Strong formulations and cutting planes for designing digital data service networks

Youngho Lee, Lu Lu and Yuping Qiu
US WEST Technologies, 4001 Discovery Drive, Boulder, CO 80303, USA

Fred Glover
Graduate School of Business, University of Colorado, Boulder 80309, USA

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This paper deals with the problem of designing a least-cost digital data service (DDS) network that connects a given set of locations through digital switching offices with bridging capabilities. We present several alternative mixed 0-1 integer programming formulations and evaluate analytically their relative strengths by comparing their respective linear programming relaxations. By exploiting the structures inherent in a particularly strong formulation, we develop several classes of valid inequalities and cutting planes in order to tighten the initial formulation. For several problems of real-world data, computational results show that the strong formulation with valid inequalities and cutting planes generates a very tight lower bound (over 98% of the optimality) and so finds an optimal solution well within an acceptable time bound.

1. Introduction

We consider a network design problem of connecting several customer locations with dedicated digital communication links provided by a regional telephone company. Such a communication network is called a digital data service (DDS) network and has the following unique features. First, to receive digital data service, each customer location must be linked directly to exactly one of the digital switching offices with bridging capabilities, called hubs. Second, if more than one hub is used for that purpose, the chosen hubs must also be interconnected by links so that all customer locations can communicate with each other. Third, the cost of each link is calculated according to the tariff charges established by the Federal Communications Commission (FCC). In addition, there is also a fixed charge for each hub used in the network. The network design problem addressed in this paper, called *problem DDS*, seeks to design a least-cost DDS network that connects a given set of customer locations.

To formulate the problem, consider the following mathematical description. Suppose that there are m customer locations, called *target nodes*, indexed by

$i \in M = \{1, \ldots, m\}$ and that there are n switching offices with bridging capabilities, called *hubs*, indexed by $j \in N = \{1, \ldots, n\}$. Each target node $i \in M$ must connect to exactly one hub $j \in N$, incurring a connection cost of c_{ij}. Each hub j that is connected to at least one target node or other hubs incurs a fixed cost b. Also, two distinct hubs j and k that are connected in the solution incur a connection cost of d_{jk}, which is symmetric (i.e. $d_{jk} = d_{kj}$). We assume that all the costs are positive and the connection costs satisfy the triangle inequality, i.e. $c_{ik} \le c_{ij} + d_{jk}$ and $d_{jl} \le d_{jk} + d_{kl}$ for all $i \in M$, j, k, $l \in N$. This triangle inequality assumption, which is verified by the real data, allows us to develop a family of valid inequalities that plays a key role in solving the problem.

An alternating sequence of distinct nodes and edges is called a *path*. If such a sequence has the first and the last node identical while the remaining nodes are distinct, then it is called a *cycle*. Let L denote the set of all cycles consisting of only hub nodes.

Let x_{ij} be a 0-1 variable, where $x_{ij} = 1$ if and only if target node $i \in M$ connects to hub $j \in N$. Let u_j be a 0-1 variable, where $u_j = 1$ if and only if hub $j \in N$ is selected. Such a hub is called an *active* hub. Finally, let y_{jk} be a 0-1 variable, where $y_{jk} = 1$ if and only if hub $j \in N$ connects to hub $k \in N$, $j < k$. Then, problem DDS can be formulated as follows.

$$\text{DDS1:} \quad \text{minimize} \quad \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} \sum_{j<k, k \in N} d_{jk} y_{jk} + \sum_{j \in N} b u_j$$

$$\text{subject to} \quad \sum_{j \in N} x_{ij} = 1, \qquad i \in M, \tag{1}$$

$$x_{ij} \le u_j, \qquad i \in M, \; j \in N, \tag{2}$$

$$y_{jk} \le u_j, \qquad j < k, \; j,k \in N, \tag{3}$$

$$y_{jk} \le u_k, \qquad j < k, \; j,k \in N, \tag{4}$$

$$\sum_{j \in N} \sum_{j<k, k \in N} y_{jk} = \sum_{j \in N} u_j - 1, \tag{5}$$

$$\sum_{j,k \in C, j<k} y_{jk} \le |C| - 1, \qquad \forall C \in L \text{ (anti-cycle inequalities)}, \tag{6}$$

$$x_{ij} \ge 0, \qquad i \in M, \; j \in N,$$

$$u_j \in \{0,1\}, \qquad j \in N,$$

$$y_{jk} \in \{0,1\}, \qquad j < k, \; j,k \in N.$$

Constraint (1) indicates that each target node connects to exactly one hub, and (2) indicates that a hub is active if some target node connects to it. Constraints (3) and

