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The Instability of Viscous Two-Layer Oscillatory Flows*

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Abstract: The instability of oscillatory flows in a two-layer fluid where the two layers differ in density and viscosity has been analysed using a perturbation method for long waves with special interest on effects of viscosity, time scale, density and depth of the fluid. The flow of a fluid with homogeneous density can be unstable, when the kinematic viscosity of the upper fluid layer is different from that of the lower one. Viscosity stratification results in unstable oscillatory flows. Two limiting cases of single-layer flow are also considered.

1. Introduction

The stability theories of steady flows of homogeneous or stratified fluids have been well documented (LIN, 1955; STUART, 1963; DRAZIN and HOWARD, 1966; NIINO, 1981; DRAZIN and REID, 1981). In particular, YIH (1967) found that a two-layer steady flow can be unstable when a fluid is stratified in viscosity, but not in density.

In the stability theory of inviscid two-layer oscillatory flows, Bernoulli's equation and a kinematic condition yield Mathieu's equation. Hence, the instability characteristics of such flows are determined in terms of the eigenvalues of Mathieu's equation. The mechanism described by Mathieu's equation is well known as the parametric excitation of oscillations (KELLY, 1965). Analysis of viscous oscillatory flows, however, are difficult, because their time-dependence precludes the method of separating the exponential time factor in the vorticity equations for disturbances. Thus the conventional Orr-Sommerfeld procedure fails. A few studies of the linear theory of viscous single-layer oscillatory flows have been published (CONRAD and CRIMINALE, 1965; KERCZEK and DAVIS, 1974; HALL, 1978). CONRAD and CRIMINALE (1965) attempted to generalize Squire's theorem for application to unsteady flows, and also to obtain a Rayleigh theorem for time-dependent flows. KERCZEK and DAVIS (1974) and HALL (1978) investigated the linear stability of viscous oscillatory flows without stratification. They found that the flow may always be stable under certain conditions, although HINO and SAWAMOTO (1975) showed that the flow can be unstable for a different range of wavenumber and time scale of the flow from that was investigated by the above mentioned authors.

A single-layer flow with a free surface excited by the lower boundary moving periodically in its own plane may be unstable. YIH (1968) examined the instability of the flow of this type. He used a perturbation method with a disturbance of small wavenumber, α, and extended Floquet's theorem to determine the criterion of instability from the kinematic condition of the free surface. He showed that the growth rate of the disturbance is of order α^2.

The interface of a two-layer fluid may have a destabilizing effect, and we have investigated the instability characteristics of viscous two-layer oscillatory flows. In Section 2, we obtain the velocity distribution of the primary oscillatory flow. In Sect. 3, we formulate the stability problem, and solve it in Sect. 4. Finally, in Sect. 5, we discuss the regions of instability in parameter spaces.

2. Primary flow

We consider two horizontal layers of fluid between two fixed rigid planes under the influence of a periodic pressure gradient, which produces an oscillatory flow. This type of flow occurs, e.g., under the influence of a reciprocating piston (see Fig. 1). The symbols $d^{(j)}$,

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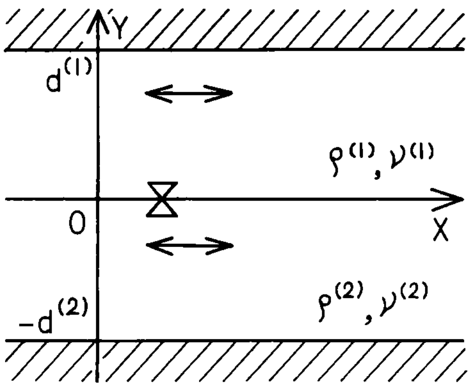

Fig. 1. A two-layer fluid system dealt with in this study. The y-axis is pointed upward. An oscillatory flow exists in both layers.

$\rho^{(j)}$, $\nu^{(j)}$, U_0 and t are the depth, denisty, kinematic viscosity, maximum velocity amplitude and time, respectively; $j=1$ denotes the upper layer and $j=2$ the lower. We consider a two-dimensional parallel flow independent of X, and introduce the non-dimensional quantities

$$\tau=\frac{U_0 t}{d^{(2)}},\quad x=\frac{X}{d^{(2)}},\quad y=\frac{Y}{d^{(2)}}. \tag{2.1}$$

The equations governing the primary flow are

$$\left\{\begin{aligned}\frac{\partial \bar{u}^{(1)}}{\partial \tau}&=-\frac{\partial \tilde{p}^{(1)}}{\partial x}+\frac{1}{R}\frac{\partial^2 \bar{u}^{(1)}}{\partial y^2}, \qquad (2.2)\\ 0&=-\frac{\partial \tilde{p}^{(1)}}{\partial y}, \qquad (2.3)\end{aligned}\right.$$

$$\left\{\begin{aligned}\frac{\partial \bar{u}^{(2)}}{\partial \tau}&=-\hat{\rho}\frac{\partial \tilde{p}^{(2)}}{\partial x}+\frac{1}{\hat{\nu}R}\frac{\partial^2 \bar{u}^{(2)}}{\partial y^2}, \qquad (2.4)\\ 0&=-\hat{\rho}\frac{\partial \tilde{p}^{(2)}}{\partial y}, \qquad (2.5)\end{aligned}\right.$$

where $\bar{u}^{(j)}(y,\tau)=\bar{u}_d{}^{(j)}(y,\tau)/U_0$, $R=U_0 d^{(2)}/\nu^{(1)}$; Reynolds number, $\hat{\rho}=\rho^{(1)}/\rho^{(2)}$, $\hat{\nu}=\nu^{(1)}/\nu^{(2)}$, and $\tilde{p}^{(j)}$ is the pressure containing the gravitational potential as

$$\left.\begin{aligned}\tilde{p}^{(1)}&=Fr^{-2}y+\bar{p}^{(1)},\\ \tilde{p}^{(2)}&=\rho^{-1}Fr^{-2}y+\bar{p}^{(2)},\\ \bar{p}^{(j)}&=\bar{p}_d{}^{(j)}/\rho^{(1)}U_0{}^2,\end{aligned}\right\} \tag{2.6}$$

where $F_r=U_0/\sqrt{gd^{(2)}}$ is the Froude number, and the subscript d indicates a dimensional quantity.

Since the primary flow is parallel in the x-direction and harmonic in time and considering (2.3, 5), we put

$$-\frac{\partial \tilde{p}^{(1)}}{\partial x}=-\hat{\rho}\frac{\partial \tilde{p}^{(2)}}{\partial x}=Pe^{i\omega\tau}, \tag{2.7}$$

which attributes a physical significance only to the real part, and where $P=i\omega$ is a constant and ω is a non-dimensional angular frequency (SCHLICHTING, 1979: pp. 436–438; BATCHELOR, 1967: pp. 353–355). The boundary conditions for $\bar{u}^{(j)}$ are

$$\bar{u}^{(1)}=0 \qquad \text{at } y=d(=d^{(1)}/d^{(2)}), \tag{2.8}$$

$$\bar{u}^{(2)}=0 \qquad \text{at } y=-1, \tag{2.9}$$

$$\bar{u}^{(1)}=\bar{u}^{(2)} \qquad \text{at } y=0, \tag{2.10}$$

$$\frac{\partial \bar{u}^{(1)}}{\partial y}=\frac{1}{\hat{\rho}\hat{\nu}}\frac{\partial \bar{u}^{(2)}}{\partial y} \qquad \text{at } y=0, \tag{2.11}$$

