Journal of the Oceanographical Society of Japan
Vol. 30, pp. 271 to 281, 1974

Effect of Stratification on Long Period Trapped Waves on the Shelf*

Kinjiro KAJIURA**

Abstract: The effect of stratification on very long-period waves trapped on a straight continental shelf of constant depth is examined for a two-layer model. There are 4 modes in this system. The characteristics of the mode with the largest phase velocity can be approximated by the barotropic mode. The mode corresponding to the barotropic shelf-wave mode is modified by the baroclinic motions significantly, and in the limit of very narrow shelf width, the mode characteristics are transformed from those of the barotropic shelf-wave to the baroclinic Kelvin wave if the long-shore wave length is larger than the internal deformation radius. In this case, the stratification has an apparent effect of increasing phase velocity of barotropic shelf-waves. The remaining two modes are dominated by baroclinic motions with significant contribution from barotropic motions: among which the one has a shelf-wave characteristics for small values of the shelf width and approaches the mode corresponding to the baroclinic Kelvin wave in shallower water for large shelf width and the other is a stationary mode. If the long-shore wave length is much shorter than the internal deformation radius, the motions in the upper and lower layers are decoupled: the surface and bottom modes analogous to those discussed by RHINES (1970) appears.

If the interface is deeper than the shelf depth, the stationary mode is absent and the characteristics of the third mode approaches those of the baroclinic double Kelvin wave mode as the shelf width increases.

1. Introduction

In 1971 the abnormal rise of sea level along the Pacific coasts of Japan was observed after the passage of a typhoon (YOSHIDA, SHOJI, and Masuzawa, 1972). Events of this kind were not uncommon (ISOZAKI, 1972) and the subsurface water temperature offshore often showed a marked change accompanying the sea level disturbance (SHOJI, private communication). Since the sea level disturbances moved slowly to the west with the phase speed of several meters per second, the generation and propagation of a continental shelf-wave (ROBINSON, 1964) were considered to have played an important role. Numerical model experiments indicated that the continental shelf-wave was indeed generated by a moving typhoon (ENDOH, 1973). Even in a two layer model, the generated shelf-wave was hardly affected by the stratification, although the disturbance of the interface remained for a long time behind the shelf-wave (SUGINOHARA, 1973).

As to the modification of shelf-wave characteristics by the density stratification, MYSAK (1967) showed that the density stratification in deeper water increases the phase velocity of continental shelf-waves significantly. On the other hand, before the concept of "shelf-wave" was advanced, there had been an attempt to explain coastal sea level disturbances, moving slowly clockwise around the Japanese Islands (SHOJI, 1961) in terms of internal Kelvin waves (YOSHIDA, 1960; KAJIURA, 1962) with the emphasis in the baroclinic motions generated by the long-shore wind stress close to the coast. From somewhat different point of view, RHINES (1970) discussed the mode waves on a sloping bottom in a stably stratified ocean and indicated the importance of a non-dimensional parameter B for low frequency waves: $(B=ND/(fL)$;

* Received July 5, 1974
** Earthquake Research Institute, University of Tokyo, Bunkyo-ku, Tokyo, 113 Japan

N, Väisälä frequency; f, Coriolis parameter; D, vertical scale; L, horizontal scale). The behavior of waves with small B is barotropic and the baroclinic motion becomes important for large B. In a two layer model of the ocean, the parameter B can be interpreted as the ratio of the Rossby's internal deformation radius to the horizontal scale of the wave. Therefore, the explanation of the slowly moving sea level disturbances in terms of the internal Kelvin wave may be valid only when the width of the varying depth near the coast is narrower than the internal deformation radius.

In the present paper, a simple analysis is made on the mode characteristics of trapped waves in a two layer ocean with a shelf in order to clarify the relative importance of the barotropic and baroclinic motions in relation to the ratios of various length scales relevant in the problem. The depth on both sides of the shelf break is assumed constant and the interface is assumed either shallower or deeper than the shelf depth. Thus, in this model, the coupling of the barotropic and baroclinic motions is possible only at the shelf break. There are 4(3) modes of trapped waves if the interface is shallower (deeper) than the shelf depth, and the characteristics of slowly moving shelf waves are expected to be modified significantly.