(4) compel hubs j and k to be active if hubs j and k are connected. Since every feasible solution to problem DDS has a *tree* structure (i.e. a solution does not contain any cycle), (5) compels the number of connections between hubs to equal one less than the number of active hubs. Constraint (6) is a standard "*cycle-breaking*" constraint. Members of this set of constraints may be generated adaptively, as needed, in the framework of the branch-and-cut algorithm. Together, these constraints ensure all variables are 0-1 integer valued, assuming values for the x_{ij} variables are obtained by an extreme point linear programming algorithm given values for the y_{jk} and u_j variables.

Remark 1

Note that if the fixed cost of the hub node is negative, then we need additional contraints, which prevent each active hub node from being a *leaf* node (i.e. a node connects to exactly one other node) in the solution tree. For each $k \in N$, we have the following constraints:

$$\sum_{i \in M} x_{ik} + \sum_{l > k, l \in N} y_{kl} + \sum_{l < k, l \neq j, l \in N} y_{lk} \geq y_{jk}, \qquad j < k, \ \ j, k \in N,$$

$$\sum_{i \in M} x_{ij} + \sum_{l > j, l \neq k, l \in N} y_{jl} + \sum_{l < j, l \in N} y_{lj} \geq y_{jk}, \qquad j < k, \ \ j, k \in N.$$

Since the fixed cost b of problem DDS is positive, the subgraph associated with the optimal (minimum) solution of model DDS1 is also *minimal* in the sense that no active hubs appear as leaf nodes. Furthermore, since connection costs satisfy the triangle inequality, the foregoing constraints can be further strengthened as follows:

$$\sum_{i \in M} x_{ik} + \sum_{l > k, l \in N} y_{kl} + \sum_{l < k, l \neq j, l \in N} y_{lk} \geq 2y_{jk}, \qquad j < k, \ \ j, k \in N,$$

$$\sum_{i \in M} x_{ij} + \sum_{l > j, l \neq k, l \in N} y_{jl} + \sum_{l < j, l \in N} y_{lj} \geq 2y_{jk}, \qquad j < k, \ \ j, k \in N.$$

This paper is organized as follows. Section 2 presents several alternate formulations of the problem and evaluates analytically the relative strength of the formulations. Section 3 reports computational results and compares the formulations empirically. Section 4 concludes the paper.

2. Valid inequalities and alternate problem formulations

Observed that problem DDS can be conceptualized as a *degree-constrained node-weighted Steiner tree problem* on a simple undirected graph in which hubs are

Steiner nodes (a node is called a Steiner node if it does not always need to be selected in the solution), and each customer location must be connected to exactly one hub node. The node-weighted Steiner tree problem with degree constraints is an extension of the standard Steiner tree problem by the addition of node-associated weights and some degree constraints. The Steiner tree problem on graphs has been extensively investigated by many researchers for the last twenty years because of its simple combinatorial structure and its numerous practical applications (see Aneja [1], Beasley [2], Beasley [3] and Hakimi [10]). Winter [22] and Maculan [14] have presented excellent surveys of various applications and algorithms. Recently, Hwang and Richards [12] have provided a comprehensive bibliography which covers various important classes of Steiner tree problems including Euclidean, rectilinear, graphic and phylogenetic problems. For the node-weighted Steiner tree problem, Segev [19] investigates a single point weighted Steiner tree problem as a special case. This special case occurs when the set of nodes, which must be included in the solution tree, consists of a single node and all Steiner nodes have negative weights. However, in this paper, we consider a node-weighted Steiner tree problem in which there are several target nodes that must be connected to exactly one of the hub (Steiner) nodes.

