^{-1} = \exp i2\pi\alpha I$. This problem has been solved in [2] and has been formulated earlier by the author in [3, 4] and by Rieffel [5]. The above formulated theorem has applications in operator theory, in the theory of approximations of dynamical systems and in C^*-algebras; it can be considered as a variant of the noncummutative constructive theory of operators (see [4]).

The proof of the theorem is divided into two stages. The first one is a new variant of the theory of periodic approximations (Theorem 3 in [1]), namely the construction of the Markov towers of periodic automorphisms (see below) and is considered in this paper. The second stage refers to the theory of AF-algebras and will be considered elsewhere.

The approximations in the theory of dynamical systems, as in the theory of equations, can be partitioned into two classes: Factor-approximations or the moment method (projective approximation) and subapproximations or the method of nets (inductive approximation). In the first case we investigate the behavior of the operators on finite-dimensional subspaces of functions or on a collection of sets and in the second case we investigate the behavior

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of the system on finite collection of points. The approximation considered below is of the second type; we approximate an automorphism by periodic automorphisms whose trajectories are included in the trajectories of the initial one, i.e., we approximate the automorphism of a random permutation. This variant is related to the author's paper [6] on the scale of an automorphism and makes use of a modification of the construction of a consistent sequence of periodic automorphisms, found there (the lemma of Sec. 3 in [6]). The important modification introduced here consists in the fact that we do not require the constancy of the minimal periods with respect to all points for each of the periodic automorphisms of the approximation; in return, the consistency of the approximation steps is stronger: Each permutation differs from the approximated automorphism only at one point of its layer and, as a function on the quotient space relative to the trajectories of a periodic automorphism with values in the (variable) group of permutations, is measurable relative to some additional partition to the trajectory partition of the subsequent term of the approximation (see below). Such a rigid requirement excludes in many respects the additional arbitrariness in the approximation which one usually has. Thus we arrive at the concept of a Markov tower or adic automorphism model, whose existence is proved in this paper. The construction of these models or approximations in concrete cases is a complex problem; in Sec. 4 we give some examples. In Sec. 2 we give the definitions and in Sec. 3 we give the proof of the fundamental theorem.

We recall that the existence of a periodic automorphism, approximating uniformly (in the sense of the theory of measure) the given one, is the content of the classical Rokhlin—Halmos lemma; the question of the sequences of periodic approximations, consistent in some sense, has been considered in different ways in a series of investigations (see, for example, the survey [7]) and the projective approximation (by multivalued mappings) has been introduced in [8]. We also mention that in our view such an approximation (multivalued; see [9]) must be an object of investigation also in hydrodynamics (in its metric variant) in close connection with the approximations, so fruitfully investigated in the works of O. A. Ladyzhesnkaya.

2. Markov Compactum and Adic Transformation

In the sequel, for each automorphism of the Lebesgue space we construct a tower of periodic automorphisms. But first we define an appropriate model space and its transform, to which we wish to reduce (i..e, to establish an isomorphism) an arbitrary pair; in other words, we define a class of canonical towers.

Let $r_0=1, r_1, \ldots$ be a sequence of natural numbers $r_i \geqslant 2$, $i=1, \ldots$; let $M_1, M_2, \ldots$ be a sequence of $\mathbb{Z}_+$-matrices, where the matrix $M_i=(m^i_{jk})$ has dimension $r_{i-1} \times r_i$, $i=1, \ldots$. We shall assume that none of these matrices has zero rows or columns. We consider the graded graph $\Gamma(\{M_i\})$ whose vertices $\mathfrak{D}=\bigcup_{i=0}^{\infty}\mathfrak{D}_i$ are partitioned into finite "levels" $\mathfrak{D}_0=\{\varnothing\}, \mathfrak{D}_1, \ldots$, and the number of elements is $|\mathfrak{D}_i|=r_i$, $i \geqslant 0$; the arcs of the graph join only vertices of adjacent levels and the number of arcs joining vertex j of the ($i-1$)-st level with vertex k of the i-th level is equal to m^i_{jk}, $k=1,\ldots,r_i$, $j=1,\ldots, r_{i+1}$, $i=0,1,\ldots$. By virtue of the conditions on the matrices, each vertex is joined at least with one vertex of the next level and (except for the vertex $\varnothing \in \mathfrak{D}_0$) of the previous level. By a path in the graph we mean a sequence of arcs $(\gamma_0, \gamma_1, \ldots)$, where γ_{i-1} and

r_i have a common vertex from the i-th level, $i=1,2,\dots$. Let $\mathfrak{X}(\Gamma)$ be the space of all the paths of the graph $\Gamma(\{M_i\}_{i=1}^{\infty})$, equipped with the weak topology. (A neighborhood of a path is the collection of all paths coinciding with the given one up to some places; obviously, $\mathfrak{X}(\Gamma)$ is a compactum).