2. Formulation of the eigen-value problem

Let us consider a straight shelf with the step type bottom topography and take the right-handed Cartesian co-ordinates as shown in Figure 1: the x-axis toward the open ocean with the origin at the shelf break and the z-axis positive upwards. The width and depth of the shelf is L and D, respectively. Assuming a two-layer fluid model, the quantities in the upper and lower homogeneous layers are distinguished by subscripts 1 and 2, respectively. Thus, D_1 and D_2 are the layer thicknesses on the shelf. In the later discussions, the subscripts 1 and 2 are also used to denote the quantities related to barotropic and baroclinic motions, respectively. The quantities in deeper water are expressed by putting "dash" on the right shoulder. The relative density difference μ in two layers is assumed very small ($\mu=$

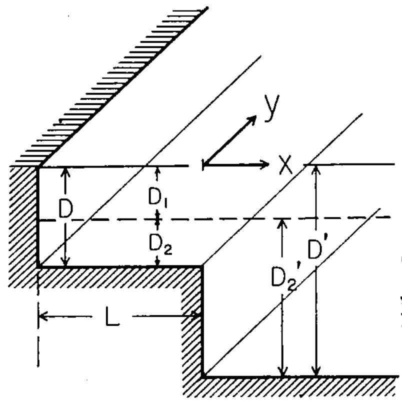

Fig. 1. Shelf topography and the co-ordinate system.

$(\rho_2-\rho_1)/\rho_2 \simeq 2\times10^{-3}$: ρ is the density).

The motion is assumed hydrostatic and non-dissipative. Then, under the usual linearizing assumptions, the motions of the barotropic and baroclinic modes are independent in the stably stratified ocean with constant depth. They are governed by the equation of the same type:

$$(\partial^2/\partial t^2+f^2+c_i^2\nabla^2)R_i=0 \qquad (2.1)$$

where t is time, ∇^2 is the two-dimensional (horizontal) Laplacian, R_i is the representative vertical displacement of each mode (i), c_i is the representative velocity. For a two-layer fluid model, $i=1$ or 2 and within the error of $O(\mu)$ we have

$$c_1^2=gD, \quad c_2^2=\mu g D_1 D_2/D \qquad (2.2)$$

with g the acceleration due to gravity.

The x and y components (U_i, V_i) of the vertual volume transport vector $\boldsymbol{U}_i$ corresponding to R_i satisfy the following equations:

$$(\partial^2/\partial t^2+f^2)U_i=-c_i^2(\partial^2/\partial t\partial x+f\partial/\partial y)R_i \qquad (2.3a)$$

$$(\partial^2/\partial t^2+f^2)V_i=-c_i^2(\partial^2/\partial t\partial y-f\partial/\partial x)R_i \qquad (2.3b)$$

The actual surface and interface displacements ζ_1, ζ_2 can be expressed by

$$\zeta_1=R_1+\mu(D_2/D)^2R_2 \qquad (2.4a)$$

$$\zeta_2=(D_2/D)(R_1-R_2) \qquad (2.4b)$$

and the volume transport vector $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ in the upper and lower layers are:

$$\boldsymbol{Q}_1=(D_1/D)\boldsymbol{U}_1+(D_2/D)\boldsymbol{U}_2 \quad (2.5a)$$

$$\boldsymbol{Q}_2=(D_2/D)(\boldsymbol{U}_1-\boldsymbol{U}_2) \quad (2.5b)$$

Exactly the same relationships from (2.1) to (2.5b) hold in deep water, provided that all relevant quantities are dashed.

At the coastal boundary, no volume flux across the boundary exists. In terms of mode solutions, this condition requires

$$U_i=0 \quad \text{at} \quad x=-L \quad (2.6)$$

The offshore boundary condition for very long distances from the shelf is

$$R_i' \to 0 \quad \text{at} \quad x \to \infty \quad (2.7)$$

since we are concerned with trapped waves only. The boundary conditions at the shelf-edge are the continuity of vertical displacements at the surface and at the interface, together with the continuity of the volume fluxes across the shelf break in the upper and the lower layers, respectively. In terms of mode solutions with $O(\mu)$ neglected compared with 1, these conditions can be transformed to

$$R_1'=R_1-\mu rr'(1-r/r')R_2 \quad (2.8)$$

$$R_2'=(1-r/r')R_1+(r/r')R_2 \quad (2.9)$$

and

$$U_1'=U_1 \quad (2.10)$$

$$U_2'=(1-r/r')U_1+(r/r')U_2 \quad (2.11)$$

where $r=D_2/D$, $r'=D_2'/D'$.

