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ON THE TORSION THEORIES OF MORITA EQUIVALENT RINGS

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Abstract

We generalize the well-known fact that for a pair of Morita equivalent rings R and S their maximal rings of quotients are again Morita equivalent: If $\tau_n(M)$ denotes the torsion theory cogenerated by the direct sum of the first $n+1$ injective modules forming part of the minimal injective resolution of M then $\alpha\tau_n(R) = \tau_n(S)$ where α is the category equivalence R-Mod $\to S$-Mod. Consequently the localized rings $R_{\tau_n(R)}$ and $S_{\tau_n(S)}$ are Morita equivalent.

1. Rings are associative with unit elements, and modules will be unitary. Consider a pair of rings R and S, which are Morita equivalent. It is well-known that the maximal left rings of quotients of R and S are again Morita equivalent ([11], Chapter X, Proposition 3.2). The maximal left ring of quotients is obtained from the family of dense left ideals. This is the Gabriel topology corresponding to the Lambek torsion theory, cogenerated by the injective envelope of the ring, considered as a left module over itself. The Lambek torsion theory is one of a sequence of hereditary torsion theories defined below. Let A be any ring and consider the minimal injective resolution of ${}_AA$:

$$0 \longrightarrow A \longrightarrow E_0 \xrightarrow{\mu_0} E_1 \xrightarrow{\mu_1} E_2 \xrightarrow{\mu_2} \cdots$$

where $E_0 = E(A)$ and each E_{i+1} is the injective envelope of Coker μ_i. The Gabriel topology corresponding to the hereditary torsion theory $\tau_n(A)$ cogenerated by the injective module $E_0 \oplus \cdots \oplus E_n$ is denoted by F_A^n. Thus in particular F_A^0 is the dense topology, and $E(A) \simeq Q_{\max}(A) = \varinjlim \mathrm{Hom}(I, A)$, $I \in F_A^0$. Now assume that R and S are two Morita equivalent rings, with bimodules ${}_RV_S$ and ${}_SW_R$ giving the equivalence of categories:

$$\alpha = W \otimes_R : R\text{-Mod} \to S\text{-Mod} \text{ and } \beta = V \otimes_S : S\text{-Mod} \to R\text{-Mod}.$$

Given any hereditary torsion theory $\tau = (\underline{T}, \underline{F})$ for R-Mod, one may use the functor α to obtain

$$\alpha\tau = \big(\alpha(\underline{T}), \alpha(\underline{F})\big),$$

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a hereditary torsion theory for S-Mod. The fact that $Q_{\max}(R)$ is Morita equivalent to $Q_{\max}(S)$ is really a consequence of the result that the functor α carries the Lambek torsion theory of R onto the Lambek torsion theory of S, that is, $\alpha\tau_0(R) = \tau_0(S)$. The purpose of this note is to investigate $\alpha\tau_n(R)$ in relation to $\tau_n(S)$. We shall establish that these theories do coincide for all $n \geq 0$. For a general hereditary torsion theory τ on R-Mod some properties such as stability or being prime, preserved by the functor α will be established.

It is not out of place to remark that more generally, questions on the transfer of properties from a ring A to a ring B where A and B are in a Morita context have been extensively considered by several authors, e.g. [3], [4], [5], [6], [8] and [10], to mention but a few. More recently [2] and [9] contain information on the torsion theories of context equivalent rings. For unexplained terminology and standard results on torsion theory see [1] and [11].

2. Throughout this note R and S are Morita equivalent rings. Thus there exists a progenerator W_R such that $S \simeq \operatorname{End} W_R$. We recall that ${}_SW$ is also a progenerator with $R \simeq \operatorname{End}_S W$, and that $W \otimes_R W^* \simeq S$, $W^* \otimes_S W \simeq R$ where $*$ denotes the dual module. Put $V = (W_R)^*$; then we have $V \simeq ({}_SW)^*$ and $W \simeq (V_S)^*$ both as bimodules. Finally the functor $W \otimes_R -$ will simply be denoted by α, and if $\tau = (\underline{T}, \underline{F})$ is any torsion theory on R-mod then clearly $\alpha\tau := (\alpha(\underline{T}), \alpha(\underline{F}))$ is a torsion theory on S-mod.

We shall need the following, well-known result:

LEMMA 2.1. *Let A be any ring and L any left A-module. Then L is $\tau_n(A)$-torsion if and only if $\operatorname{Ext}^i_A(L', A) = 0$ for all $i \leq n$ and all (cyclic) submodules L' of L.*

PROOF. [11], Chapter VI, Proposition 6.9. ■

THEOREM 2.2. *The functor α carries $\tau_n(R)$ onto $\tau_n(S)$, that is $\alpha\tau_n(R) = \tau_n(S)$.*

PROOF. Fix n and for simplicity write σ for $\alpha\tau_n(R)$. Let ${}_RN$ be $\tau_n(R)$-torsion. Since α is an equivalence of categories, a (cyclic) submodule of $M = W \otimes_R N$ is of the form $W \otimes N'$ where we can assume that N' is a submodule of N. Because W_R is projective the first isomorphism below holds:

$$\operatorname{Ext}^i_S(W \otimes N', S) \simeq \operatorname{Ext}^i_R(N', \operatorname{Hom}_S(W, S)) \simeq \operatorname{Ext}^i_R(N', V).$$

Since ${}_RV$ is finitely generated projective, there is a natural number t such that $R^t \simeq V \otimes V'$ for some suitable left R-module V'. But using the above Lemma, we have for $i \leq n$:

$$0 = \sum \oplus \operatorname{Ext}^i_R(N', R) \simeq \operatorname{Ext}^i_R(N', R^t) \simeq \operatorname{Ext}^i_R(N', V) \oplus \operatorname{Ext}^i_R(N', V')$$

