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On Nonlinear Planetary Waves: A Class of Solutions Missed by the Traditional Quasi-Geostrophic Approximation*

Toshio YAMAGATA**

Abstract: Weakly nonlinear quasi-geostrophic planetary waves on a beta-plane and topographic waves over a linearly inclined bottom are examined by use of shallow water equations for a small beta parameter. Long solitary wave solutions missed by the use of the traditional quasi-geostrophic approximation are found in a channel ocean with neither a sheared current nor a curved (non-linearly inclined) bottom topography. The solutions are missed in the traditional approach because the irrotational motion associated with the geostrophic divergence is neglected by the quasi-geostrophic approximation. Another example which calls attention to the limitation of the traditional quasi-geostrophic approximation is the nonlinear evolution of divergent planetary eddies whose scale is much larger than the Rossby's radius of deformation. Some aspects of a new evolution equation are briefly discussed.

1. Introduction

Solitary waves of planetary scale have been discussed by CLARKE (1971), SMITH (1972), GRIMSHAW (1977), BOYD (1977), ODULO and PELINOVSKIY (1978), and MALANOTTE RIZZOLI and HENDERSHOTT (1980). Solitary planetary waves in zonal shear flows were studied by LONG (1964), LARSEN (1965), BENNEY (1966), CLARKE (1971), MAXWORTHY and REDEKOPP (1976), REDEKOPP (1977), HUKUDA (1979) and FLIERL (1979). Among them, MAXWORTHY and REDEKOPP (1976) applied the shear soliton theory to the Great Red Spot observed in the Jovian atmosphere. FLIERL (1979) presented a two-dimensional baroclinic shear soliton which is similar to observed atmospheric or oceanic isolated eddies. Recently, MILES (1979) has laid the variational foundations of the planetary shear soliton with no critical layers.

In these studies, authors except CLARKE (1971), SMITH (1972), GRIMSHAW (1977) and BOYD (1977) adopted the potential vorticity equation or the quasi-geostrophic potential vorticity equation (Q-G.P.V.E.) as a model equation; they were obliged to include sheared currents or curved (non-linearly inclined) bottom topography in their systems in order to obtain the Korteweg-de Vries (K-dV) dynamics.

CLARKE (1971), SMITH (1972) and GRIMSHAW (1977) discussed the K-dV dynamics of ageostrophic and/or semigeostrophic long waves over a continental shelf. In particular CLARKE (1971) noticed the significance of the lateral divergence in the long wavelength limit. Solitary equatorial waves were discussed by BOYD (1977, 1980). These four authors adopted the shallow water equations in their analyses; this is natural since they examined those ageostrophic and/or semigeostrophic waves in which irrotational motion is important.

The present article starts from shallow water equations, which are less restrictive than the Q-G.P.V.E., and discusses the limitations of the Q-G.P.V.E. for long quasi-geostrophic waves. The major difference from the traditional approach lies in the significance attached to the irrotational motion due to lateral divergence which is neglected by the traditional quasi-geostrophic approximation.

The format of the present work is as follows: Section 2 presents the K-dV equation which governs the evolution of long divergent quasi-geostrophic planetary waves of odd cross-channel mode. The deficiencies in the Q-G.P.V.E. are discussed in detail. In Section 3, long topographic waves over a linearly inclined bottom are discussed, and it is shown that the K-dV

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** Research Institute for Applied Mechanics, Kyushu Univ., 6-10-1 Hakozaki, Higashi-ku 812, Japan
Present Affiliation: Geophysical Fluid Dynamics Program, Princeton University, P.O. Box 308, Princeton, N.J. 08540, U.S.A.

dynamics holds when the free surface is not included. In Section 4, the role of lateral shear of mean currents and/or curved bottom topography is discussed. This section clarifies the special nature of a linear bottom slope and a beta-plane without mean sheared currents. In Section 5, a new two-dimensional nonlinear evolution equation* is derived which governs the motion on a lateral scale much larger than the Rossby's radius of deformation. This section again shows that one should be cautious when using the traditional quasi-geostrophic approximation. The final section is a summary.

2. Long planetary waves in a shallow layer of homogeneous fluid

2.1. Formulation

The nonlinear shallow water equations** on a mid-latitude beta-plane are

$$\begin{aligned} &u_t+uu_x+vu_y-fv=-g\zeta_x\,,\\ &v_t+uv_x+vv_y+fu=-g\zeta_y\,,\\ &\{(H+\zeta)u\}_x+\{(H+\zeta)v\}_y+\zeta_t=0\,. \end{aligned} \qquad (2.1.1)$$

Here the Coriolis parameter f is given by

$$f=f_0+\beta y\,, \qquad (2.1.2)$$

and the mean fluid depth H is assumed to be constant. The other notations follow conventional usage. We adopt a channel ocean with north-south extent L. Let L, $(\beta L)^{-1}$, U and $g^{-1}fUL$ denote a length scale, a time scale, a velocity scale and a scale of surface elevation, respectively. Then we obtain the nondimensional equations:

$$\begin{aligned} &\delta u_t+R_0(uu_x+vu_y)-(1+\delta y)v=-\zeta_x\,,\\ &\delta v_t+R_0(uv_x+vv_y)+(1+\delta y)u=-\zeta_y\,,\\ &u_x+v_y+F\{\delta\zeta_t+R_0(\zeta u)_x+R_0(\zeta v)_y\}=0\,, \end{aligned} \qquad (2.1.3)$$

where $\delta(=\beta Lf_0^{-1})$ is the beta parameter, F $(=f_0^2L^2g^{-1}H^{-1})$ is the rotational Froude number and $R_0(=Uf_0^{-1}L^{-1})$ is the Rossby number. Although CLARKE (1971) assumed $\delta\sim O(1)$, we assume $\delta\ll O(1)$ hereafter. The present assumption is valid for meso-scale motions which have a north-south extent L smaller than $O(10^3\,\text{km})$ in mid-latitude. We further assume $R_0=\gamma\delta^2$ where γ is of the order of unity. The latter assumption is made in order to discuss weakly nonlinear waves and is valid for flows of a few cm s^{-1} velocity in mid-latitude. The rotational Froude number F is assumed to be $O(1)$ parameter except for Section 5.