The solutions for (2.2) to (2.12) are

$$\bar{u}^{(1)}=\frac{1}{2}[(A^{(1)}\cosh qy + B^{(1)}\sinh qy+1)e^{i\omega\tau}+c.c.], \tag{2.12}$$

$$\bar{u}^{(2)}=\frac{1}{2}[(A^{(2)}\cosh\sqrt{\hat{\nu}}qy + B^{(3)}\sinh\sqrt{\hat{\nu}}qy+1)^{i\omega\tau}+c.c.], \tag{2.13}$$

where

$$\begin{aligned}&q=\beta(1+i),\quad \beta=\sqrt{\omega R/2},\\ &A^{(1)}=\frac{1}{X}\left(\sinh q\sqrt{\hat{\nu}}+\frac{1}{\hat{\rho}\sqrt{\hat{\nu}}}\sinh qd\right),\\ &B^{(1)}=-\frac{1}{\hat{\rho}\sqrt{\hat{\nu}}X}(\cosh q\sqrt{\hat{\nu}}-\cosh qd),\\ &A^{(2)}=\frac{1}{X}\left(\sinh q\sqrt{\hat{\nu}}+\frac{1}{\hat{\rho}\sqrt{\hat{\nu}}}\sinh qd\right),\\ &B^{(2)}=\frac{1}{X}(\cosh q\sqrt{\hat{\nu}}-\cosh qd),\\ &X=-\cosh qd\sinh q\sqrt{\hat{\nu}}-\frac{1}{\hat{\rho}\sqrt{\hat{\nu}}}\sinh qd\cosh q\sqrt{\hat{\nu}},\end{aligned} \tag{2.14}$$

and c.c. denotes the complex conjugate.

3. Formulation of the stability problem

When $u_d{}^{(j)}$ and $v_d{}^{(j)}$ denote the velocity components in the X- and Y-directions, respectively, and $p_d{}^{(j)}$ denotes the pressure in the disturbed field, and when

$$u^{(j)}=u_d^{(j)}/U_0,\ v^{(j)}=v_d^{(j)}/U_0,$$
$$p^{(j)}=p_d^{(j)}/\rho^{(1)}U_0^2, \quad (3.1)$$

with $j=1, 2$, the momentum equations are written as

$$\frac{\partial u^{(j)}}{\partial \tau}+u^{(j)}\frac{\partial u^{(j)}}{\partial x}+v^{(j)}\frac{\partial u^{(j)}}{\partial y}=-\rho_j\frac{\partial p^{(j)}}{\partial x}+\frac{1}{\nu_j R}\Delta u^{(j)}, \quad (3.2)$$

$$\frac{\partial v^{(j)}}{\partial \tau}+u^{(j)}\frac{\partial v^{(j)}}{\partial x}+v^{(j)}\frac{\partial v^{(j)}}{\partial y}=-\frac{1}{F_r^2}-\rho_j\frac{\partial p^{(j)}}{\partial y}+\frac{1}{\nu_j R}\Delta v^{(j)}, \quad (3.3)$$

$$\frac{\partial u^{(j)}}{\partial x}+\frac{\partial v^{(j)}}{\partial y}=0, \quad (3.4)$$

where

$$\rho_j=\begin{cases}1 & \text{for } j=1\\ \hat{\rho} & \text{for } j=2\end{cases},\quad \nu_j=\begin{cases}1 & \text{for } j=1\\ \hat{\nu} & \text{for } j=2\end{cases},$$
$$\Delta\equiv\partial^2/\partial x^2+\partial^{2*}\partial y^2.$$

Decomposing the dependent variables into a primary part and a perturbation part, we have

$$u^{(j)}=\bar{u}^{(j)}(y,\tau)+\hat{u}^{(j)}(x,y,\tau),$$
$$v^{(j)}=\hat{v}^{(j)}(x,y,\tau),$$
$$p^{(j)}=\bar{p}^{(j)}+\hat{p}^{(j)}, \quad (3.5)$$

where the caret (^) denotes the perturbed quantities. Substituting (3.5) into (3.2)-(3.4), subtracting the terms for the primary flow from those equations, and neglecting the quadratic terms for perturbations, we have

$$\frac{\partial \hat{u}^{(j)}}{\partial \tau}+\bar{u}^{(j)}\frac{\partial \hat{u}^{(j)}}{\partial x}+\hat{v}^{(j)}\frac{\partial \bar{u}^{(j)}}{\partial y}=-\rho_j\frac{\partial \hat{p}^{(j)}}{\partial x}+\frac{1}{\nu_j R}\Delta u^{(j)}, \quad (3.6)$$

$$\frac{\partial \hat{v}^{(j)}}{\partial \tau}+\bar{u}^{(j)}\frac{\partial \hat{v}^{(j)}}{\partial x}=-\rho_j\frac{\partial \hat{p}^{(j)}}{\partial y}+\frac{1}{\nu_j R}\Delta \hat{v}^{(j)}, \quad (3.7)$$

$$\frac{\partial \hat{u}^{(j)}}{\partial x}+\frac{\partial \hat{v}^{(j)}}{\partial y}=0. \quad (3.8)$$

We introduce the stream function $\psi^{(j)}$ with $j=1,2$ as

$$\hat{u}^{(j)}=\frac{\partial \psi^{(j)}}{\partial y},\quad \hat{v}^{(j)}=-\frac{\partial \psi^{(j)}}{\partial x}. \quad (3.9)$$

With (3.9), the momentum equations are written as

$$\psi_{y\tau}^{(j)}+\bar{u}^{(j)}\psi_{xy}^{(j)}-\bar{u}_y^{(j)}\psi_x^{(j)}=-\rho_j\hat{p}_x^{(j)}+\frac{1}{\nu_j R}\Delta\psi_y^{(j)}, \quad (3.10)$$

$$\psi_{x\tau}^{(j)}+\bar{u}^{(j)}\psi_{xx}^{(j)}=\rho_j\hat{p}_y^{(j)}+\frac{1}{\nu_j R}\Delta\psi_x^{(j)}, \quad (3.11)$$

where the subscript denotes the partial derivative. We look for a disturbance of the form

$$\psi^{(j)}=\phi^{(j)}(y,\tau)e^{i\alpha x}, \quad (3.12)$$

$$\hat{p}^{(j)}=f^{(j)}(y,\tau)e^{i\alpha x}, \quad (3.13)$$

and substitute (3.12, 13) into (3.10, 11) to obtain

$$\phi_{y\tau}^{(j)}+i\alpha\bar{u}^{(j)}\phi_y^{(j)}-i\alpha\bar{u}_y^{(j)}\phi^{(j)}=-i\alpha\rho_j f^{(j)}+\frac{1}{\nu_j R}(\phi_{yyy}^{(j)}-\alpha^2\phi_y^{(j)}), \quad (3.14)$$

$$i\alpha\phi_\tau^{(j)}-\alpha^2\bar{u}^{(j)}\phi^{(j)}=\rho_j f_y^{(j)}+\frac{i\alpha}{\nu_j R}(\phi_{yy}^{(j)}-\alpha^2\phi^{(j)}). \quad (3.15)$$

Eliminating $f^{(j)}$ from (3.14, 15), we have

$$\left(\frac{\partial}{\partial\tau}+i\alpha\bar{u}^{(j)}\right)\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)\phi^{(j)}-i\alpha\bar{u}_{yy}^{(j)}\phi^{(j)}=\frac{1}{\nu_j R}\left(\frac{\partial^2}{\partial y^2}-\alpha^2\right)^2\phi^{(j)}. \quad (3.16)$$

With the displacement of the interface η, the kinematic condition is

$$\left(\frac{\partial}{\partial\tau}+\bar{u}^{(j)}(0,\tau)\frac{\partial}{\partial x}\right)\eta=-\psi_x^{(j)}, \quad (3.17)$$

where $j=1$ or 2. Using

$$\eta=h(\tau)e^{i\alpha x}, \quad (3.18)$$

(3.17) becomes

$$\left[\frac{d}{d\tau}+i\alpha\bar{u}^{(j)}(0,\tau)\right]h(\tau)=-i\alpha\phi^{(j)}(0,\tau). \quad (3.19)$$

The boundary conditions at the rigid boundaries are

(1) $\phi^{(1)}(d,\tau)=0\,,$ (3.20)

(2) $\phi_y{}^{(1)}(d,\tau)=0\,,$ (3.21)

(3) $\phi^{(2)}(-1,\tau)=0\,,$ (3.22)

(4) $\phi_y{}^{(2)}(-1,\tau)=0\,.$ (3.23)

The boundary conditions of the continuity of $\hat{u}^{(j)}$ and $\hat{v}^{(j)}$ at the interface are

(5) $\phi^{(1)}(0,\tau)=\phi^{(2)}(0,\tau)\,,$ (3.24)

and

(6) $$\bar{u}_y{}^{(1)}(0,\tau)h+\phi_y{}^{(1)}(0,\tau) =\bar{u}_y{}^{(2)}(0,\tau)h+\phi_y{}^{(2)}(0,\tau), \quad (3.25)$$

respectively (cf. YIH, 1967, 1968).