Definition 1. By a Markov compactum with parameters $\{r_i,\ M_i;\ i=1,\dots,\}$ we mean $\mathfrak{X}(\Gamma)$, $\Gamma=\Gamma(\{M_i\})$. If $r_i\equiv r$ and $M_i\equiv M\ i\geqslant 1$, then the Markov compactum is said to be stationary.

In topological dynamics one considers frequently stationary Markov compacta and their two-sided shifts. We shall consider arbitrary Markov compacta and their transformations of another kind. Various properties of the compactum $\mathfrak{X}(\Gamma)$ and of the measures on them, in dependence on the asymptotic behavior of the matrices M_i, will be considered elsewhere; they are related, in particular, to the properties of AF-algebras constructed over the graph Γ, as over the Bratteli diagram [10].

We restrict ourselves to the case when the elements of M_i consist of zeros and ones. In this case the graph does not have multiple arcs and a path is a succession of vertices; this assumption does not diminish at all the generality since by introducing new vertices one can get rid of multiplicities. In this case the Markov compactum acquires a usual description: it is a closed subset in the compactum $\prod_{i=1}^{\infty}\mathfrak{D}_i$, consisting of all sequences $\{d_i\}$ for which $m^i_{d_i d_{i+1}}=1$.

We give a direct description of the structure of a Markov compactum.

Proposition 1. Every Markov compactum is a totally disconnected separable compactum X with a distinguished sequence of finite partitions $\{\eta_i;\ i\geqslant 1\}$, possessing the following properties:

1) $\bigvee_{i=1}^{\infty}\eta_i=\varepsilon$

2) for each $\kappa\geqslant 1$ and any set $\mathfrak{D}$ of the partition element η_κ the correspondence $\mathfrak{D}\ni x\mapsto\{\eta_i(x);\ i\neq\kappa\}$ defines a homeomorphism

$$\mathfrak{D}\simeq \mathfrak{D}\Big/\bigvee_{i=1}^{\kappa-1}\eta_i \times \mathfrak{D}\Big/\bigvee_{\kappa+1}^{\infty}\eta_i .$$

Here $\bigvee$ denotes the product of the partitions, i.e., the partition consisting of all possible intersections of the multiplied partitions; ε is the partitioning into isolated points. Condition 1) means that for $x\neq y$ there exists η_i such that $\eta_i(x)\neq\eta_i(y)$, where $\eta(x)$ is the element of η containing x. In condition 2) the symbol $\mathfrak{D}/\xi$ denotes the factorization of $\mathfrak{D}$ with respect to the partitioning and the meaning of the condition is the following: If $x,y\in\mathfrak{D}$, then there exists $z\in\mathfrak{D}$ for which the projection onto $\mathfrak{D}/\eta_i$ for $i<\kappa$ is the same as for x and onto $\mathfrak{D}/\eta_i$ for $i>\kappa$ is the same as for y.

If one denotes the number of elements of η_i by r_i and the intersection matrix of η_{i-1} and η_i by M_i, then the compactum X from this proposition becomes $\mathfrak{X}(\Gamma(M_i))$; conversely, denoting by η_i the partitioning of $\mathfrak{X}$ into the classes of paths passing through a given vertex of the i-th level, we can see that the conditions of the proposition holds. We shall use the definition from the proposition.

A subset of a Markov compactum $\mathfrak{X}$, that is measurable with respect to $\bigvee_{i=1}^{\kappa} \eta_i$ for some κ, will be said to be cylindrical; these subsets form an algebra, denoted by $\mathfrak{B}$; a transformation leaving $\bigvee_{i=1}^{\kappa} \eta_i$ invariant and $\bigvee_{i=\kappa}^{\infty} \eta_i$ fixed is κ-cylindrical. Each κ-cylindrical invertible transformation of a Markov compactum is a periodic transformation whose trajectories lie on the elements of the partition $\bigvee_{i=\kappa}^{\infty} \eta_i$. By a transformation of class $\mathfrak{P}$ we mean a homeomorphism T for which there exists a nowhere dense closed invariant set such that, in the complement of any of its open neighborhood, T is a k-cylindrical (periodic) transformation, where k depends on the neighborhood. Thus, the transformation of class $\mathfrak{P}$ are limits of k-cylindrical (periodic) transformations in the sense of uniform convergence on any closed set lying in the complement of some nowhere dense set.