It follows that $\operatorname{Ext}^i_S(W \otimes N', S) = 0$ for all $i \leq n$ and all submodules of $W \otimes N$ and so $W \otimes N$ is $\tau_n(S)$-torsion. This means that $\sigma \leq \tau_n(S)$. Conversely, let ${}_SM$ be $\tau_n(S)$-torsion. Writing $M = W \otimes_R N$ for suitable ${}_RN$, we have

$$\operatorname{Ext}^i_R(N', V) \simeq \operatorname{Ext}^i_S(W \otimes N', S) = 0$$

for all $i \leq n$ and all submodules N' of N. Now using the fact that ${}_RV$ is a progenerator, we write $V = R^s \otimes K$ for suitable integer s and module ${}_RK$, to deduce that $\mathrm{Ext}^i_R(N', R) = 0$. Therefore N is $\tau_n(R)$-torsion, hence M is σ-torsion. It follows that $\sigma = \tau_n(S)$. ∎

COROLLARY 2.3. *For any $n = 0, 1, 2, \ldots$ the rings $R_{\tau_n(R)}$ and $S_{\tau_n(S)}$ are Morita equivalent.*

Next, we shall use the following criterion of reflexivity. Let A be a prime Goldie ring with $Q(A)$ its total ring of quotiens. If I is an essential left ideal then I is reflexive if and only if A/I can be embedded in a direct product of copies of $Q(A)/A$, that is, if and only if R/I is $\tau_1(A)$-torsion free.

COROLLARY 2.4. *Let R be a prime Goldie ring, I a two sided ideal in R such that ${}_RI$ is reflexive. Then the corresponding ideal WIV of S is left reflexive.*

PROOF. It is well-known that S is also prime Goldie ([7], 3.5.10). Now it is easy to see that if M is an R-bimodule such that ${}_RM$ is τ-torsion (respectively τ-torsionfree), then $W \otimes M \otimes V$ is $\alpha\tau$-torsion (respectively $\alpha\tau$-torsionfree), where τ is any hereditary torsion theory on R-Mod. Taking $\tau = \tau_1(R)$, the fact that ${}_RI$ is reflexive is equivalent to R/I being τ-torsionfree. Hence $W \otimes R/I \otimes V$ is $\alpha\tau$-torsionfree, and by Theorem 2.2, $\alpha\tau = \tau_1(S)$. It follows that WIV is reflexive as a left S-module. ∎

3. In this final section we collect some properties of torsion theories which are preserved under the equivalence α. For the definition and properties of prime torsion theories see [1].

THEOREM 3.1. *Let τ be an arbitrary hereditary torsion theory on R-*Mod.

(1) *If R is a prime Goldie ring and τ is the Goldie torsion theory on R-*Mod *then $\alpha\tau$ is the Goldie torsion theory on S-*Mod.
(2) *If the Gabriel topology corresponding to τ has a basis consisting of an ideal then the same is true for $\alpha\tau$.*
(3) *If τ is stable then so is $\alpha\tau$.*
(4) *Let $G(R)$ stand for the Goldie torsion theory on R-*Mod. *If $G(R) \leq \tau$ and τ is prime then $\alpha\tau$ is also prime.*

PROOF. (1) Observe that in the present situation, the Goldie and Lambek torsion theories are the same, and $\alpha\tau_0(R) = \tau_0(S)$.

(2) Since W_R is finitely generated projective, the functor $\alpha = W \otimes_R$- commutes with direct products. Now the hypothesis stated on τ is equivalent to the τ-torsion class being closed under direct products (see [11]). Thus if $W \otimes_R M_i$, where $i \in I$, are in the torsion class of $\alpha\tau$ then from $\prod(W \otimes M_i) \simeq W \otimes \prod M_i$, we deduce that $\prod(W \otimes M_i)$ is in the torsion class of $\alpha\tau$.

(3) According to [1], Chapter III, 11.2, for $\alpha\tau$ to be stable it is necessary and sufficient that the $\alpha\tau$-torsion submodule of any injective left S-module M be a direct summand. Let N be the $\alpha\tau$-torsion submodule of M and write $M = W \otimes M'$,

$N = W \otimes N'$ for some suitable left R-modules $N' \subseteq M'$. Now

$$M/N \simeq (W \otimes M')/(W \otimes N') \simeq W \otimes (M'/N')$$

is $\alpha\tau$-torsionfree. Therefore N' is the τ-torsion submodule of M'. But M' is R-injective, hence from the assumption that τ is stable, we deduce that N' is a direct summand of M'. It follows that N is a direct summand of M

(4) From the assumption that τ is prime, there exists a cocritical module whose injective envelope E cogenerates τ. Since any category equivalence preserves injective objects and minimal injective resolutions it is clear that $\alpha\tau$ is cogenerated by the injective module $W \otimes E$. If we show that $W \otimes E$ is $\alpha\tau$-cocritical then $\alpha\tau$ will be prime. Let N be a nonzero submodule of $W \otimes E$, and choose $X \subset E$ such that $N = W \otimes X$. Thus $W \otimes E/W \otimes X \simeq W \otimes (E/X)$. Since a cocritical module is uniform, its injective envelope is indecomposable, hence E, being indecomposable, is also the injective envelope of X. But then by [1], Chapter II, 7.4, E/X is τ-torsion, from which we have that $W \otimes (E/X)$ is $\alpha\tau$-torsion. Therefore $W \otimes E$ is $\alpha\tau$-cocritical, as desired. ■

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