The continuity of the tangential stress at the interface is

(7) $$\bar{u}_{yy}{}^{(1)}(0,\tau)h(\tau)+\phi_{yy}{}^{(1)}(0,\tau)+\alpha^2\phi^{(1)}(0,\tau) =\frac{1}{\hat{\rho}\hat{\nu}}\{\bar{u}_{yy}{}^{(2)}(0,\tau)h(\tau) +\phi_{yy}{}^{(2)}(0,\tau)+\alpha^2\phi^{(2)}(0,\tau)\} \quad (3.26)$$

(cf. YIH, 1967, 1968). The boundary condition for continuity of normal stress at the interface is

(8) $$i\alpha F_r{}^{-2}h-\frac{1}{R}(\phi_{yyy}{}^{(1)}-3\alpha^2\phi_y{}^{(1)}) +\phi_{y\tau}{}^{(1)}+i\alpha\bar{u}^{(1)}\phi_y{}^{(1)}-i\alpha\bar{u}_y{}^{(1)}\phi^{(1)} =\frac{1}{\hat{\rho}}\left\{i\alpha F_r{}^{-2}h-\frac{1}{\hat{\nu}R}(\phi_{yyy}{}^{(2)}-3\alpha^2\phi_y{}^{(2)}) +\phi_{y\tau}{}^{(2)}+i\alpha\bar{u}^{(2)}\phi_y{}^{(2)} -i\alpha\bar{u}_y{}^{(2)}-i\alpha\bar{u}_y{}^{(2)}\phi^{(2)}\right\} \quad (3.27)$$

(cf. YIH, 1967, 1968).

4. Solution of the problem

We assume the solution to be in the form of pseudo-periodic function

$$\phi^{(j)}(y,\tau)=e^{\theta\tau}\chi^{(j)}(y,\tau)\,, \quad (4.1)$$

$$h(\tau)=e^{\theta\tau}H(\tau)\,, \quad (4.2)$$

where θ is a growth or decay rate of the disturbance, and $\chi^{(j)}$ and H are periodic functions of τ. We use a perturbation method after YIH (1968) with a small wavenumber, and write

$$\chi^{(j)}=\phi_0{}^{(j)}(y,\tau)+\alpha\phi_1{}^{(j)}(y,\tau) +\alpha^2\phi_2{}^{(j)}(y,\tau)+\dots, \quad (4.3)$$

$$H=h_0(\tau)+\alpha h_1(\tau)+\alpha^2h_2(\tau)+\dots, \quad (4.4)$$

$$\theta=\theta_0+\alpha\theta_1+\alpha^2\theta_2+\dots, \quad (4.5)$$

where $\phi^{(j)}$'s and h's are periodic functions of τ, and θ's are real constants.

We substitute (4.3)-(4.5) into (3.16)-(3.27), expand the resultant equations by α, and equate the coefficients of like powers of α. The kinematic condition (3.19) becomes

$$\frac{dh_0}{d\tau}+\theta_0h_0=0\,. \quad (4.6)$$

Since h_0 must be periodic in τ and θ_0 are a real constant, we have

$$\theta_0=0 \quad (4.5)$$

and without the loss of generality $h_0=1$ (see YIH, 1968).

Equations (3.16) become

$$\frac{\partial}{\partial\tau}\phi_{0yy}{}^{(j)}=\frac{1}{\nu_jR}\phi_{0yyyy}{}^{(j)}, \quad (4.8)$$

and the boundary conditions become

(1) $\phi_0{}^{(1)}(d,\tau)=0,$ (2) $\phi_{0y}{}^{(1)}(d,\tau)=0,$

(3) $\phi_0{}^{(2)}(-1,\tau)=0,$ (4) $\phi_{0y}{}^{(2)}(-1,\tau)=0,$

(5) $\phi_0{}^{(1)}(0,\tau)=\phi_0{}^{(2)}(0,\tau),$

(6) $\bar{u}_y{}^{(1)}(0,\tau)+\phi_{0y}{}^{(1)}(0,\tau) =\bar{u}_y{}^{(2)}(0,\tau)+\phi_{0y}{}^{(2)}(0,\tau),$

(7) $\bar{u}_{yy}{}^{(1)}(0,\tau)+\phi_{0yy}{}^{(1)}(0,\tau) =\frac{1}{\hat{\rho}\hat{\nu}}(\bar{u}_{yy}{}^{(2)}(0,\tau)+\phi_{0yy}{}^{(2)}(0,\tau))\,,$

(8) $$-\frac{1}{R}\phi_{0yyy}{}^{(1)}(0,\tau)+\phi_{0y\tau}(0,\tau) =\frac{1}{\hat{\rho}}\left[-\frac{1}{\hat{\nu}R}\phi_{0yyy}{}^{(2)}+\phi_{0y\tau}{}^{(2)}(0,\tau)\right]. \quad (4.9)$$

The solution of this system of equations is

$$\phi_0{}^{(j)}(y,\tau)=\frac{1}{2}\{[A_0{}^{(j)}+B_0{}^{(j)}y +C_0{}^{(j)}\cosh\sqrt{\nu_j}qy +D_0{}^{(j)}\sinh\sqrt{\nu_j}qy]e^{i\omega\tau}+c.c.\}, \quad (4.10)$$

where the coefficients $A_0{}^{(j)}$, $B_0{}^{(j)}$, $C_0{}^{(j)}$ and $D_0{}^{(j)}$ are determined with the boundary conditions (4.9), and the explicit forms are shown in the Appendix 1.

To the next approximation, (3.19) becomes

$$\frac{dh_1}{d\tau}+\theta_1=-i[\bar{u}^{(j)}(0,\tau)+\phi_0^{(j)}(0,\tau)], \quad (4.11)$$

where $j=1$ or 2. Because $\bar{u}^{(j)}$, $\phi_0^{(j)}$ and h_1 are periodic in τ, and θ_1 must be a real constant, we have $\theta_1=0$, and

$$h_1=-i\int\{\bar{u}^{(1)}(0,\tau)+\phi_0^{(1)}(0,\tau)\}d\tau$$
$$=-\frac{1}{2\omega}\{A_He^{i\omega\tau}-A_H{}^*e^{-i\omega\tau}\}, \quad (4.12)$$

where

$$A_H=A_0^{(1)}+C_0^{(1)}+A^{(2)}+1, \quad (4.13)$$

and $A_H{}^*$=complex conjugate of A_H. Equation (3.16) becomes

$$\phi_{1yyyy}^{(j)}-\nu_jR\phi_{1yy\tau}^{(j)}$$
$$=i\nu_jR(\bar{u}^{(j)}\phi_{0yy}^{(j)}-\bar{u}_{yy}^{(j)}\phi_0^{(j)}), \quad (4.14)$$

and the boundary conditions become

(1) $\phi_1^{(1)}(d,\tau)=0$, (2) $\phi_{1y}^{(1)}(d,\tau)=0$,

(3) $\phi_1^{(2)}(-1,\tau)=0$, (4) $\phi_{1y}^{(2)}(-1,\tau)=0$,

(5) $\phi_1^{(1)}(0,\tau)=\phi_1^{(2)}(0,\tau)$,

(6) $\bar{u}_y^{(1)}(0,\tau)h_1+\phi_{1y}^{(1)}(0,\tau)$
$=\bar{u}_y^{(2)}(0,\tau)h_1+\phi_{1y}^{(2)}(0,\tau)$,