An example of a transformation of class $\mathfrak{P}$ of a very general form, to which we shall restrict ourselves in the sequel, is obtained in the following manner.

We index the elements of the partitions η_i by natural numbers from 1 to r_i, $i=1,\ldots$ Thus, we can introduce a lexicographic ordering on the set of elements $\bigvee_{i=1}^{k} \eta_i$ for all k assuming $(d_1,\ldots,d_\kappa) \succ (d'_1,\ldots,d'_\kappa)$, if $d_{\kappa-i} = d'_{\kappa-i}$ $i=0,1,\ldots,i_0$; $d_{\kappa-i_0} > d'_{\kappa-i_0}$. On the space $\mathfrak{X}$ there emerges a partial ordering in which the sequences coinciding from some place are comparable; the order type in each class of comparable elements is either $\mathbb{Z}^{\cdot}$, or $\mathbb{Z}_+$, or $\mathbb{Z}_-$, or $\overline{n,m}$. A Markov compactum together with the ordering of the partitions is said to be minimal if all except two classes are ordered by type $\mathbb{Z}$, one by type $\mathbb{Z}_+$ and one by type $\mathbb{Z}_-$. Minimality can be expressed in a simple manner as a condition on the matrices M_i:

Proposition 2. There exists a sequence $\kappa_1 < \kappa_2 < \ldots$ such that the matrices $M_{\kappa_i} \cdot M_{\kappa_i+1} \cdot \ldots \cdot M_{\kappa_{i+1}-1}$ do not have zero elements $i=1,2,\ldots$. Clearly, it does not depend on the ordering on η; by the change of the indexing only the one-sided orbits are changed.

Definition 2. By an adic transformation of a Markov compactum with ordered partitions η_i, we mean a transformation T which assigns to an element x the element x' which is directly larger in the sense of the ordering. (If x' does not exist, then T is not defined.) If $\mathfrak{X}$ is minimal, the we define an adic transformation by combining the semiorbits $\mathbb{Z}_+$ and $\mathbb{Z}_-$.

Proposition 3. An adic transformation of a minimal Markov compactum belongs to the class $\mathfrak{P}$.

The proof follows directly from the definition of the ordering, the exceptional and inverse set in this case consisting of one point (the end of the orbits of type $\mathbb{Z}_-$ and the origin of the orbits $\mathbb{Z}_+$, respectively).

Remark. The trajectory partition of an adic transformation is the tail partition of $\mathfrak{X}$ (i.e., $\bigcap_{\kappa=1}^{\infty} \bigvee_{i=\kappa}^{\infty} \eta_i$, where the symbol $\bigcap$ is the set theoretic intersection).

Thus, an adic shift is the limit of cylindrical transformations, i.e., of the substitutions of the first coordinates of the sequence. It can be visualized as a tower of successive substitutions of the elements of the partitions $\eta_1, \eta_2, \ldots$. It is the model to which we reduce an arbitrary ergodic automorphism of the Lebesgue space (see the examples in Sec. 4).

3. Fundamental Theorem

THEOREM. Let (X, m) be a Lebesgue space and let S be its ergodic automorphism with invariant measure m; $\mathfrak{A}_0$ is the countable algebra of measurable sets in S, invariant with respect to X. There exists a set of full measure $X_0 \subset X$, $X_0 \in \mathfrak{A}_0$ and an automorphism $S' = S \bmod 0$, $S'X_0 = X_0$ as well as a minimal Markov compactum $\mathfrak{X}$ (with ordered elements of the partitions (η_i) $i \geq 1$), such that $(X_0, S', \mathfrak{A}_0)$ and $(\mathfrak{X}, T, \mathfrak{B})$ are isomorphic, i.e., one has a mapping $t : \hat{X}_0 \longrightarrow \mathfrak{X}$, defined everywhere on X_0 and $t^{-1}\mathfrak{B} = \mathfrak{A}_0$ ($\mathfrak{B}$ is the algebra of cylindrical sets) and $tS't^{-1} = T$, where T is an adic transformation.

Our problem consists in the construction of a sequence of partitions X on η_i with the required properties. But we start with the construction of periodic approximations, which will correspond to cylindrical periodic transformations in the image.