Several recent advances have been made in the development of branch-and-cut algorithms for combinatorial optimization problems. (See, for example, Crowder et al. [8], Hoffman and Padberg [11], Martin [15], Padberg and Rinaldi [17], Van Roy and Wolsey [18], Wolsey [23].) In particular, Fischetti [9], and Chopra and Rao [5] presented the study of polyhedral structure of the (node-weighted) Steiner tree problems, and Chopra and Gorres [4], and Chopra et al. [6] showed the promising computational results of the branch-and-cut approach based on the polyhedral results. As emphasized by Sherali [20], at the heart of these approaches is a sequence of linear programming problems that speed up tightening of the lower (upper) bounds. The success of these approaches strongly depends on the strength or tightness of the linear programming representations employed. In this research vein, this paper presents the key role played by tight linear programming representations generated through a reformulation process, which augments the initial model by defining appropriate additional variables and constraints, and generates suitable valid inequalities and strong cutting planes. Next, we address some families of valid inequalities for problem DDS, which effectively strengthen initial formulations.

Note that if the optimal solution requires at least two active hubs, than each active hub must be connected to other active hubs. Hence, we have the following result.

PROPOSITION 1

If $\sum_{j \in N} u_j \geq 2$, then the following inequalities are valid for problem DDS:

$$\sum_{l<j,\, l \in N} y_{lj} + \sum_{j<k,\, k \in N} y_{jk} \geq u_j, \qquad j \in N. \tag{7}$$

PROPOSITION 2

Suppose that $(\hat{x}, \hat{y}, \hat{u})$ is an optimal solution to problem DDS with $\sum_{j \in N} \hat{u}_j \geq 2$. Then $(\hat{x}, \hat{y}, \hat{u})$ satisfies

$$\sum_{i \in M} \hat{x}_{ij} + \sum_{l<j, l \in N} \hat{y}_{lj} + \sum_{j<k, k \in N} \hat{y}_{jk} \geq 3\hat{u}_j, \qquad j \in N. \tag{8}$$

Proof

Suppose that there exists an optimal solution $(\tilde{x}, \tilde{y}, \tilde{u})$ to problem DDS that violates the inequality (8). Then, since the optimal solution to problem DDS has a tree structure, (8) is reduced to

$$\sum_{i \in M} \tilde{x}_{ij} + \sum_{l<j, l \in N} \tilde{y}_{lj} + \sum_{j<k, k \in N} \tilde{y}_{jk} = 2,$$

and so it suffices to consider the following two cases.

(i) Suppose that $\sum_{i \in M} \tilde{x}_{ij} = 0$. Then, the active hub j is connected only to other active hubs, say s and t. However, since $d_{sj} + d_{jt} \geq d_{st}$ and $b > 0$, we can reduce the total cost by connecting hub s and hub t, which still maintains feasibility. This contradicts the fact that $(\tilde{x}, \tilde{y}, \tilde{u})$ is an optimal solution.

(ii) Suppose that $\sum_{i \in M} \tilde{x}_{ij} = 1$. Then, a single customer is connected to active hub j and hub j must be connected to another hub, say t. However, since $c_{ij} + d_{jt} + b > c_{it}$, we can reduce the total cost by connecting this customer directly to hub t, which is a contradiction. This completes the proof. □

PROPOSITION 3

Suppose that $(\hat{x}, \hat{y}, \hat{u})$ is an optimal solution to problem DDS with $\sum_{j \in N} \hat{u}_j \geq 2$. Then $(\hat{x}, \hat{y}, \hat{u})$ satisfies

$$\sum_{j \in N} \hat{u}_j \leq K, \tag{9}$$

where $K = \min\{n, \max(1, m-2)\}$.

Proof

The number of edges in the solution tree is

$$m + \sum_{j \in N} \hat{u}_j - 1 = \left\{ \sum_{j \in N} \left(\sum_{i \in M} \hat{x}_{ij} + \sum_{l<j, l \in N} \hat{y}_{lj} + \sum_{j<k, k \in N} \hat{y}_{jk} \right) + m \right\} / 2.$$

Now, from (8) in proposition 2, we have that

$$m + \sum_{j \in N} \hat{u}_j - 1 \geq \left\{ \sum_{j \in N} 3\hat{u}_j + m \right\} / 2 .$$

It follows that $\sum_{j \in N} \hat{u}_j \leq m - 2$. This completes the proof. □

Motivated by the observation that the solution to problem DDS has a tree structure, we develop an alternative formulation with additional interesting features. We conceive of a flow that enters a single hub from an artificial hub of supply $\sum_{j \in N} u_j$, denoted by hub 0, and seek to send the flow to other appropriate hubs to satisfy the connectivity conditions. Toward this end, define f_{jk} to be a directed flow from hub j to hub k, $j \neq k$, and introduce an artificial hub which must send the flow. With these flow variables, we obtain the following formulation for problem DDS.