(7) $\bar{u}_{yy}^{(1)}(0,\tau)h_1+\phi_{1yy}^{(1)}(0,\tau)$
$=\frac{1}{\hat{\rho}\hat{\nu}}\{\bar{u}_{yy}^{(2)}(0,\tau)h_1+\phi_{1yy}^{(2)}(0,\tau)\}$,

(8) $iF_r{}^{-2}-\frac{1}{R}\phi_{1yyy}^{(1)}(0,\tau)+\phi_{1y\tau}(0,\tau)$
$+i\bar{u}^{(1)}(0,\tau)\phi_{0y}^{(1)}(0,\tau)$
$-i\bar{u}_y^{(1)}(0,\tau)\phi_0^{(1)}(0,\tau)$
$=\frac{1}{\hat{\rho}}\Big\{iF_r{}^{-2}-\frac{1}{\hat{\nu}R}\phi_{1yyy}^{(2)}(0,\tau)$
$+\phi_{1y\tau}^{(2)}(0,\tau)+i\bar{u}^{(2)}(0,\tau)\phi_{0y}^{(2)}(0,\tau)$
$-i\bar{u}_y^{(2)}(0,\tau)\phi_0^{(2)}(0,\tau)\Big\}$. (4.15)

Using the method of undetermined coefficients (e.g., WYLIE, 1975: pp. 53–59) with (2.12), (2.13) and (4.10), the particular solutions of (4.14) for $\phi_1^{(j)}$ are

$$\phi_{1p}^{(1)}=i\nu_jR\,[\sum_{k=1}^{10}I_k^{(j)}(y)+e^{2i\omega\tau}\sum_{k=11}^{14}I_k^{(j)}(y)$$
$$+\sum_{k=1}^{10}I_k^{(j)*}(y)+e^{-2i\omega\tau}\sum_{k=11}^{14}I_k^{(j)*}(y)], \quad (4.16)$$

where the asterisk denotes a complex conjugate, and $I_k^{(j)}$ ($j=1,2$; $k=1-14$) are given in the Appendix 2. Because the boundary conditions (6)–(8) in (4.15) are independent of τ, the complementary solutions must be composed of τ-independent terms and $e^{\pm 2i\omega\tau}$-dependent terms. Therefore, we have the complementary solutions

$$\phi_{1c}^{(j)}=A_1^{(j)}+B_1^{(j)}y+C_1^{(j)}y^2+D_1^{(j)}y^3$$
$$+E_1^{(j)}V_1^{(j)}+F_1^{(j)}W_1^{(j)}+E^{1(j)*}V_1^{(j)*}$$
$$+F_1^{(j)*}W_1^{(j)*}+G_1^{(j)}(\tau)+K_1^{(j)}(\tau)y, \quad (4.17)$$

where

$$\left.\begin{aligned}&V_1^{(j)}=\cosh(\sqrt{2\nu_j}\,qy)e^{2i\omega\tau},\\&W_1^{(j)}=\sinh(\sqrt{2\nu_j}\,qy)e^{2i\omega\tau},\\&G_1^{(j)}(\tau)=G_1^{(j)}e^{2i\omega\tau}+G_1^{(j)*}e^{-2i\omega\tau},\\&K_1^{(j)}(\tau)=K_1^{(j)}e^{2i\omega\tau}+K_1^{(j)*}e^{-2i\omega\tau},\end{aligned}\right\} \quad (4.18)$$

and $A_1^{(j)}$, $B_1^{(j)}$, $C_1^{(j)}$, $D_1^{(j)}$, $E_1^{(j)}$, $F_1^{(j)}$, $G_1^{(j)}$ and $K_1^{(j)}$ are constants to be determined with the boundary conditions (4.15); these are presented fully in KAMACHI (1980), and the explicit form of $A_1^{(1)}$ is shown in the Appendix 3, because only the explicit form of $A_1^{(1)}$ is necessary for the calculation of the growth rate θ_2 in (4.25).

As mentioned above, we have

$$\phi_1^{(j)}=\phi_{1p}^{(j)}+\phi_{1c}^{(j)}, \quad (4.19)$$

and put

$$\phi_1^{(j)}\equiv\Phi_1^{(j)}(y)+\phi_{11}^{(j)}(y)e^{2i\omega\tau}$$
$$+\phi_{12}^{(j)}(y)e^{-2i\omega\tau}, \quad (4.20)$$

where

$$\left.\begin{aligned}&\Phi_1^{(j)}(y)=\Pi_1^{(j)}(y)+A_1^{(j)}+B_1^{(j)}y\\&\quad+C_1^{(j)}y^2+D_1^{(j)}y^2,\\&\Pi_1^{(j)}(y)=iR(\sum_{k=1}^{10}I_k^{(j)}(y)+\sum_{k=1}^{10}I_k^{(j)*}(y)),\\&\phi_{11}^{(j)}=i\nu_jR\sum_{k=11}^{14}I_k^{(j)}(y)+E_1^{(j)}\cosh\\&\quad\times(\sqrt{2\nu_j}\,qy)+F_1^{(j)}\sinh(\sqrt{2\nu_j}\,qy)\\&\quad+G_1^{(j)}+K_1^{(j)}y,\\&\phi_{12}^{(j)}=i\nu_jR\sum_{k=11}^{14}I_k^{(j)*}(y)+E_1^{(j)*}\cosh\\&\quad\times(\sqrt{2\nu_j}\,q^*y)+F_1^{(j)*}\sinh(\sqrt{2\nu_j}\,q^*y)\\&\quad+G_1^{(j)*}+K_1^{(j)*}y.\end{aligned}\right\} \quad (4.21)$$

Now, according to (3.19),

$$\frac{dh_2}{d\tau}=-\theta_2-i\bar{u}^{(j)}(0,\tau)h_1-i\phi_1^{(j)}(0,\tau), \quad (4.22)$$

where $j=1$, though j may be 1 or 2. Because h_2 must be periodic in τ,

$$-\theta_2-\{i\bar{u}^{(1)}(0,\tau)h_1\}_{\tau\text{-}ind.}-i\Phi_1^{(1)}(0)=0, \quad (4.23)$$

where the suffix "τ-ind." indicates the terms independent of τ in $i\bar{u}^{(j)}(0,\tau)h_1$, i.e.

$$\{i\bar{u}^{(1)}(0,\kappa)h_1\}_{\tau\text{-}ind.}\equiv-\frac{i}{4\omega}[A_H(A^{(1)*}+1)-A_H^*(A^{(1)}+1)]. \quad (4.24)$$

Then we have

$$\theta_2=\frac{i}{4\omega}[A_H(A^{(1)*}+1)-A_H^*(A^{(1)}+1)]-i\Pi_1^{(1)}(0)-iA_1^{(1)}. \quad (4.25)$$

Therefore, using (2.14), (4.13), (4.21) and the Appendices, we can calculate the value of θ_2. The flow is unstable for $\theta_2>0$, neutral for $\theta_2=0$, and stable for $\theta_2<0$. Considering the values of θ_2, we discuss the stability characteristics in Sect. 5.