2.2. K-dV Equation

We adopt, a coordinate system moving with the long wave speed c_0 and introduce the long length scale and time scales:

$$\begin{aligned} &X=\delta(x-c_0t-\delta c_1t)\,,\\ &T=\delta^3t\,, \end{aligned} \qquad (2.2.1)$$

where c_1 denotes the $O(\delta)$ correction of the long wave speed determined later. These scales are introduced by considering the Morikawa-Gardner transformation and the assumption that $R_0=\gamma\delta^2$.

Then we can write (2.1.3) in the form:

$$\begin{aligned} &-\delta^2c_0u_X-\delta^3c_1u_X+\delta^4u_T+\gamma\delta^2(\delta uu_X+vu_y)\\ &\qquad-(1+\delta y)v=-\delta\zeta_X\,,\\ &-\delta^2c_0v_X-\delta^3c_1v_X+\delta^4v_T+\gamma\delta^2(\delta uv_X+vv_y)\\ &\qquad+(1+\delta y)u=-\zeta_y\,,\\ &\delta u_X+v_y+F\{-\delta^2c_0\zeta_X-\delta^3c_1\zeta_X+\delta^4\zeta_T\\ &\qquad+\gamma\delta^3(\zeta u)_X+\gamma\delta^2(\zeta v)_y\}=0\,. \end{aligned} \qquad (2.2.2)$$

We seek an asymptotic solution of (2.2.2) with the form:

$$\begin{pmatrix} u\\ v\\ \zeta\end{pmatrix}=\begin{pmatrix} u^{(0)}\\ v^{(0)}\\ \zeta^{(0)}\end{pmatrix}+\delta\begin{pmatrix} u^{(1)}\\ v^{(1)}\\ \zeta^{(1)}\end{pmatrix}+\delta^2\begin{pmatrix} u^{(2)}\\ v^{(2)}\\ \zeta^{(2)}\end{pmatrix}+\cdots. \qquad (2.2.3)$$

Substituting (2.2.3) into (2.2.2) yields, to $O(\delta^2)$,

$$\zeta_{yyX}{}^{(0)}-\left(F+\frac{1}{c_0}\right)\zeta_X{}^{(0)}=0\,, \qquad (2.2.4)$$

with boundary conditions:

$$\zeta_X{}^{(0)}=0 \text{ at } y=0 \text{ and } 1\,. \qquad (2.2.5)$$

The solution is

$$\zeta^{(0)}=A(T,X)\sin m\pi y\,, \qquad (2.2.6)$$

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* FLIERL (1980) recently discussed a similar problem.

** Strictly speaking, the approximations to obtain (2.1.1) should be improved step by step as we proceed to higher order nonlinear problems by a small parameter expansion. Here (2.1.1) is taken as the model equation. The shallow water equations are less restrictive than the traditional quasi-geostrophic potential vorticity equation and they are sufficient for our purpose of revealing the source of nonlinearity missed by the traditional approach.

with the phase speed of the long wave:

$$c_0=\frac{-1}{m^2\pi^2+F}\ (m=1,2,3,\ldots). \quad (2.2.7)$$

Here m denotes the mode number in the cross-channel direction. Proceeding to $O(\delta^3)$, we get

$$\begin{aligned}\zeta_{yyX}{}^{(1)}-\left(F+\frac{1}{c_0}\right)\zeta_X{}^{(1)}\\ =2A_X(m\pi\cos m\pi y+Fy\sin m\pi y)\\ -A_X\frac{c_1}{c_0{}^2}\sin m\pi y,\end{aligned} \quad (2.2.8)$$

with boundary conditions:

$$\zeta_X{}^{(1)}=-A_Xc_0m\pi\cos m\pi y \quad \text{at } y=0 \text{ and } 1. \quad (2.2.9)$$

The forced solution is

$$\zeta^{(1)}=-c_0m\pi A\cos m\pi y+Ay\sin m\pi y+\sum_{\substack{n=1\\(\neq m)}}^{\infty}p_nA\sin n\pi y, \quad (2.2.10)$$

where

$$p_n=\begin{cases}0 & \text{for } m+n \text{ even},\\ \dfrac{-16mnF}{(m^2-n^2)^3\pi^4} & \text{for } m+n \text{ odd}.\end{cases}$$

The phase speed correction c_1 is determined in the process of removing the linear resonance at the present order; we obtain

$$c_1=Fc_0{}^2. \quad (2.2.11)$$

Notice that the above phase speed correction vanishes for the nondivergent case ($F=0$). This is not surprising because the solution (2.2.6) is an exact solution of the nondivergent shallow water equations. Proceeding further to $O(\delta^4)$, we obtain

$$\zeta_{yyX}{}^{(2)}-\left(F+\frac{1}{c_0}\right)\zeta_X{}^{(2)}=\mathbf{F}, \quad (2.2.12)$$

with the boundary condition:

$$\begin{aligned}\zeta_X{}^{(2)}=c_0y\zeta_{yX}{}^{(0)}-c_0\zeta_{yX}{}^{(1)}-c_1\zeta_{yX}{}^{(0)}\\ -\gamma\zeta_y{}^{(0)}\zeta_{yX}{}^{(0)} \quad \text{at } y=0 \text{ and } 1,\end{aligned} \quad (2.2.13)$$

where $\mathbf{F}$ is shown in the appendix. Applying the solvability condition and integrating by parts by use of (2.2.13), we find the evolution equation for A:

$$A_T+a_1A_X+a_2AA_X+a_3A_{XXX}=0, \quad (2.2.14)$$

where the values of the coefficients a_i ($i=1$-3) are shown in the appendix. Equation (2.2.14) reduces to the well-known K-dV equation by use of the transformation:

$$\begin{pmatrix}X\\T\end{pmatrix}\to\begin{pmatrix}X-a_1T\\T\end{pmatrix}. \quad (2.2.15)$$