DDS2: minimize $\sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} \sum_{j<k, k \in N} d_{jk} y_{jk} + \sum_{j \in N} b u_j$

subject to

$$\sum_{j \in N} x_{ij} = 1, \quad i \in M, \tag{10}$$

$$\sum_{i \in M} x_{ij} \leq m u_j, \quad j \in N, \tag{11}$$

$$\sum_{j \in N} f_{0j} = \sum_{j \in N} u_j, \tag{12}$$

$$f_{0j} - \sum_{j \neq k, k \in N} (f_{jk} - f_{kj}) = u_j, \quad j \in N, \tag{13}$$

$$f_{0j} \leq n y_{0j}, \quad j \in N, \tag{14}$$

$$f_{jk} \leq n y_{jk}, \quad j, k \in N,\ j < k, \tag{15}$$

$$f_{kj} \leq n y_{jk}, \quad j, k \in N,\ j < k, \tag{16}$$

$$\sum_{j \in N} y_{0j} = 1, \tag{17}$$

$$x_{ij} \geq 0, \quad i \in M,\ j \in N,$$

$$f_{jk} \geq 0, \quad j \neq k,\ j \in N \cup \{0\},\ k \in N,$$

$$u_j \in \{0,1\}, \quad j \in N,$$

$$y_{jk} \in \{0,1\}, \quad j < k,\ j \in N \cup \{0\},\ k \in N.$$

Remark 2

The constraints (12)–(17) guarantee the connectivity of the solution because the variable f_{jk} provides the flow between the active hubs.

We now compare the relative strength of the foregoing two models. Toward this end, let $\overline{P}$ denote the linear programming (LP) relaxation of model P and let $v(P)$ denote the optimal objective function value of a given model P. Then, we have the following result.

PROPOSITION 4

There exists an optimal solution $(\bar{x}, \bar{y}, \bar{f}, \bar{u})$ to $\overline{DDS2}$ of the form

$$\bar{x}_{ij} = \begin{cases} 1, & \text{if } j = t, \ \ t = \arg\min\{c_{ij} : j \in N\} \ \ i \in M, \\ 0, & \text{otherwise,} \end{cases}$$

$$\bar{y}_{jk} \equiv 0, \qquad j < k, \ j, k \in N,$$

$$\bar{f}_{jk} \equiv 0, \qquad j \neq k, \ j, k \in N,$$

$$\bar{u}_j = \sum_{i \in M} \bar{x}_{ij} / m, \quad j \in N,$$

$$\bar{f}_{0j} = \sum_{i \in M} \bar{x}_{ij} / m, \quad j \in N,$$

$$\bar{y}_{0j} = \sum_{i \in M} \bar{x}_{ij} / m, \quad j \in N.$$

Proof

Suppose that there exists an optimal solution to $\overline{DDS2}$ such that $\bar{y}_{jk} > 0$ for some $j < k$, $j, k \in N$. Then, without losing feasibility, we can put $\bar{y}_{jk} = 0$ for all $j < k$, $j, k \in N$, which in turn reduces the total cost. This contradicts the fact that $(\bar{x}, \bar{y}, \bar{f}, \bar{u})$optimal to $\overline{DDS2}$. Moreover, from constraints (15) and (16), we have that $\bar{f}_{jk} \equiv 0, \forall j \neq k, j, k \in N$, and the rest of the result follows by solving the following trivial linear programming problem.