The result obtained is

$$\Theta=\theta_2\alpha^2+O(\alpha^3), \quad (4.26)$$

and the instability so far discussed is a kind of secular instability, since the growth rate is of the order of α^2.

5. Results and discussion

The parameters which have appeared in the present stability calculation are R, ω, $\hat{\nu}$, d, F_r and $\hat{\rho}$. In this section we discuss instability characteristics in the parameter spaces. In the figures presented in Sect. 5, diagonally lined regions indicate that the flow is unstable and blank regions indicate that it is stable.

Figure 2 shows the region of instability in the ω-R plane, when $d=1.0$, $\hat{\nu}=0.01$ and $\hat{\rho}=1.0$. The non-dimensional frequency is

$$\omega=\frac{\omega_d d^{(2)}}{U_0}=\frac{d^{(2)}/U_0}{1/\omega_d}=\frac{\tau_w}{\tau_v},$$

where τ_w is a time scale of wavy disturbances propagating in the direction normal to the walls and τ_v is a time scale for the diffusion of vorticity. In the figure, the values of ω for the neutral curves decrease with R, because the flow has time scale of wavy disturbance propagation that is smaller than that of the diffusion of vorticity when the inertial forces are much more dominant compared with the viscous ones. When the values of ω are very small, the flow is always stable. Use of (2.12, 13, 14) as $\omega\to0$ leads to $A^{(1)}=-1+O(\omega)$, $B^{(1)}=O(\omega)$, $A^{(2)}=-1+O(\omega)$ and $B^{(2)}=O(\omega)$. Then we have $\bar{u}^{(j)}=\frac{1}{2}[O(\omega)e^{i\omega\tau}+c.c.]$, where $j=1, 2$. The velocities of the primary flows therefore are very small, and the two-layer liquid is statically stable.

Figure 3 shows the dependence of the regions of unstable disturbance on $\hat{\nu}$ and R, when $\hat{\rho}=1.0$, $F_r=1.0$, $d=1.0$ and $\omega=0.1$. When $0.0<\hat{\nu}<1.0$ (the lower fluid is more viscous than the upper), and $\hat{\rho}=1.0$ (homogeneous density), the oscillatory flow can be unstable. The flow is neutral to $O(\alpha^2)$ when $\hat{\nu}=0.0$. These two cases correspond to a single-layer flow. We shall discuss the two cases separately.

To see what happens if the density and kine-

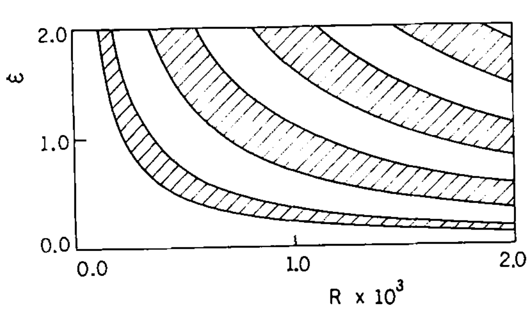

Fig. 2. Instability characteristics. $d=1.0$, $\hat{\nu}=0.01$ and $\hat{\rho}=1.0$. The flow is unstable in the diagonally-lined region.

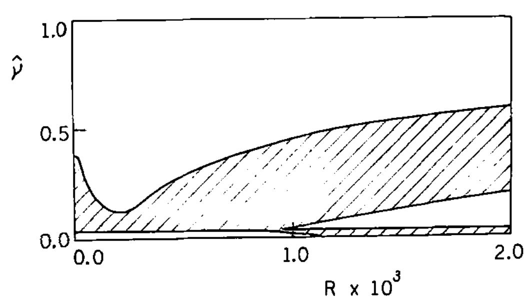

Fig. 3. Instability characteristics. $\hat{\nu}=1.0$, $d=1.0$ and $\omega=0.1$. The flow is unstable in the diagonally-lined region.

matic viscosity of one layer become equal to those of the other (i.e., $\hat{\rho}=\hat{\nu}=1.0$), consider the result (4.25). Recalling (2.14), (4.13), (4.21), and the Appendices 1, 2 and 3, it is seen that all coefficients of $\phi_0^{(j)}$ and $\phi_1^{(j)}$ are zero and $\theta_2=0$. When $\hat{\rho}=1.0$, therefore, θ_2 has a factor $(\hat{\nu}-1)$. When $\hat{\rho}=\hat{\nu}=1.0$, $\phi_0^{(j)}$ and $\phi_1^{(j)}$ are zero. Therefore the flow is a simple oscillatory flow without disturbances, and our perturbation theory does not contain the stability theory for the single layer oscillatory flow as a limiting case. When $\hat{\nu}<1.0$, therefore, the instability characteristics are different from those of KERCZEK and DAVIS (1974), HINO and SAWAMOTO (1975) and HALL (1978). The unstable disturbance in the present theory, therefore, may be a 'soft wave' in an oscillatory flow; the soft wave in a steady flow was discussed by YIH (1963, 1967).

The limiting case of $\hat{\nu}=0.0$ and $\hat{\rho}=1.0$ will now be discussed. When $\hat{\rho}=1.0$ and $\hat{\nu}=0.0$ on using (2.14), $A^{(1)}$, $B^{(1)}$ and $A^{(2)}$ are of $O(1)$, but $B^{(2)}=0$. There, $\bar{u}^{(2)}=\frac{1}{2}[(A^{(2)}+1)e^{i\omega\tau}+c.c.]$, and this solution does not satisfy the boundary condition (2.9). The value $\hat{\nu}=0$ is also a singularity of the basic equation (2.4). We cannot, therefore, discuss the case of $\hat{\nu}=0$ based on the present formulation.

KERCZEK and DAVIS (1974) investigated a 'fully' unsteady flow in the range of small wavenumbers. HINO and SAWAMOTO (1975) investigated a quasi-steady flow in the range of large wavenumbers, because the Strouhal number investigated by them is 0.05. The single-layer oscillatory flows may be stable in the range of small wavenumbers (KERCZEK and DAVIS, 1974), though the flow can be unstable in the range of large wavenumbers (HINO and SAWAMOTO, 1975). The difference in characteristics of each of these workers instability treatment is due to differences in the range of wavenumbers and the time scale of the primary flow considered. The stability characteristics of the two-layer 'fully' oscillatory flow is different from that of the single-layer one, which may be stable in the range of small wavenumbers (KERCZEK and DAVIS, 1974). This indicates that the viscous stratification (i.e., $\hat{\nu}<1.0$ and $\hat{\rho}=1.0$) has a destabilizing effect. YIH (1967), however, showed that steady two-layer flows are always unstable when $\hat{\rho}=1.0$, $\hat{\nu}<1.0$ and $d=1.0$. This indicates that the oscillatory flows have a stabilizing effect, because the oscillatory flow has a zero mean velocity of the primary flow.

Figure 4 shows the dependence of the regions of unstable disturbance on d and R when $\hat{\rho}=1.0$, $\omega=0.1$ and $\hat{\nu}=0.01$. In Fig. 4, the instability characteristic above the line $d=0.57$ is different from that below the line. We call the regions above and below this line the upper and lower regions respectively. When $\hat{\rho}<O(1)$, the upper region disappears and only the lower one exists. The unstable disturbance in the lower region is therefore similar to that of surface wave type in the oscillatory flow (YIH, 1968), because the flow with $\hat{\rho}<O(1)$ is a limiting case with a free surface. The unstable disturbance in the upper region is a soft wave ($\hat{\rho}=O(1)$ and $\hat{\nu}<O(1)$) in the oscillatory flow.

Figure 5 shows that the dependence of the regions of unstable disturbance on F_r and R, when $\hat{\rho}=0.5$. This parameter dependence is similar to that shown in YIH (1968).