Then we find

$$A_T+a_2AA_X+a_3A_{XXX}=0. \quad (2.2.16)$$

Since a_2 is non-zero when m is odd, (2.2.16) reduces to the canonical form:

$$\eta_T+6\eta\eta_\xi+\eta_{\xi\xi\xi}=0, \quad (2.2.17)$$

where the transformation is adopted such that

$$\begin{pmatrix}A\\X\\T\end{pmatrix}\to\begin{pmatrix}6a_3{}^{1/3}a_2{}^{-1}\eta\\a_3{}^{1/3}\xi\\T\end{pmatrix}. \quad (2.2.18)$$

The progressive wave solution which moves with the speed λ in the ξ-direction and satisfies the condition $\eta(\xi\to\pm\infty)\to0$ is obtained for positive λ by integrating (2.2.18) twice with respect to ξ. The solution is

$$\eta=\frac{\lambda}{2}\mathrm{sech}^2\left\{\frac{1}{2}\sqrt{\lambda}(\xi-\lambda T)\right\}. \quad (2.2.19)$$

Since both a_2 and a_3 are negative, (2.2.19) represents a solitary wave of surface elevation. This can be explained as follows. The magnitude of the long wave speed becomes large as the fluid depth increases (see (2.2.7)). Hence an anticyclonic eddy with surface elevation has a tendency to steepen if the nonlinearity is considered. The weak dispersion cancels that effect, thus enabling the solitary wave of surface elevation to exist. If m is even, there is no net surface elevation. Hence we cannot obtain the K-dV equation in such a case. This is because the eigensolution (2.2.6) is antisymmetric. CLARKE (1971) asssumed $\delta\sim O(1)$, and so the eigensolution is not antisymmetric when m is odd. This is the reason why the solitary solution was obtained for even m by CLARKE (1971).

For a non-divergent case ($F=0$) the coefficient a_2 also vanishes. This is not surprising; the linear solution (2.2.6) is an exact solution of the nonlinear shallow water equations on a beta-planê. In this case the motion is purely rotational. Hence we see that the lateral convergence in front of the wave crest and the lateral divergence behind it are prerequisites for obtaining the K-dV dynamics of long planetary waves.

2.3. Deficiencies of the Q-G.P.V.E. in the long wavelength limit

Now let us start tentatively from the barotropic quasi-geostrophic potential vorticity equation, which is written as

$$\delta(\nabla^2-F)\zeta_t+R_0 J(\zeta,\nabla^2\zeta)+\delta\zeta_x=0\,. \quad (2.3.1)$$

Here J denotes the Jacobian operator and ∇^2 denotes the two-dimensional Laplacian operator. Focusing our attention on the case*:

$$R_0=\gamma\delta^2, \quad (2.3.2)$$

as before and introducing the same time and length scales (with $c_1=0$) as in the preceding section, we obtain

$$\begin{aligned}&(\delta^2\partial_T-c_0\partial_X)(\delta^2\partial^2{}_{XX}+\partial^2{}_{yy}-F)\zeta\\&\quad+\gamma\delta(\zeta_X\partial_y-\zeta_y\partial_X)(\delta^2\partial^2{}_{XX}+\partial^2{}_{yy})\zeta\\&\quad+\zeta_X=0\,,\end{aligned} \quad (2.3.3)$$

where ∂_T, ∂_X and ∂_y denote partial derivatives. Expanding ζ in terms of δ, we have the lowest order equations such as (2.2.4) and (2.2.5). Therefore the solution is identical with (2.2.6) and (2.2.7). To $O(\delta)$, we have an equation:

$$\zeta_{yyX}{}^{(1)}-\left(F+\frac{1}{c_0}\right)\zeta_X{}^{(1)}=0\,, \quad (2.3.4)$$

with boundary conditions:

$$\zeta_X{}^{(1)}=0 \text{ at } y=0 \text{ and } 1\,. \quad (2.3.5)$$

At this point we should choose the solution forced by the lower order solution. Thus we decide to adopt

$$\zeta^{(1)}=0\,. \quad (2.3.6)$$

To $O(\delta^2)$, we obtain

$$\zeta_{yyX}{}^{(2)}-\left(F+\frac{1}{c_0}\right)\zeta_X{}^{(2)}=\left(\frac{1}{c_0{}^2}A_T-A_{XXX}\right)\sin m\pi y\,, \quad (2.3.7)$$

with boundary conditions:

$$\zeta_X{}^{(2)}=0 \text{ at } y=0 \text{ and } 1\,. \quad (2.3.8)$$

The solvability condition yields the evolution equation for A:

$$A_T-c_0{}^2A_{XXX}=0\,. \quad (2.3.9)$$

Thus we have a linear dispersive equation regardless of the value of m. As we have seen in the preceding section, the consequences are quite misleading when m is odd. The root cause of this problem lies in the fact that, in traditional quasi-geostrophic dynamics, the irrotational motion associated with the nonlinear terms in the integrated form of the mass concervation equation is neglected. Thus if we adopt the traditional quasi-geostrophic equation on a beta-plane, we might conclude erroneously that a sheared current is essential for the existence of solitary planetary waves even for a divergent case (cf. REDEKOPP, 1977).