If B is an arbitrary measurable set, $mB > 0$, in X, then each ergodic automorphism can be represented as an integral over the derivative S_B relative to the subset B (see [11]). We fix an automorphism and we form partitions $\zeta(B) = \zeta(B, S)$ and $\xi(B) = \xi(B, S)$ in the following manner. $\zeta(B)$ consists of at most a countable number of elements $B_0 = B$, $B_1 = SB_0 \setminus B_0, \ldots, B_k = SB_{k-1} \setminus \bigcup_{i=0}^{k-1} B_i$, and the elements of $\zeta(B)$ are ordered in the natural manner; the element $\xi(B)(x)$, containing the point $x \in B$, consists of all points of the form $S^i x$, where $S^i x \in B_i$; ζ and ξ are mutually complementary. If $r_B(x) = |\xi(B)(x)|$, since S is ergodic, we have $r_B(x) < \infty$ almost everywhere; we denote $r_{B,S} = \operatorname{supess} r_B(x)$. Let $S^{(B)}x = Sx$ if $Sx \in \xi(B)(x)$ and $S^{(B)}x = S^{-r_B(x)}x$ if $Sx \notin \xi(B)(x)$; $S^{(B)}$ is a periodic automorphism with period depending on the point.

LEMMA. Let $\mathfrak{A}_0$ be an everywhere dense algebra of measurable sets, invariant with respect to the ergodic automorphism S. There exists a sequence of sets $A_1 \supset A_2 \supset \ldots$, $m(\bigcap_i A_i) = 0$, $A_i \in \mathfrak{A}_0$, for which $r_{A_i} < \infty$ for all $i = 1, \ldots$

Proof. We select an arbitrary $A_1' \in \mathfrak{A}_0$, $mA_1' < \frac{1}{2}$. We denote $A_1 = A_1' \cup A_1''$, where A_1'' is the union of all but a finite number of elements of the partition $\zeta(A_1')$, having the measure $< \frac{1}{2} mA_1'$. Since $\mathfrak{A}_0$ is invariant, then $\zeta(A_1')$ is measurable relative to $\mathfrak{A}_0$ and therefore A_1'' and A_1 lies in $\mathfrak{A}_0$. We note that $r_{A_1, S} < \infty$. We consider $S_1 = S_{A_1}$, the derivative of S on A_1 and $A_2' \subset A_1$, $nA_2' < \frac{1}{2} mA_1$. We apply the same method to A_1, S_1 and A_2' as for X, S, A_1' in the first step. As a result, we construct $A_2 \subset A_1$, $r_{A_2, S} < \infty$ and, therefore $r_{A_2, S} < \infty$. Since $mA_1 < \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, we have $mA_2 < \frac{3}{8} + \frac{3}{16} = \left(\frac{3}{4}\right)^2$. Continuing this process, we obtain a system of sets $A_1 \supset A_2 \ldots$ from $\mathfrak{A}_0, mA_n < \left(\frac{3}{4}\right)^n$ and $r_{A_i, S} < \infty$, which is what we intended to prove.

We denote $S_n = S^{(A_n)}$; the required periodic automorphisms have been constructed. Obviously, $\lim_n m\{x : S_n x \neq Sx\} = 0$. Now we construct a sequence of refinements $\zeta(A_n)$ which brings in the structure of a Markov compactum.

Let $\rho_n \nearrow \varepsilon$ be an increasing sequence of finite partitions (here ε is the partition into points $\bmod 0$), consisting of sets from $\mathfrak{A}_0$ and generating $\mathfrak{A}_0$. We identify $X/\xi(A_n)$ and A_n, associating to an element its unique point from A_n. Let

$$\pi_n : X/\xi(A_n) \to A_n.$$

Obviously, $\pi_n \pi_{n-1} = \pi_n$ for all n since $A_{n-1} \subset A_n$.