With respect to the stability theory for a flexible boundary, it is well known that there is a threefold classification of unstable disturbances (BENJAMIN, 1960, 1963, 1966; LANDAHL, 1962;

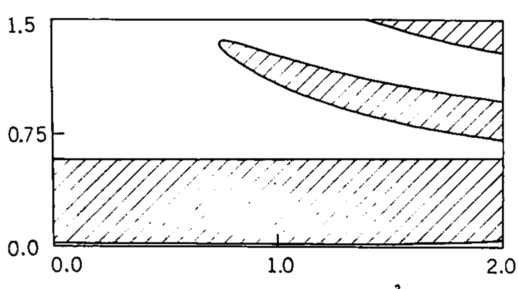

Fig. 4. Instability characteristics. $\hat{\rho}=1.0$, $\omega=0.1$ and $\hat{\nu}=0.01$. The flow is unstable in the diagonally-lined region.

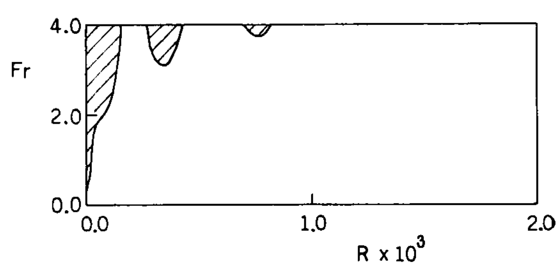

Fig. 5. Instabilily characteristics. $\hat{\rho}=0.5$, $d=0.5$, $\hat{\nu}=1.0$ and $\omega=1.0$. The flow is unstable in the diagonally-lined region,

TAKEMATSU, 1968, 1969, 1970; TURNER, 1973: pp. 92–94). The effect of viscosity is negligible in Class C disturbances (e.g., Kelvin-Helmhltz type; BENJAMIN (1963, 1966). The regions of unstable disturbances appear at small values of R in Fig. 5, and the viscosity has a destabilizing effect. Thus, the instability may not be classified into Class C type.

Figure 6 shows the dependence of the regions of unstable disturbance on $\hat{\rho}$ and R when $\hat{\nu}=0.01$, $F_r=1.0$, $d=0.5$ an $\omega=0.1$. When $\hat{\rho}=1.0$, in the figure, the flow is always unstable. The instability in the upper region in Fig. 6 is due to the soft wave in the oscillatory flow. At the limit $\hat{\rho}\to 0.0$, the flow may be the oscillatory with free surface waves similar to those treated by YIH (1968) and may be a flow of Class B type (after BENJAMIN, 1966). When $\hat{\nu}\to 1.0$, the regions of unstable disturbance in the figure disappear. In other words, the unstable disturbance in the lower region in Fig. 6 is a soft wave of surface wave type ($\hat{\rho}<O(1)$, cf. YIH, 1968) in the oscillatory flow which is unstable at $\hat{\nu}<O(1)$ (cf. YIH, 1967). Because the fluid viscosity has a destabilizing effect, the instability is not of Class B type (cf. Table 1 of BENJAMIN, 1966), although the fluid viscosity has a stabilizing

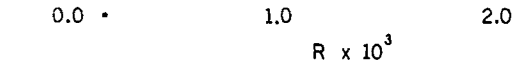

Fig. 6. Instability characteristics. $\hat{\nu}=0.01$, $F_r=1.0$, $d=0.5$ and $\omega=0.1$. The flow is unstable in the diagonally-lined region.

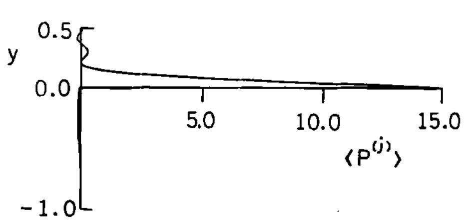

Fig. 7. Distribution of the values of the production term. $d=0.5$, $\hat{\nu}=0.01$, $\hat{\rho}=1.0$, $\omega=0.1$ and $R=1500$.

effect on Class B type disturbances at large Reynolds numbers.

To estimate the distribution of the kinetic energy production term for the disturbances, we derive the kinetic energy equation for them. The equations are

$$\frac{\partial}{\partial\tau}E^{(j)}+\bar{u}^{(j)}\frac{\partial}{\partial x}E^{(j)}=P^{(j)}+T^{(j)}-\varepsilon^{(j)}$$

where

$$E^{(j)}=\frac{1}{2}\{(\hat{u}_1^{(j)})^2+(\hat{u}_2^{(j)})^2\},$$

$$P^{(j)}=-\hat{u}_1^{(j)}\hat{u}_2^{(j)}\frac{\partial\bar{u}^{(j)}}{\partial y},$$

$$T^{(j)}=-\frac{\partial}{\partial x_l}\left[\hat{u}_l^{(j)}\rho_j p^{(j)}-\frac{2}{\nu_j R}\hat{u}_k^{(j)}S_{kl}\right],$$

$$\varepsilon^{(s)}=\frac{2}{\nu_j R}S_{kl}^2,\quad S_{kl}=\frac{1}{2}\left(\frac{\partial\hat{u}_k^{(j)}}{\partial x_l}+\frac{\partial\hat{u}_l^{(j)}}{\partial x_k}\right),$$

$$k=1,2,\quad l=1,2,$$

$$(\hat{u}_1^{(j)},\ \hat{u}_2^{(j)})\equiv(\hat{u}^{(j)},\ \hat{v}^{(j)}),$$

$$(x_1,\ x_2)\equiv(x,\ y).$$

We rewrite $P^{(j)}$ as

$$\begin{aligned}P^{(j)}&=i\alpha\phi^{(j)}\phi_y^{(j)}\bar{u}_y^{(j)}\\&=i\alpha e^{2\theta_2\alpha^2\tau}[\chi^{(j)}\chi_y^{(j)}\bar{u}_y^{(j)}]\\&\equiv i\alpha e^{2\theta_2\alpha^2\tau}[P_1(y)+P_2(y,\tau)],\end{aligned}$$

where $P_1(y)$ is a function independent of τ in $\chi^{(j)}\chi_y^{(j)}\bar{u}_y^{(j)}$, and $P_2(y,\tau)$ is periodic in τ. Then we put

$$\begin{aligned}\langle P^{(j)}(y)\rangle&\equiv\frac{1}{2\pi}\int_0^{2\pi}[P_1(y)+P_2(y,\tau)]d(\omega\tau)\\&=P_1(y)\end{aligned}$$

and $\langle P^{(j)}\rangle$ is an approximate value of the time mean of the production term $P^{(j)}$. We evaluated the y-contribution to $\langle P^{(j)}\rangle$, and plotted the values of $\langle P^{(j)}\rangle$ against y for $d=0.5$, $\hat{\nu}=0.01$, $\hat{\rho}=1.0$, $\omega=0.1$ and $R=1500$ in Fig. 7. It can be seen that the values of the production term have an extremum at the interface, although KERCZEK and DAVIS (1974) showed that the production term has a negative minimum value in the outer region of the boundary layer. This indicates that the interface of fluid viscosity has a destabilizing effect.

In conclusion, the unstable disturbances do not belong to any of the three classes A, B and C developed in the instability theory for a flexible boundary, but are of the soft wave type for oscillatory flows. The disturbances are also of the surface wave type, which are unstable for two-layer oscillatory flows in a viscous fluid.

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Appendix 1.