3. Long topographic waves in a shallow layer of homogeneous fluid

In this section the assumption of a constant fluid depth is adopted, and in order to elucidate the point at issue, we choose a linear depth profile:

$$H=H_0-\alpha y\,, \quad (3.1)$$

and assume the Coriolis parameter f is constant. Introducing the nondimensional value $\delta(=\alpha L/H_0)$ (which corresponds to the beta parameter in Section 2) and nondimensionalising the shallow water equation by use of L, $(f\delta)^{-1}$, U and $g^{-1}fLU$ as a length scale, a time scale, a velocity scale and a scale of surface elevation respectively, we obtain the nondimensional equations:

$$\begin{aligned}&\delta u_t+R_0(uu_x+vu_y)-v=-\zeta_x\,,\\&\delta v_t+R_0(uv_x+vv_y)+u=-\zeta_y\,,\\&(1-\delta y)u_x+(1-\delta y)v_y-\delta v\\&\qquad+F\{\delta\zeta_t+R_0(\zeta u)_x+R_0(\zeta v)_y\}=0\,.\end{aligned} \quad (3.2)$$

CLARKE (1971), SMITH (1972) and GRIMSHAW

* Strictly speaking, (2.3.1) is valid for $\delta\lesssim R_0$. The condition is derived from the beta-plane approximation (see, CHARNEY (1973)). Violation of the above condition does not affect the point at issue.

(1977) assumed $\delta \sim O(1)$ and derived the K-dV equation for ageostrophic long topographic waves after expanding variables by powers of $R_0^{1/2}$. Here we assume $\delta \ll O(1)$, and thus consider quasi-geostrophic long topographic waves. We also assume $R_0 \sim O(\delta^2)$ as before. Hence, introducing long length and time scales as in Section 2 and expanding dependent variables by powers of δ, we obtain, to $O(\delta^2)$, the same equation and boundary conditions as (2.2.4) and (2.2.5). Thus the solution is given by (2.2.6) and (2.2.7). To $O(\delta^3)$, we have

$$\zeta^{(1)} = -c_0 m\pi A \cos m\pi y + \frac{1}{2} A y \sin m\pi y + \sum_{\substack{n=1 \\ (\neq m)}}^{\infty} q_n A \sin n\pi y\,, \quad (3.3)$$

where

$$q_n = \begin{cases} 0 & \text{for} \quad m+n \quad \text{even}\,, \\ \dfrac{8m^3 n}{(m^2-n^2)^3 \pi^2} & \text{for} \quad m+n \quad \text{odd}\,. \end{cases}$$

The phase speed correction is now determined as

$$c_1 = -\frac{1}{2} c_0^2 m^2 \pi^2. \qquad (3.4)$$

Proceeding to $O(\delta^4)$ and applying the solvability condition, we finally obtain the evolution equation for A:

$$A_T + b_1 A_X + b_2 A A_X + b_3 A_{XXX} = 0\,, \quad (3.5)$$

where the values of the coefficients b_i (i=1–3) are shown in the appendix. Note here that the coefficient b_2 has a positive sign. Thus we have a solitary wave of surface depression or low pressure in contrast to the case of long planetary waves in which we have a positive surface displacement. Also note that the coefficient b_2 does not vanish at the limit $F \to 0$. Therefore the presence of the free surface is not necessary to obtain the solitary solution. This is because the lateral convergence in front of the wave trough and the lateral divergence behind it are provided by the existence of the bottom slope. This also explains why we have a solitary wave of low pressure in contrast to the case of long planetary waves. If m is even, there is no net convergence in front of the wave trough and no net divergence behind it. Hence we cannot have the K-dV equation in such a case.

4. Long topographic waves in a laterally sheared current

In this section we incorporate a laterally sheared current of $O(\delta^{-1})$ and consider the general topography of $O(\delta)$. Then the non-dimensional shallow water equations are

$$\begin{aligned} &\delta u_t + \underline{\delta u_m u_x} + \gamma\delta^2(u u_x + v u_y) + \underline{\delta v u_{my}} - v \\ &\qquad = -\zeta_x\,, \\ &\delta v_t + \underline{\delta u_m v_x} + \gamma\delta^2(u v_x + v v_y) + u = -\zeta_y\,, \quad (4.1) \\ &(1-\delta h)u_x + (1-\delta h)v_y - \delta h_y v + F\{\delta\zeta_t \\ &\qquad + \underline{\delta\zeta_m u_x} + \underline{\delta\zeta_x u_m} + \underline{\delta\zeta_{my} v} + \underline{\delta\zeta_m v_y} \\ &\qquad + \gamma\delta^2(\zeta u)_x + \gamma\delta^2(\zeta v)_y\} = 0\,, \end{aligned}$$

where the underlined parts are added in this section. We assume that the mean current is in geostrophic balance: $u_m = -\zeta_{my}$. If the weak dispersion is to be balanced by the nonlinearity, we must introduce the slow space and time scales

$$X = \delta^{1/2}(x - c_0 t)\,, \quad T = \delta^{3/2} t\,. \qquad (4.2)$$

Note here that the above scales are different from those for $u_m = 0$ and $h_y = 1$ in the preceding section. The difference is due to the fact that the ratio of the wave amplitude to the mean current, in other words the effective amplitude of the wave, is now $O(\delta)$.