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} b u_j \\ \text{subject to} \quad & \sum_{j \in N} x_{ij} = 1, \qquad i \in M, \\ & \sum_{i \in M} x_{ij} \le m u_j, \quad j \in N, \\ & x_{ij} \ge 0, \qquad i \in M, \ \ j \in N, \\ & u_j \ge 0, \qquad j \in N. \end{aligned}$$

Letting $(\bar{x}, \bar{u})$ be the optimal solution of the above linear programming problem, we have that $\bar{f}_{0j} = \sum_{i \in M} \bar{x}_{ij}/m$, $j \in N$ and $\bar{y}_{0j} = \sum_{i \in M} \bar{x}_{ij}/m$, $j \in N$. □

Note that if some $\bar{u}_j$ of an optimal solution $(\bar{x}, \bar{y}, \bar{f}, \bar{u})$ to $\overline{DDS2}$ is fractional, then $(\bar{x}, \bar{y}, \bar{f}, \bar{u})$ violates the constraints (2) of $\overline{DDS1}$. That is, the constraints of $\overline{DDS1}$ cut off the optimal solution of $\overline{DDS2}$. Furthermore, it can easily be shown that there always exists a feasible solution to $\overline{DDS2}$ corresponding to a given feasible solution to $\overline{DDS1}$. Hence, DDS1 is a stronger formulation in the sense that DDS1 generates a tighter LP relaxation and so provides a tighter lower bound for problem DDS.

PROPOSITION 5

$$v(\overline{DDS1}) \geq v(\overline{DDS2}).$$

Glover [13] and McCrady [16] developed an alternative formulation based on a different flow concept. We present the model in order to compare the relative strength of models and select the best model for implementation. Toward this end, we can define the value $\sum_{i \in M} x_{ij}$ to be flow that enters hub j from the nodes $i \in M$, and can transmit this flow through other hubs to satisfy the desired connectivity conditions. For this, we introduce a master sink, which must receive flow from a single hub. Accordingly, let f_{j0} denote the flow from hub j to the master sink hub, denoted by hub 0, and let y_{j0} be a 0-1 variable where $y_{j0} = 1$ if and only if the flow f_{j0} is positive. Then, we have the following formulation.

DDS3:

$$\text{minimize} \quad \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} \sum_{j<k, k \in N} d_{jk} y_{jk} + b \left(\sum_{j \in N} \sum_{j<k, k \in N} y_{jk} + 1 \right)$$

$$\text{subject to} \quad \sum_{j \in N} x_{ij} = 1, \qquad i \in M,$$

$$\sum_{i \in M} x_{ij} + \sum_{j \neq k, k \in N} (f_{kj} - f_{jk}) - f_{j0} = 0, \qquad j \in N,$$

$$f_{jk} \leq m y_{jk}, \qquad j < k, \ j, k \in N,$$

$$f_{kj} \leq m y_{jk}, \qquad j < k, \ j, k \in N,$$

$$\sum_{j \in N} f_{j0} = m,$$

$$f_{j0} = m y_{j0}, \qquad j \in N,$$

$$x_{ij} \geq 0, \qquad i \in M, \ j \in N,$$

$$f_{jk} \geq 0, \qquad j \neq k, \ j \in N, \ k \in N \cup \{0\},$$

$$y_{jk} \in \{0, 1\}, \qquad j \neq k, \ j \in N, \ k \in N \cup \{0\}.$$

Remark 3

Note that model DDS3 reveals that node-associated variables u_j, $j \in N$, are not necessary for formulating problem DDS. Also note that model DDS3 is a *compact* formulation in the sense that model DDS3 is a polynomial size formation, while model DDS1 requires an exponential number of "anti-cycle" constraints (6). Although model DDS2 is also a polynomial size formulation with "superfluous" variables u_j, $j \in N$, model DDS3 contains only the necessary variables. However, as in the case of the symmetric traveling salesman problem, the compact formulation is not necessarily a strong formulation that provides a tight lower bound to the problem. The following proposition shows that the LP relaxation of model DDS3 provides the same (weak) lower bound as model DDS2 to the problem DDS.

PROPOSITION 6

$$v(\overline{DDS2}) = v(\overline{DDS3}).$$

Proof

Let $(\bar{x}, \bar{y}, \bar{f}, \bar{u})$ be an optimal solution to $\overline{DDS3}$. In the same way as for proposition 3, it is readily verified that the optimal solution of $\overline{DDS3}$ is of the form

$$\bar{x}_{ij} = \begin{cases} 1, & \text{if } j = t, \quad t = \arg\min\{c_{ij} : j \in N\} \quad i \in M, \\ 0, & \text{otherwise,} \end{cases}$$