The explicit forms of $A_0^{(j)}$, $B_0^{(j)}$, $C_0^{(j)}$ and $D_0^{(j)}$ are as follows;

$$
\begin{aligned}
A_0^{(1)} = \frac{1}{Y_0}\Big[& -(\hat{\rho}\hat{\nu}-1)B^{(1)}\{\hat{\rho}(\sqrt{\hat{\nu}}\,q\cosh q\sqrt{\hat{\nu}} \\
& -\sinh q\sqrt{\hat{\nu}}) - \sqrt{\hat{\nu}}\,(\hat{\rho}-(\hat{\rho}-1)\cosh q\sqrt{\hat{\nu}}) \\
& \times(\sinh qd - qd\cosh qd)\} \\
& -(\hat{\rho}-1)\sqrt{\hat{\nu}}\,A^{(1)}\{\hat{\rho}(2-2\cosh q\sqrt{\hat{\nu}} \\
& +q\sqrt{\hat{\nu}}\sinh q\sqrt{\hat{\nu}}) \\
& -\sqrt{\hat{\nu}}\sinh q\sqrt{\hat{\nu}}(\sinh qd - qd\cosh qd) \\
& +(\cosh q\sqrt{\hat{\nu}}-1)(1+qd\sinh qd-\cosh qd)\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
B_0^{(1)} = \frac{1}{Y_0}\Big[& (\hat{\rho}\hat{\nu}-1)B^{(1)}\{\cosh q\sqrt{\hat{\nu}}(\cosh qd-1) \\
& -\hat{\rho}\cosh qd\,(\cosh q\sqrt{\hat{\nu}}-1)\}
\end{aligned}
$$

$$
\begin{aligned}
&+(\hat{\rho}-1)A^{(1)}\{-\sqrt{\hat{\nu}}\sinh q\sqrt{\hat{\nu}}(1-\cosh qd)\\
&-\sinh qd\,(1-\cosh q\sqrt{\hat{\nu}})\}],
\end{aligned}
$$

$$
\begin{aligned}
C_0^{(1)}=&\frac{1}{Y_0}[-(\hat{\rho}\hat{\nu}-1)B^{(1)}\{\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}}\\
&\times(\sinh qd-qd\cosh qd)\\
&+\hat{\rho}\cosh qd(\sinh q\sqrt{\hat{\nu}}-q\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}})\}\\
&+(\hat{\rho}-1)A^{(1)}\{(1-\cosh q\sqrt{\hat{\nu}})\\
&\times(1+(2\hat{\rho}-1)\cosh qd)\\
&-\sqrt{\hat{\nu}}\sinh qd\sinh q\sqrt{\hat{\nu}}\\
&+q\sqrt{\hat{\nu}}(\hat{\rho}+d)\cosh qd\sinh q\sqrt{\hat{\nu}}\}],
\end{aligned}
$$

$$
\begin{aligned}
D_0^{(1)}=&\frac{1}{Y_0}[(\hat{\rho}\hat{\nu}-1)B^{(1)}\{\hat{\rho}\sinh qd\\
&\times(\sinh q\sqrt{\hat{\nu}}-q\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}})\\
&-\hat{\rho}\sqrt{\hat{\nu}}+(\hat{\rho}-1)\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}}\\
&+\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}}(\cosh qd-qd\sinh qd)\}\\
&+(\hat{\rho}-1)A^{(1)}\{\sqrt{\hat{\nu}}(1-2\hat{\rho})\sinh qd\\
&\times(1-\cosh q\sqrt{\hat{\nu}})\\
&-\hat{\nu}(1+\hat{\rho}q)\sinh qd\sin q\sqrt{\hat{\nu}}\\
&+\hat{\nu}\sinh q\sqrt{\hat{\nu}}(\cosh qd-qd\sinh qd)\}],
\end{aligned}
$$

$$
\begin{aligned}
A_0^{(2)}=&\frac{1}{Y_0}[\hat{\rho}(\hat{\rho}\hat{\nu}-1)B^{(1)}\{(1+(\hat{\rho}-1)\cosh qd)\\
&\times(\sinh q\sqrt{\hat{\nu}}-q\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}})\\
&+\sqrt{\hat{\nu}}(\sinh qd-qd\cosh qd)\}\\
&-(\hat{\rho}-1)A^{(1)}\{\sinh qd\,(\hat{\rho}(\sinh q\sqrt{\hat{\nu}}\\
&-q\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}})-q\sqrt{\hat{\nu}}d)\\
&-\sqrt{\hat{\nu}}(1-\cosh qd)(2+\hat{\rho}(-1+\cosh q\sqrt{\hat{\nu}}\\
&-q\sqrt{\hat{\nu}}\sinh q\sqrt{\hat{\nu}}))\}],
\end{aligned}
$$

$$
\begin{aligned}
B_0^{(2)}=&\frac{\hat{\rho}\sqrt{\hat{\nu}}q}{Y_0}[(\hat{\rho}\hat{\nu}-1)B^{(1)}\{\cosh q\sqrt{\hat{\nu}}\\
&\times(-1+\cosh qd)\\
&+\hat{\rho}\cosh qd(1-\cosh q\sqrt{\hat{\nu}})\}\\
&+(\hat{\rho}-1)A^{(1)}\{\sinh qd\,(-1+\cosh q\sqrt{\hat{\nu}})\\
&+\sqrt{\hat{\nu}}\sinh q\sqrt{\hat{\nu}}(-1+\cosh qd)\}],
\end{aligned}
$$

$$
\begin{aligned}
C_0^{(2)}=&\frac{1}{Y_0}[-(\hat{\rho}\hat{\nu}-1)\hat{\rho}B^{(1)}\{\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}}\\
&\times(\sinh qd-qd\cosh qd)\\
&+\hat{\rho}\cosh qd\,(\sinh q\sqrt{\hat{\nu}}-q\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}})\}\\
&-(\hat{\rho}-1)A^{(1)}\{\sqrt{\hat{\nu}}\cosh qd\,(-\hat{\rho}+(\hat{\rho}-2)\\
&\times\cosh q\sqrt{\hat{\nu}})-\sinh qd\,(\hat{\rho}\sinh q\sqrt{\hat{\nu}}\\
&-q\sqrt{\hat{\nu}}(\hat{\rho}+d)\cosh q\sqrt{\hat{\nu}})+\sqrt{\hat{\nu}}\cosh q\sqrt{\hat{\nu}}\\
&-\sqrt{\hat{\nu}}(-\hat{\rho}+(\hat{\rho}-1)\cosh q\sqrt{\hat{\nu}})\}],
\end{aligned}
$$

$$
\begin{aligned}
D_0^{(2)}=&\frac{1}{Y_0}[(\hat{\rho}\hat{\nu}-1(B^{(1)}\{\hat{\rho}-\hat{\rho}\sqrt{\hat{\nu}}\sinh qd\\
&\times\sinh q\sqrt{\hat{\nu}}+\hat{\rho}\cosh qd[(\hat{\rho}-1)\\
&+q\sqrt{\hat{\nu}}(\hat{\rho}+d)\sinh q\sqrt{\hat{\nu}}-\hat{\rho}\cosh q\sqrt{\hat{\nu}}]\}\\
&+(\hat{\rho}-1)A^{(1)}\{-\hat{\rho}\sinh qd\,(1+q\sqrt{\hat{\nu}}\sinh q\sqrt{\hat{\nu}}\\
&-\cosh q\sqrt{\hat{\nu}})+\sqrt{\hat{\nu}}\sinh q\sqrt{\hat{\nu}}\\
&\times((\hat{\rho}-2)-qd\sinh qd-(\hat{\rho}-2)\cosh qd)\}],
\end{aligned}
$$

where

$$
\begin{aligned}
Y_0=&-2\hat{\rho}\sqrt{\hat{\nu}}\\
&-2\hat{\rho}\sqrt{\hat{\nu}}(\hat{\rho}-1)\cosh qd\\
&+2\sqrt{\hat{\nu}}(\hat{\rho}-1)\cosh q\sqrt{\hat{\nu}}\\
&+\hat{\rho}(1+\hat{\nu})\sinh qd\sinh q\sqrt{\hat{\nu}}\\
&-q\sqrt{\hat{\nu}}(\hat{\rho}+d)\sinh qd\cosh q\sqrt{\hat{\nu}}\\
&-\hat{\rho}q\hat{\nu}(\hat{\rho}+d)\sinh q\sqrt{\hat{\nu}}\cosh qd\\
&+2\sqrt{\hat{\nu}}(\hat{\rho}^2-\hat{\rho}+1)\cosh qd\cosh q\sqrt{\hat{\nu}}.
\end{aligned}
$$

Appendix 2.