Expanding variables by powers of $\delta^{1/2}$, we obtain, to $O(\delta^{3/2})$,

$$(c_0 - u_m)\zeta_{yyX}^{(0)} + (u_{myy} - h_y - F c_0)\zeta_X^{(0)} = 0\,, \quad (4.3)$$

with boundary conditions:

$$\zeta_X^{(0)} = 0 \text{ at } y=0 \text{ and } 1\,. \qquad (4.4)$$

The solution is obtained in the form:

$$\zeta^{(0)} = A(T, X)\phi(y)\,, \qquad (4.5)$$

where ϕ is the solution of the regular eigen value problem:

$$(c_0 - u_m)\phi_{yy} + (u_{myy} - h_y - F c_0)\phi = 0\,, \quad (4.6)$$

with the boundary conditions:

$$\phi = 0 \text{ at } y=0 \text{ and } 1\,. \qquad (4.7)$$

Here we assume for simplicity that c_0 is outside the range of u_m: i.e., critical layers do not exist. Proceeding to $O(\delta^{5/2})$, we find

$$(c_0-u_m)\zeta_{yyX}{}^{(1)} \\ +(u_{myy}-h_y-Fc_0)\zeta_X{}^{(1)}=\mathbf{G}\,, \quad (4.8)$$

with boundary conditions:

$$\zeta_X{}^{(1)}=(u_m-c_0)\zeta_{yX}{}^{(0)} \text{ at } y=0 \text{ and } 1\,, \quad (4.9)$$

where $\mathbf{G}$ is shown in the Appendix. Applying the solvability condition, we obtain the evolution equation for A:

$$A_T+d_1A_X+d_2AA_X+d_3A_{XXX}=0\,, \quad (4.10)$$

where $d_i\,(i=1-3)$ are shown in the Appendix. Generally, the coefficient d_2 takes a non-zero value, and so incorporation of $O(\delta^{-1})$ mean shear and/or general curved topography changes the length and time scales of the solitary solution. For the special case of no mean shear and no curved bottom slope, as discussed in the preceding section, we need to introduce the longer length and time scales in order to keep a balance between the phase dispersion and the nonlinearity.

As is easily verified, the Q-G.P.V.E. approach leads to the evolution equation for A such as

$$A_T+d_2AA_X+d_3A_{XXX}=0\,. \quad (4.11)$$

The above equation is identical to that obtained by REDEKOPP (1977). Equation (4.11) is also equivalent to (4.10) except for the phase speed correction which can be removed by the Galilei transformation. Thus we can conclude that the Q-G.P.V.E. is accurate enough to predict the nonlinear evolution of long waves in a general sheared current of $O(\delta^{-1})$ and/or over curved bottom topography of $O(\delta^{-1})$; the results of Section 2 and Section 3 reflect the special properties of the conditions of no mean shear and no curved bottom topography.

5. Nonlinear planetary and topographic eddies whose scale is much larger than the radius of deformation

In this section we consider the evolution of nonlinear planetary eddies whose scale is much larger than the Rossby's radius of deformation. It is again shown that the simple use of Q-G.P.V.E. is misleading. The significance of a new evolution equation is discussed in MATSUURA and YAMAGATA (1982) along with its application to anticyclonic eddies off the Pacific coast of Central America (STUMPF and LEGECKIS, 1977). Its application to the Jovian atmosphere will be reported elsewhere.

The basic equations are given by (2.1.1). The variables are nondimensionalised in the same way as in Section 2 except for a time scale which is now $(\beta L)^{-1}F$. In particular we focus our attention on the case:

$$F=\varepsilon\delta^{-1}\,, \quad (5.1)$$

where ε is of the order of unity. We assume $R_0=\gamma\delta^2$ as before. The planetary wave dispersion appears at $O(\delta)$, and so we introduce a long time scale such as

$$T=\delta t\,. \quad (5.2a)$$

We adopt a coordinate system moving with the long wave speed c as in the preceding sections:

$$X=x-ct\,. \quad (5.2b)$$

Then the nondimensional equations are written as

$$\begin{aligned} &-\varepsilon^{-1}\delta^2cu_X+\varepsilon^{-1}\delta^3u_T+\gamma\delta^2(uu_X+vu_y) \\ &\qquad -(1+\delta y)v=-\zeta_X\,, \\ &-\varepsilon^{-1}\delta^2cv_X+\varepsilon^{-1}\delta^3v_T+\gamma\delta^2(uv_X+vv_y) \\ &\qquad +(1+\delta y)u=-\zeta_y\,, \qquad (5.3) \\ &u_X+v_y-\delta c\zeta_X+\delta^2\zeta_T+\varepsilon\gamma\delta\{(\zeta u)_X \\ &\qquad +(\zeta v)_y\}=0\,. \end{aligned}$$

We seek an asymptotic solution similar to (2.2.3). To $O(\delta^0)$, we obtain

$$\begin{aligned} -v^{(0)}&=-\zeta_X{}^{(0)}\,, \\ u^{(0)}&=-\zeta_y{}^{(0)}\,, \qquad (5.4) \\ u_X{}^{(0)}&+v_y{}^{(0)}=0\,. \end{aligned}$$

Therefore $\zeta^{(0)}$ denotes the geostrophic streamfunction. To $O(\delta^1)$, we obtain

$$\begin{aligned} &-v^{(1)}-yv^{(0)}=-\zeta_X{}^{(1)}\,, \\ &u^{(1)}+yu^{(0)}=-\zeta_y{}^{(1)}\,, \qquad (5.5) \\ &u_X{}^{(1)}+v_y{}^{(1)}-c\zeta_X{}^{(0)} \\ &\quad +\varepsilon\gamma\{(\zeta^{(0)}u^{(0)})_X+(\zeta^{(0)}v^{(0)})_y\}=0\,. \end{aligned}$$

Eliminating $\zeta^{(1)}$ by cross-differentiation and using the equation of continuity, we obtain

$$(c+1)\zeta_X{}^{(0)}=0, \tag{5.6}$$

namely, we obtain $c=-1$ as the long wave speed at this stage. Thus the solution can be written in the form:

$$\zeta^{(0)}=\zeta^{(0)}(X,y,T),\quad X=x+t, \tag{5.7}$$

which represents non-dispersive planetary eddies (WHITE, 1977; MEYERS, 1979). To $O(\delta^2)$, we obtain

$$\begin{aligned}&\varepsilon^{-1}u_X{}^{(0)}+\gamma(u^{(0)}u_X{}^{(0)}+v^{(0)}u_y{}^{(0)})\\&\qquad -v^{(2)}-yv^{(1)}=-\zeta_X{}^{(2)},\\&\delta^{-1}v_X{}^{(0)}+\gamma(u^{(0)}v_X{}^{(0)}+v^{(0)}v_y{}^{(0)})\\&\qquad +u^{(2)}+yu^{(1)}=-\zeta_y{}^{(2)},\\&u_X{}^{(2)}+v_y{}^{(2)}+\zeta_X{}^{(1)}+\zeta_T{}^{(0)}\\&\qquad +\varepsilon\gamma\{(\zeta^{(1)}u^{(0)})_X+(\zeta^{(0)}u^{(1)})_X\\&\qquad +(\zeta^{(0)}v^{(1)})_y+(\zeta^{(1)}v^{(0)})_y\}=0.\end{aligned} \tag{5.8}$$