$$\bar{y}_{jk} \equiv 0, \qquad j < k, \quad j, k \in N,$$

$$\bar{f}_{jk} \equiv 0, \qquad j \neq k, \quad j, k \in N,$$

$$\bar{f}_{j0} = \sum_{i \in M} \bar{x}_{ij}, \qquad j \in N,$$

$$\bar{y}_{j0} = \sum_{i \in M} \bar{x}_{ij} / m, \quad j \in N.$$

This completes the proof. □

The foregoing propositions provide a foundation for generating a strong formulation susceptible to solution by available commercial branch-and-bound codes. In particular, the constraints (2)–(5) can be incorporated into the model DDS2 as cutting planes. Also note that an inequality of the form $y_{0j} \leq u_j$, $j \in N$, is also a valid inequality to the problem DDS. Furthermore, by proposition 3, constraint (14) can be tightened as $f_{0j} \leq K y_{0j}$, and the constraints (15) and (16) can be tightened as follows:

$$f_{jk} + f_{kj} \leq (K - 1) y_{jk}, \quad j < k, \quad j, k \in N.$$

Accordingly, we can obtain another equivalent formulation for the problem DDS by including all of these inequalities in model DDS2. Let us denote this model by model DDS4.

$$
\begin{aligned}
\text{DDS4:}\quad & \text{minimize} && \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} + \sum_{j \in N} \sum_{j<k, k \in N} d_{jk} y_{jk} + \sum_{j \in N} b u_j \\
& \text{subject to} && \sum_{j \in N} x_{ij} = 1, && i \in M, \\
& && x_{ij} \le u_j, && i \in M,\ j \in N, \\
& && y_{jk} \le u_j, && j < k,\ j,k \in N, \\
& && y_{jk} \le u_k, && j < k,\ j,k \in N, \\
& && y_{0j} \le u_j, && j \in N, \\
& && \sum_{j \in N} \sum_{j<k, k \in N} y_{jk} = \sum_{j \in N} u_j - 1, \\
& && \sum_{j \in N} f_{0j} = \sum_{j \in N} u_j, \\
& && f_{0j} - \sum_{k \ne j, k \in N} (f_{jk} - f_{kj}) = u_j, && j \in N, \\
& && f_{0j} \le k y_{0j}, && j \in N, \\
& && f_{jk} + f_{kj} \le (K-1) y_{jk}, && j < k,\ j,k \in N, \\
& && \sum_{j \in N} y_{0j} = 1, \\
& && x_{ij} \ge 0, && i \in M,\ j \in N, \\
& && f_{jk} \ge 0, && j \ne k,\ j \in N \cup \{0\},\ k \in N, \\
& && u_j \in \{0,1\}, && j \in N, \\
& && y_{jk} \in \{0,1\}, && j < k,\ j \in N \cup \{0\},\ k \in N.
\end{aligned}
$$

Remark 4

It can easily be proved that if $\sum_{j \in N} u_j \ge 3$, then there exists an optimal solution to the problem DDS which satisfies the following inequality:

$$\sum_{l<j, l \in N} y_{lj} + \sum_{j<k, k \in N} y_{jk} \ge 2 y_{0j}, \quad j \in N. \tag{18}$$

Note that constraints (7), (8), (9) and (18) can be utilized to strengthen the formulation for the case when $\sum_{j \in N} u_j \ge 3$. This observation motivates the following two-step solution procedure for problem DDS.

Step 1 {Enumeration Phase}: Find the minimum solution by enumerating the feasible solution of problem DDS for the cases where $\sum_{j \in N} u_j = 1$ and $\sum_{j \in N} u_j = 2$.

Step 2 {Optimization Phase}: Solve the augmented model, called DDS4E, which is obtained by adding the constraints (7), (8), (9) and (18) to model DDS4. Then, compare the solutions of step 1 and step 2, and so generate the optimal solution to problem DDS.

3. Computational results

In this section, we report on our computational results using a set of suitable real-world test problems in order to evaluate the relative strength of the models developed in section 2. Toward this end, we examine three models, DDS2, DDS4 and DDS4E, over ten real-data problems. We use the CPLEX/MIP package [7] to solve the LP relaxation of these models and to find the optimal solution of the test problems. All the reported computation times include model generation time and run time on a SUN SPARC station 2. In all the test problems, the fixed charge cost assosiated with each hub is $b = \$41.00$, and the connection cost consists of two parts, a fixed cost and a unit cost per mile, as shown in table 1.