Let $\eta_1 = \zeta(A_1; S) \vee \rho_1$. We define by induction the pattitions η_k :

$$\eta_k = \pi_{k-1}^{-1}\left[\zeta(A_k; S_{A_{k-1}}) \vee \pi_{k-1}\left(\bigvee_{i=1}^{k-1} \eta_i \vee \rho_k\right)\right]$$

$k = 2, 3, \ldots$. The elements of η_k inherit the ordering of $\zeta(A_k; S_{A_{k-1}})$. We verify that the required properties hold. Since $\zeta(A_k; S) = \zeta(A_1; S) \vee \pi_1^{-1}(\zeta(A_2; S_{A_1})) \ldots \pi_{k-1}^{-1}(\zeta(A_k; S_{A_{k-1}}))$, we have $\zeta(A_k; S) \prec \bigvee_{i=1}^{k} \eta_i$ and, therefore, the last partition is complementary to $\xi(A_k; S)$, i.e., to the trajectory partition of the periodic automorphism S_k . Now, $\rho_k \prec \bigvee_{i=1}^{k} \eta_i$, since $\rho_k \prec \zeta(A_k; S) \vee \eta_k$; consequently, $\bigvee_{i=1}^{\infty} \eta_i \succ \lim \rho_k = \varepsilon$. By construction, the sequence ${}^{k}\{\eta_i\}$ is Markov and in one element $\mathfrak{D} \in \eta_k$ the automorphism S_{k-1} has the same periods for all points, the elements of $\bigvee_{i=1}^{k-1} \eta_i$ being moved while the elements of $\bigvee_{i=1}^{\infty} \eta_i$ remain fixed. Finally, all elements of all η_k lie in $\mathfrak{A}_0$.

Now we define a Markov compactum $\mathfrak{X}$ with respect to the partitions $\{\eta_i\}$ and we reflect X into the space $\prod_{k=1}^{\infty} X/\eta_k = \tilde{\mathfrak{X}}$ considered as the infinite product of the finite sets X/η_k with the weak topology. From what has been said it is clear that the mapping $\theta : x \to (\eta_1(x), \ldots) \in \tilde{\mathfrak{X}}$ is a $\bmod 0$ monomorphism and the closure of the image of a set of complete measure is a Markov compactum $\mathfrak{X} \subset \tilde{\mathfrak{X}}$; the image $\mathfrak{A}_0$ is the algebra of cylindrical sets; the Markov property of $\mathfrak{X}$ follows from the Markov property of $\{\eta_k\}_1^{\infty}$, where the elements η_k are ordered, and the transformation S goes into the adic shift as one can see from the construction. Finally we verify the minimality of $\mathfrak{X}$; for this we note that from the erogodicity of S there follows that for any elements $\mathfrak{D} \in \eta_k$ and $E \in \eta_s$ there exists $n = n(k, s)$ for which $m(S^n \mathfrak{D} \cap E) > 0$; this means that the matrix of the interactions η_k and η_s consists of positive elements (see Proposition 2, Sec. 2). The theorem is proved.

Remarks

1. Our method of construction of $\{\eta_k\}$ is not economical since the refinement of ζ_k is not accompanied by the lengthening of the trajectories of S_k ; as far as possible, $\bigvee_1^k \eta_i$ must be "almost" an independent complement to ξ_k . The construction of economical realizations is a complex problem (see the examples).

2. By the theorem of [12], there exist an invariant algebra $\mathfrak{A}_0$ and a relatively ergodic automorphism, consisting of strictly ergodic sets. If in the theorem one starts from this algebra, then the corresponding adic transformation will be strictly ergodic.

3. The asymptotic properties of the matrices of the intersections of the partitions $\{\eta_k\}$ contain ample information about the metric invariants of the automorphism of the scale type (see [6]).

4. The theorem can be easily generalized to nonergodic automorphisms; in this case the Markov compactum will not be minimal.

5. The theorem is valid for any countable group G of automorphisms for which the Rokhlin—Halmos lemma holds.

4. Examples

1. Let S be an ergodic automorphism with rational spectrum, i.e., with a spectrum which is the union of finite subgroups of roots of unity of order $p_1, p_2, \ldots$; $p_i \mid p_{i+1}$; $p_{i+1}/p_i = r_i$. In this case the adic realization is the following. The compactum $\mathfrak{X} = \varprojlim \mathbb{Z}/p_i\mathbb{Z}$ and the transformation T is given by $Tx = x+1$, addition being considered in the additive group $\mathfrak{X}$. Since the Markov compactum $\mathfrak{X}$ is $\prod_{i=1}^{\infty} \mathbb{Z}/r_i\mathbb{Z}$, the matrices $M_i = \{m^i_{jk}\}$, $m^i_{jk} \equiv 1$. If $r_i \equiv r$, then $\mathfrak{X}$ is the additive group of integer r-adic numbers which explains the term "adic automorphism."