Eliminating $\zeta^{(2)}$ by cross-differentiation and using the equation of continuity and (5.7) we obtain

$$\begin{aligned}&\zeta_T{}^{(0)}-\varepsilon^{-1}\nabla^2\zeta_X{}^{(0)}+\gamma J(\nabla^2\zeta^{(0)},\zeta^{(0)})\\&\qquad -\varepsilon\gamma\zeta^{(0)}\zeta_X{}^{(0)}+2y\zeta_X{}^{(0)}=0,\end{aligned} \tag{5.9}$$

where $\nabla^2=\partial^2{}_{XX}+\partial^2{}_{yy}$ and $J(A,B)=\partial_X A\partial_y B-\partial_y A\partial_X B$. The above equation reduces to

$$\xi_T+\gamma J(\nabla^2\xi,\xi)-\varepsilon\gamma\xi\xi_X+y\xi_X=0, \tag{5.10}$$

where $\xi=\zeta^{(0)}-(\gamma\varepsilon)^{-1}y$. If we start from the traditional quasi-geostrophic potential vorticity equation, we obtain

$$\xi_T+\gamma J(\nabla^2\xi,\xi)=0 \tag{5.11}$$

instead of (5.10). Thus the use of the Q-G.P.V.E. again leads to an erroneous result. For divergent topographic waves, however, the evolution equation resulting from the shallow water equations is identical to that resulting from the Q-G.P.V.E. This is because $O(\delta)$ motion is in geostrophic balance and is laterally nondivergent; the convergence (divergence) due to the topography is exactly cancelled by the divergence (convergence) due to the free surface for the divergent topographic waves.

It is interesting that (5.9) is a kind of two-dimensional K-dV equation. From the result in Section 2 we expect that an eddy with surface elevation (anticyclonic eddy) will keep its identity longer than an eddy with surface depression (cyclonic eddy). This point will be discussed in detail in MATSUURA and YAMAGATA (1982). It is also interesting that (5.11) has an arbitrary solution with radial symmetry. The evolution of this type of solution, which is now governed by (5.10), is shown in MATSUURA and YAMAGATA (1982). The initial pattern is found to show little changes for a simulation time interval; this might suggest the existence of a two-dimensional soliton in (5.9) or (5.10). The reader is referred to MATUURA and YAMAGATA (1982) as to the detailed numerical results.

6. Summary

First, using shallow water equations, we have derived the K-dV equation governing the zonally elongated quasi-geostrophic divergent planetary waves of odd cross-channel mode on a beta-plane without a mean current. The solitary solution is found to be of an elevation type. The nonlinear term of the evolution equation is missed by the use of the Q-G.P.V.E. This is because the source of nonlinearity is due to the irrotational motion which is neglected in the traditional quasi-geostrophic approximation. A similar result is obtained for long quasi-geostrophic topographic waves over a linearly inclined bottom. The solitary solution (in the latter case), however, is of a low pressure type because the irrotational motion is mainly due to the inclined bottom topography. The effects of $O(\delta^{-1})$ mean shear and/or curved bottom topography mask the above stated dynamics; the traditional quasi-geostrophic approximation gives a correct result for the nonlinear evolution of long waves, except for the correction term of the phase velocity.

Second, another example which calls attention to the deficiencies of the Q-G.P.V.E. has been presented concerning the nonlinear evolution of divergent planetary eddies whose scale is much larger than the Rossby's radius of deformation. A new evolution equation, which is a kind of two-dimensional K-dV equation, is derived. The nonlinearity in the equation is derived from the irrotational motion associated with the movement of the free surface. It is therefore suggested that an eddy with surface elevation will keep its identity longer than an eddy with surface depression; the traditional quasi-geostrophic approximation misses this point.

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Appendix

This appendix presents explicit forms of **F**, **G**, $a_i\,(i=1\text{-}3)$, $b_i\,(i=1\text{-}3)$ and $d_i\,(i=1\text{-}3)$:

$$\begin{aligned}\mathbf{F}=&\frac{1}{c_0}\zeta_{yyT}{}^{(0)}-\frac{F}{c_0}\zeta_T{}^{(0)}+\Big\{-4y\zeta_{yX}{}^{(0)}\\&-y^2\zeta_{yyX}{}^{(0)}+\frac{1}{c_0}y^2\zeta_X{}^{(0)}+2\zeta_{yX}{}^{(1)}+y\zeta_{yyX}{}^{(1)}\\&+\frac{2c_1}{c_0}\zeta_{yX}{}^{(0)}-\frac{1}{c_0}y\zeta_X{}^{(1)}+\frac{c_1}{c_0}y\zeta_{yyX}{}^{(0)}\\&-\frac{c_1}{c_0}\zeta_{yyX}{}^{(1)}\Big\}+F\Big\{\frac{c_1}{c_0}\zeta_X{}^{(1)}+y\zeta_X{}^{(1)}\\&+\frac{c_1}{c_0}y\zeta_X{}^{(0)}\Big\}+\frac{\gamma}{c_0}\{2y\zeta_y{}^{(0)}\zeta_{yX}{}^{(0)}\\&-2y\zeta_X{}^{(0)}\zeta_{yy}{}^{(0)}-\zeta_y{}^{(0)}\zeta_{yX}{}^{(1)}-\zeta_y{}^{(1)}\zeta_{yX}{}^{(0)}\\&+\zeta_X{}^{(0)}\zeta_{yy}{}^{(1)}+\zeta_{yy}{}^{(0)}\zeta_X{}^{(1)}-\zeta_X{}^{(0)}\zeta_y{}^{(0)}\\&+c_0\zeta_{yy}{}^{(0)}\zeta_{yX}{}^{(0)}\}_y+\frac{\gamma}{c_0}\{\zeta_y{}^{(0)}\zeta_{yX}{}^{(0)}\\&-\zeta_X{}^{(0)}\zeta_{yy}{}^{(0)}\}-F\gamma\Big\{\zeta_y{}^{(0)}\zeta_{yX}{}^{(0)}+\zeta^{(0)}\zeta_{yyX}{}^{(0)}\\&-\frac{1}{c_0}\zeta^{(0)}\zeta_X{}^{(0)}\Big\}-\zeta_{XXX}{}^{(0)},\end{aligned}\tag{A-1}$$

$$\begin{aligned}\mathbf{G}=&\zeta_{yyT}-F\zeta_T{}^{(0)}-u_{my}(h_y+Fc_0)\zeta_X{}^{(0)}\\&-u_{myy}(c_0-u_m)\zeta_{yX}{}^{(0)}-u_{myy}u_{my}\zeta_X{}^{(0)}\\&+h(h_y+Fc_0)\zeta_X{}^{(0)}+h_y\{(c_0-u_m)\zeta_{yX}{}^{(0)}\\&+u_{my}\zeta_X{}^{(0)}\}-F\{\zeta_m(h_y+Fc_0)\zeta_X{}^{(0)}\\&+u_m(u_m-c_0)\zeta_{yX}{}^{(0)}-u_mu_{my}\zeta_X{}^{(0)}\}\\&-\gamma\{\zeta_y{}^{(0)}\zeta_{yX}{}^{(0)}-\zeta_X{}^{(0)}\zeta_{yy}{}^{(0)}\}_y\\&-(c_0-u_m)\zeta_{XXX}{}^{(0)},\end{aligned}\tag{A-2}$$

$$\begin{aligned}a_1=&2c_0\Big\{-c_0{}^2\sum_{\substack{n=1\\(\neq m)}}^{\infty}p_nmn\pi^2((-1)^{m+n}-1)\\&+\frac{3}{4}c_0{}^2m^2\pi^2-c_0\sum_{\substack{n=1\\(\neq m)}}^{\infty}p_n\frac{mn(m^2+n^2)}{(m^2-n^2)^2}\\&\times((-1)^{m+n}-1)+\frac{3}{4}c_0+\frac{1}{2}c_1F\\&+c_0\sum_{\substack{n=1\\(\neq m)}}^{\infty}p_n\frac{mn}{n^2-m^2}((-1)^{m+n}-1)\\&+\frac{1}{2}(Fc_0-1)\sum_{\substack{n=1\\(\neq m)}}^{\infty}p_nr_n+\frac{1}{4}c_0{}^2F\\&+\Big(\frac{1}{6}-\frac{1}{4m^2\pi^2}\Big)Fc_0\Big\},\end{aligned}\tag{A-3}$$

$$a_2=\frac{2\gamma Fc_0(1-(-1)^m)(m^3\pi^3+2m^2\pi^2+2F)}{m\pi(m^2\pi^2+F)}\ (\leq 0),\tag{A-4}$$

$$a_3=-c_0{}^2\quad(<0),\tag{A-5}$$

$$\begin{aligned}b_1=&c_0\Big\{-\frac{1}{4}c_0+\frac{3}{4}Fc_1+\frac{1}{4}c_1m^2\pi^2\\&-c_0m^2\pi^2\Big(\frac{1}{2}-\frac{3}{4m^2\pi^2}+\frac{1}{2}c_0\\&+\sum_{\substack{n=1\\(\neq m)}}^{\infty}\frac{m^2r^2{}_n}{n^2-m^2}\Big)+4c_0{}^2m^3\pi^2\sum_{\substack{n=1\\(\neq m)}}^{\infty}\frac{nr_n}{n^2-m^2}\\&-4m^3c_0\sum_{\substack{n=1\\(\neq m)}}^{\infty}\frac{nr_n}{(n^2-m^2)^2}\Big\},\end{aligned}\tag{A-6}$$

$$\begin{aligned}b_2=&\frac{2\gamma c_0}{3}(1-(-1)^m)\\&\times\{4c_0m^3\pi^3-2m\pi+Fc_0m\pi)\}\ (\geq 0),\end{aligned}\tag{A-7}$$

$$b_3=-c_0{}^2\quad(<0),\tag{A-8}$$

$$\begin{aligned}d_1=&\Big\{\Big\langle\frac{\phi_{yy}\phi}{c_0-u_m}\Big\rangle-F\Big\langle\frac{\phi^2}{c_0-u_m}\Big\rangle\Big\}^{-1}\\&\times\Big\{-\Big\langle\frac{Fc_0u_{my}\phi^2}{c_0-u_m}\Big\rangle-\langle u_{myy}\phi\phi_y\rangle\\&-\Big\langle\frac{u_{myy}u_{my}\phi^2}{c_0-u_m}\Big\rangle+\Big\langle\frac{hh_y\phi^2}{c_0-u_m}\Big\rangle\\&+\Big\langle\frac{hFc_0\phi^2}{c_0-u_m}\Big\rangle+\langle h_y\phi\phi_y\rangle\\&-\Big\langle\frac{F\zeta_mh_y\phi^2}{c_0-u_m}\Big\rangle-\Big\langle\frac{c_0F^2\zeta_m\phi^2}{c_0-u_m}\Big\rangle\\&+\langle Fu_m\phi\phi_y\rangle+\Big\langle\frac{Fu_mu_{my}\phi^2}{c_0-u_m}\Big\rangle\Big\},\end{aligned}\tag{A-9}$$