Table 1

Connection cost of test problems.

Mileage	Fixed cost ($)	Unit cost per mile ($)
0	30.00	0.00
1–15	125.00	1.20
16–1000	130.00	1.50

The number of hubs in the problem is either $n = 10$ or $n = 20$, while the numbers of target nodes are $m = 10$, 15, 20, 25 and 30, respectively. Table 2 shows the problem sizes corresponding to the foregoing three models.

In the computational experiments, we examine the CPU time and also the tightness of the LP solution (table 3) measured by the ratio of $v(\overline{P})/v(P)$, which is the relative gap between the optimal LP solution and the optimal integer solution. Table 3 shows the superiority of DDS4E over DDS4 and DDS2 in providing a tight lower bound for the test problem. In all the test problems, the optimal solution has at least three active hubs. The CPU time for the enumeration phase in the two-step procedure using model DDS4E is less than 0.5 second in all the cases. The results indicate that DDS4E generates a much tighter lower bound (LP solution), solution bounds (always over 98%), and runs much faster than DDS4, especially for large size problems, while model DDS2 is practically unsolvable for large size problems such as problems 3–10. In particular, it also shows that valid inequalities and cutting planes (7), (8), (9) and (18) in model DDS4E play a key role in providing effectively tight lower bounds for the test problems.

Table 2

Problem sizes of test problems.

Problem (m, n)	DDS2			DDS4			DDS4E		
	Int. var.	Cont. var.	Constr.	Int. var.	Cont. var.	Constr.	Int. var.	Cont. var.	Constr.
Problem 1 (10,10)	65	200	132	65	200	278	65	200	309
Problem 2 (15,10)	65	250	137	65	250	333	65	250	364
Problem 3 (20,10)	65	300	142	65	300	388	65	300	429
Problem 4 (25,10)	65	350	147	65	350	443	65	350	474
Problem 5 (30,10)	65	400	152	65	400	498	65	400	529
Problem 6 (10,20)	230	600	452	230	600	843	230	600	904
Problem 7 (15,20)	230	700	457	230	700	948	230	700	1009
Problem 8 (20,20)	230	800	462	230	800	1053	230	800	1114
Problem 9 (25,20)	230	900	467	230	900	1158	230	900	1219
Problem 10 (30,20)	230	1000	472	230	1000	1263	230	1000	1324

Table 3

Computational results of test problems.

Problem (m, n)	DDS2		DDS4		DDS4E	
	Ratio (%)	CPU time (min)	Ratio (%)	CPU time (min)	Ratio (%)	CPU time (min)
Problem 1 (10,10)	67.2	964.4	86.6	0.7	98.6	0.04
Problem 2 (15,10)	72.3	4853.4	89.1	1.4	98.5	0.1
Problem 3 (20,10)	77.5	$\gg$ 14000	91.6	1.9	98.3	0.2
Problem 4 (25,10)	79.8	$\gg$ 14000	92.9	4.3	98.7	0.3
Problem 5 (30,10)	80.0	$\gg$ 14000	93.6	13.7	98.7	0.5
Problem 6 (10,20)	65.4	$\gg$ 40000	85.1	100.8	98.3	0.6
Problem 7 (15,20)	69.5	$\gg$ 40000	88.4	16.8	98.3	0.7
Problem 8 (20,20)	70.4	$\gg$ 40000	91.1	357.1	98.3	1.1
Problem 9 (25,20)	71.9	$\gg$ 40000	93.7	405.9	98.3	1.3
Problem 10 (30,20)	70.2	$\gg$ 40000	93.1	30415.0	98.5	7.0

4. Concluding remarks

This paper has presented several mixed zero-one integer programming formulations for a digital data service network design problem (DDS) and has incorporated some classes of valid inequalities and cutting planes for strengthening these formulations. Especially the strong formulation DDS4E generates a very tight lower bound and so finds an optimal solution within an effort that is well within acceptable time standards for practitioners. In particular, the lower bounds obtained by solving LP relaxation of the model DDS4E are over 98% of the optimality for the test problems. These strong formulations can be naturally extended for cases where the fixed charge is negative and/or depends on the location of the hubs. The facial structure (facets and strong valid inequalities) of the problem DDS appears to be a viable candidate to be further studied in the framework of a suitable branch-and-cut procedure.

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