The explicit forms of $I_k^{(j)}$ ($j=1,2$, $k=1\text{-}14$) are shown as follows;

$$I_1^{(j)}=\frac{C_0^{(j)}}{4\nu_j q^2}\cosh\sqrt{\nu_j}\,qy,$$

$$I_2^{(j)}=\frac{D_0^{(j)}}{4\nu_j q^2}\sinh\sqrt{\nu_j}\,qy,$$

$$I_3^{(j)}=\frac{A^{(j)*}}{4\nu_j q^2}\left(\frac{4}{\sqrt{\nu_j}\,q^*}B_0^{(j)}-A_0^{(j)}\right)\cosh\sqrt{\nu_j}\,q^*y,$$

$$I_4^{(j)}=\frac{B^{(j)*}}{4\nu_j q^2}\left(\frac{4}{\sqrt{\nu_j}\,q^*}B_0^{(j)}-A_0^{(j)}\right)\sinh\sqrt{\nu_j}\,q^*y,$$

$$I_5^{(j)}=\frac{1}{4\nu_j q^2}A^{(j)*}B_0^{(j)}y\cosh\sqrt{\nu_j}\,q^*y,$$

$$I_6^{(j)}=\frac{B^{(j)*}B_0^{(j)}}{4\nu_j q^2}y\sinh\sqrt{\nu_j}\,q^*y,$$

$$I_7^{(j)}=\frac{i}{32\nu_j\beta^2}(A^{(j)*}C_0^{(j)}+B^{(j)*}D_0^{(j)})\cosh s_j y,$$

$$I_8^{(j)}=\frac{i}{32\nu_j\beta^2}(B^{(j)*}C_0^{(j)}+A^{(j)*}D_0^{(j)})\sinh s_j y,$$

$$I_9^{(j)}=\frac{i}{32\nu_j\beta^2}(A^{(j)*}C_0^{(j)}-B^{(j)*}D_0^{(j)})\cos s_j y,$$

$$I_{10}^{(j)}=\frac{i}{32\nu_j\beta^2}(A^{(j)*}D_0^{(j)}-B^{(j)*}C_0^{(j)})i\sin s_j y,$$

$$
\begin{aligned}
I_{11}^{(j)}=&\frac{1}{4\nu_j q^2}\left\{\frac{2}{\sqrt{\nu_j}\,q}(-A^{(j)}B_0^{(j)})\right.\\
&\left.-(C_0^{(j)}-A_0^{(j)}A^{(j)})\right\}\cosh\sqrt{\nu_j}\,qy,
\end{aligned}
$$

$$I_{12}^{(j)}=\frac{1}{4\nu_j q^2}\left\{\frac{2}{\sqrt{\nu_j}q}(-B^{(j)}B_0^{(j)})-(D_0^{(j)}-A_0^{(j)}B^{(j)})\right\}\sinh\sqrt{\nu_j}qy\,,$$

$$I_{13}^{(j)}=-\frac{1}{4\nu_j q^2}(-A^{(j)}B_0^{(j)})y\cosh\sqrt{\nu_j}qy\,,$$

$$I_{14}^{(j)}=-\frac{1}{4\nu_j q^2}(-B^{(j)}B_0^{(j)})\,y\sinh\sqrt{\nu_j}qy\,,$$

where

$$s_j=2\sqrt{\nu_j}\beta.$$

Appendix 3.

The explicit form of $A_1^{(1)}$ is as follows.

$$A_1^{(1)}=\frac{1}{Y_1}[-2\hat{\nu}(\hat{\rho}\hat{\nu}+4d+3\hat{\rho}d^2)B_{11}-2\hat{\nu}d\times(\hat{\rho}\hat{\nu}-2d-\hat{\rho}d^2)B_{12}-2d^2(3\hat{\nu}+4\hat{\rho}\hat{\nu}d+d^2)B_{13}-2d^2(\hat{\nu}+2\hat{\rho}\hat{\nu}d+d^2)B_{14}-2d^2(\hat{\nu}+4\hat{\rho}\hat{\nu}d+d^2)B_{15}+4\hat{\nu}d^2(1+\hat{\rho}d)B_{16}+\hat{\rho}\hat{\nu}d^2(d^2-\hat{\nu})B_{17}+\frac{\hat{\nu}Rd^3}{3}(2\hat{\rho}\hat{\nu}+d)B_{18}]\,,$$

where

$$Y_1=-2\hat{\nu}(\hat{\rho}\hat{\nu}+4d+3\hat{\rho}d^2)+d^2(d^2-6\hat{\nu}-8\hat{\rho}\hat{\nu}d),$$

$$B_{11}=-\Pi_1^{(1)}(d),\qquad B_{12}=-\Pi_{1y}^{(1)}(d),$$

$$B_{13}=-\Pi_1^{(2)}(-1),\ B_{14}=-\Pi_{1y}^{(2)}(-1),$$

$$B_{15}=-\Pi_1^{(1)}(0)+\Pi_1^{(2)}(0),$$

$$B_{16}=-\Gamma-\Pi_{1y}^{(1)}(0)+\Pi_{1y}^{(2)}(0),$$

$$B_{17}=-\Lambda-\Pi_{1yy}^{(1)}(0)+\frac{1}{\hat{\rho}\hat{\nu}}\Pi_{1yy}^{(2)}(0)\,,$$

$$B_{18}=-\chi+\frac{1}{R}\Pi_{1yyy}^{(1)}(0)-\frac{1}{\hat{\nu}R}\Pi_{1yyy}^{(2)}(0),$$

$$\Gamma=\frac{1}{4\omega}(\hat{\rho}\hat{\nu}-1)\{A_Hq^*B^{(1)*}-A_H^*qB^{(1)}\},$$

$$\Lambda=\frac{1}{4\omega}\left\{A_H^*q^2\left(A^{(1)}-\frac{1}{\hat{\rho}}A^{(2)}\right)-A_Hq^{*2}\left(A^{(1)*}-\frac{1}{\hat{\rho}}A^{(2)*}\right)\right\},$$

$$\chi=\frac{i}{4}[(A^{(1)}+1)(B_0^{(1)*}+q^*D_0^{(1)*})+(A^{(1)*}+1)(B_0^{(1)}+qD_0^{(1)})-qB^{(1)}(A_0^{(1)*}+C_0^{(1)*})-q^*B^{(1)*}(A_0^{(1)}+C_0^{(1)})]-\frac{i}{4\hat{\rho}}[(A^{(2)}+1)(B_0^{(2)*}+\sqrt{\hat{\nu}}q^*D_0^{(2)*})+(A^{(2)*}+1)(B_0^{(2)}+\sqrt{\hat{\nu}}qD_0^{(2)})-\sqrt{\hat{\nu}}qB^{(2)}(A_0^{(2)*}+C_0^{(2)*})-\sqrt{\hat{\nu}}q^*B^{(2)}(A_0^{(2)}+C_0^{(2)})]-\left(1-\frac{1}{\hat{\rho}}\right)iF_r^{-2}.$$

粘性二層振動流の安定性

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要旨: 密度および粘度成層の存在する二層振動流の安定性を，特に不安定特性への粘性，時間スケール，密度，流体の深さの影響に注目して，摂動展開を用いて調べた．密度一様の振動流は，上下層の粘性が異なる時に不安定になりうることがわかった．粘度成層は振動流を不安定化する．一層の流体への極限について，二通りの場合が調べられている．

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