$$\begin{aligned}d_2=&\gamma\Big\{\Big\langle\frac{\phi_{yy}\phi}{c_0-u_m}\Big\rangle-F\Big\langle\frac{\phi^2}{c_0-u_m}\Big\rangle\Big\}^{-1}\\&\times\Big\{\Big\langle\frac{\phi_y{}^3}{c_0-u_m}\Big\rangle-\Big\langle\frac{\phi\phi_y\phi_{yy}}{c_0-u_m}\Big\rangle\Big\},\end{aligned}\tag{A-10}$$

$$d_3=-\Big\{\Big\langle\frac{\phi_{yy}\phi}{c_0-u_m}\Big\rangle-F\Big\langle\frac{\phi^2}{c_2-u_m}\Big\rangle\Big\}^{-1}\langle\phi^2\rangle,\tag{A-11}$$

where

$$r_n=\begin{cases} 0 \quad \text{for} \quad m+n \quad \text{even}\,, \\ \dfrac{-8mn}{(m^2-n^2)^2\pi^2} \quad \text{for} \quad m+n \quad \text{odd}\,, \end{cases}$$

and $<>$ denotes the integral in the interval $0\leq y\leq 1$.

References

BENNEY, D. J. (1966): Long nonlinear waves in fluid flows. J. Math. Phys. (Cambridge, MA), **45**, 52-63.

BOYD, J. P. (1977): Solitary waves on the equatorial betaplane. *In*, Review Papers of Equatorial Oceanography FINE Workshop Proceedings, ed. by D. MOORE. Nova Univ., Dania, Florida, 13 pp.

BOYD, J. P. (1980): Equatorial solitary waves. Part I: Rossby solitons. J. Phys. Oceanogr., **10**, 1699-1717.

CHARNEY, J. G. (1973): Planetary fluid dynamics. *In*, Dynamic Meteorology, ed. by P. MOREL, D. Reidel, Dordrecht, pp. 97-351.

CLARKE, R. A. (1971): Solitary and cnoidal planetary waves. Geophys. Fluid Dyn., **2**, 343-354.

FLIERL, G. R. (1979): Baroclinic solitary waves with radial symmetry. Dyn. Atmos. Oceans, **3**, 15-38.

FLIERL, G. R. (1980): Introduction to coherent features. Woods Hole Oceanogr. Inst. Tech. Rep. WHOI-80-53, 1-33.

GRIMSHAW, R. (1977): Nonlinear aspects of long shelf waves. Geophys. Astrophys. Fluid Dyn., **8**, 3-16.

HUKUDA, H. (1979): Solitary Rossby waves in a two-layer system. Tellus, **31**, 161-169.

LARSEN, L. H. (1965): Comments on "solitary waves in the Westerlies". J. Atmos. Sci., **22**, 222-224.

LONG, R. R. (1964): Solitary waves in the Westerlies. J. Atmos. Sci., **21**, 197-200.

MALANOTTE RIZZOLI, P. and M. C. HENDERSHOTT (1980): Solitary Rossby waves over variable relief and their stability. Part I: the analytical theory. Dyn. Atmos. Oceans, **4**, 247-260.

MATSUURA, T. and T. YAMAGATA (1982): On the evolution of nonlinear planetary eddies larger than the radius of deformation. J. Phys. Oceanogr., **12**, 440-456.

MAXWORTHY, T. and L. G. REDEKOPP (1976): A solitary wave theory of the Great Red Spot and other observed features in the Jovian atmosphere. Icarus, **29**, 261-271.

MEYERS, G. (1979): On the annual Rossby wave in the tropical North Pacific Ocean. J. Phys. Oceanogr., **9**, 663-674.

MILES, J. W. (1979): On solitary Rossby waves. J. Atmos. Sci., **36**, 1236-1238.

ODULO, A. B. and Ye. N. PELINOVSKIY (1978): Nonlinear topographic Rossby waves. Oceanology, **18**, 9-11.

REDEKOPP, L. G. (1977): On the theory of solitary Rossby waves. J. Fluid Mech., **82**, 725-745.

STUMPF, H. G. and R. V. LEGECKIS (1977): Satellite observations of mesoscale eddy dynamics in the eastern tropical Pacific Ocean. J. Phys. Oceanogr., **7**, 648-658.

SMITH, R. (1972): Nonlinear Kelvin and continental-shelf waves. J. Fluid Mech., **52**, 376-391.

WHITE, W. B. (1977): Annual forcing of baroclinic long waves in the tropical North Pacific Ocean. J. Phys. Oceanogr., **7**, 50-61.

非線型惑星波について
—準地衡流近似を使うと見失われる一連の解—

山　形　俊　男*

要旨: 微小なベータ係数の場合の準地衡流惑星波および海底斜面上の地形性波の弱非線型論を浅水波近似のもとで議論する．準地衡流近似を採用すると見失われる長波の孤立波解が，水平シアー流も傾斜が一様でない海底地形も必要とせずに存在しうることを示す．この原因は地衡流の発散に起因する渦なし運動の存在による．準地衡流近似の安易な使用に注意を喚起する別の例として，変形半径よりも大きな準地衡流渦の非線型発展を論じる．新しく得られた発展方程式（二次元のK-dV方程式と呼びうるもの）の重要性にも簡単に触れる．

* 〒812 福岡市東区箱崎6-10-1
九州大学応用力学研究所海洋流体力学部門
現滞在先: 地球流体力学プログラム，プリンストン大学，私書箱 308，プリンストン，ニュージャージー州 08540，アメリカ合衆国