2. Let $S_\alpha x = (x+\alpha) \bmod 1$, where $x \in [0,1)$. If $\alpha \in [0,1)$ is irrational, then S_α is ergodic and its adic realization can be given in the following manner. Let $\alpha = (n_1, n_2, \ldots)$ be the expansion of α into a continuous fraction; $\widetilde{\mathfrak{X}} = \prod_{i=1}^{\infty} N_i$, $r_i = |N_i| = n_i + 1$; we consider the matrices $M_i = \{m^i_{jk}\}$ i of order $r_i \times r_{i+1}$: $m^i_{jk} = 1$ for $j \geqslant 2$; $m_{1k} = 0$ for $k \leqslant n_{i+1}$ $m_1 r_{i+1} = 1$; $i = 1, 2, \ldots$. In other words,

$$M_i = \begin{pmatrix} 0 \ldots\ldots 0 & 1 \\ 1 \ldots\ldots\ldots & 1 \\ 1 \ldots 1 \ldots\ldots & 1 \end{pmatrix}.$$

The Markov compactum $\mathfrak{X} \subset \widetilde{\mathfrak{X}}$ is constructed on $\{r_i\}$ and $\{M_i\}$. The adic shift in it is isomorphic to the shift S_α ; more precisely: there exists an isomorphism of the space with measure (on * there is a unique measure, invariant relative to the adic shift) reducing S_α to an adic shift. Indeed, in this case the isomorphism is the Borel isomorphism of the topological spaces (circumference and $\mathfrak{X}$). This model is the modification of the constructions from [2, 13]; however, there, the constructions are used for other purposes. At the suggestion of the author, M. and L. Gandel'sman have established the isomorphism which has been mentioned above.

Another adic realization of the rotation of the circumference is the folllowing: $\alpha_0 = \alpha$, $\alpha_i = 1 - \{\frac{1}{\alpha - 1}\}$, $r_i = n_i = [\frac{1}{\alpha_{i-1}}] + 1$, $i = 1, 2, \ldots$; M_i is a matrix of order $r_i \times r_{i+1}$, $M_i = (m^i_{jk})$, $m^i_{jk} = 1$ for all j, k except $(1, r_{i+1})$; $m_{1, r_{i+1}} = 0$, i.e.,

$$M_i = \begin{pmatrix} 1 \ldots\ldots 1 & 0 \\ 1 \ldots\ldots 1 & 1 \\ \vdots & \vdots \\ 1 \ldots\ldots\ldots & 1 \end{pmatrix}.$$

Thus, in each matrix M_i there is exactly one zero (see Sec. 1). $\mathfrak{X}(\Gamma(\{r_i\}, \{M_i\})$ gives the required compactum.

3. A shift on tori and so much more the automorphisms with continuous spectrum involve more complications. Here the Perron algorithm and the generalized expansions into a continuous fraction are useful. It is interesting to pose inverse problems, i.e., to investigate the ergodic properties of an addic shift in given Markov compacta and graphs. The Pascal graph, the Young graph generate examples of shifts about which one does not know anything.

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EXISTENCE AND UNIQUENESS THEOREMS FOR A-REGULAR GENERALIZED SOLUTIONS OF THE FIRST BOUNDARY-VALUE PROBLEM FOR $(A,\vec{0})$-ELLIPTIC EQUATIONS

A. V. Ivanov UDC 517.9

For second-order quasilinear degenerate elliptic equations, having the structure of $(A,\vec{0})$-elliptic equations in a bounded domain $\Omega\subset R^n$, $n\geq 2$, one establishes theorems of existence and uniqueness for the generalized solutions of the first boundary-value problem, bounded together with their A-derivatives of first order and also of first and second order. The case of linear second-order $(A,\vec{0})$-elliptic equations are separately considered.

In the early sixties, O. A. Ladyzhenskaya and N. N. Ural'tseva have constructed the theory of solvability of boundary-value problems for quasilinear second-order uniformly elliptic and parabolic equations [1, 2]. These results have formed the necessary foundation for the subsequent development of the theory of boundary-value problems for quasilinear elliptic and parabolic equations. The overwhelming majority of the subsequent investigations in this area is based to a certain extent on the mentioned results of Ladyzhenskaya and Ural'tseva. In this respect, the present paper is not an exception; we investigate the question of the existence and uniqueness of the regular solutions of the first boundary-value problem for a class of quasilinear degenerate second-order elliptic equations. The results obtained here are new even for the case of linear equations with a nonnegative characteristic form. Other results regarding the existence and the uniqueness of regular solu-

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