Assuming the sinusoidal wave motion traveling along the shelf

$$R_i=Z_i(x)\exp[i(my+\omega t)] \quad (2.12)$$

with m the positive wave number in the y-direction and ω the angular frequency, (2.1) is transformed into

$$d^2Z_i/dx^2-K_i^2Z_i=0 \quad (2.13)$$

with

$$K_i^2=(f^2-\omega^2)/c_i^2+m^2 \quad (2.14)$$

Since we are concerned with the slowly moving long-period wave only, the wave period $(2\pi/\omega)$ is assumed considerably larger than the inertial period $(2\pi/f)$ and $(\omega/f)^2$ is neglected compared with unity. Thus, K_i is independent of ω. It is mentioned that for $(mc_i/f)^2 \ll 1$, K_i is the reciprocal of the Rossby's deformation radius $L_{Ri}(=c_i/f)$.

Now taking real and positive K_i and K_i', we write the solutions to (2.13) as follows:

$$Z_i=A_i(\sinh K_ix+\alpha_i\cosh K_ix) \quad (2.15)$$

$$Z_i'=B_i\exp[-K_i'x] \quad (2.16)$$

(2.16) automatically satisfies the condition (2.7), and the condition (2.6) requires formally that

$$(X_iT_i-1)\alpha_i=(X_i-T_i) \quad (2.17)$$

where

$$X_i=(\omega K_i)/(fm), \quad T_i=\mathrm{Tanh}(K_iL) \quad (2.18)$$

From (2.15) and (2.16), it is possible to classify the wave form in the x-direction, Z_i, according to the values of α_i. For $\alpha_i<0$, $|Z_i|$ has the maximum at the coastal boundary and decreases offshore somewhat like the Kelvin wave. For $\alpha_i=0$, Z_i vanishes at the shell break so that Z_i' is always zero. On the other hand, for $\alpha_i>0$, $|Z_i|$ has local maxima at the coastal boundary and at the shelf break. In particular, for $0<\alpha_i<1$, Z_i changes sign on the shelf (shelf wave type) and, for $\alpha_i>1$, Z_i keeps the same sign throughout (double peak type). The case when $\alpha_i=1$ corresponds to a so-called "double Kelvin wave" for which the vertical displacement decreases exponentially on both sides of the shelf break.

Since α_i is determined by (2.17), the wave form can be found from the relative magnitude of X_i and T_i as follows:

double peak type ($\alpha_i>1$) for $X_i>1/T_i$

Kelvin wave type ($\alpha_i<0$) for $1/T_i>X_i>T_i$

shelf wave type ($0<\alpha_i<1$) for $T_i>X_i$

double Kelvin wave type ($\alpha_i=1$) for $T_i=1$ and $X_i\neq 1$

For the case when $T_i = X_i = 1$, α_i is indeterminate in (2.17). However, this case corresponds to the ordinary Kelvin waves in shallower water.

The boundary conditions from (2.8) to (2.11) at the shelf break give after some manipulations:

$$a_{11}A_1 + a_{12}A_2 = 0 \qquad (2.19)$$

and

$$a_{21}A_1 + a_{22}A_2 = 0 \qquad (2.20)$$

where

$$a_{11} = (1 - X_1')\alpha_1 - (D/D')(\alpha_1 + X_1) \qquad (2.21a)$$

$$a_{22} = (1 - X_2')\alpha_2 - (r/r')(\alpha_2 + X_2) \qquad (2.21b)$$

$$a_{12} = -(1 - X_1')\mu r^2 (r'/r - 1)\alpha_2 \qquad (2.21c)$$

$$a_{21} = -(r'/r - 1)[(\alpha_1 + X_1)\{\mu r'(1 - r)\}^{-1} - (1 - X_2')\alpha_1] \qquad (2.21d)$$

In these equations, the following abbreviations are used.

$$X_1' = k_1' X_1,\ X_2' = k_2' X_1, \text{ and } X_2 = k_2 X_1$$

with

$$k_1' = K_1'/K_1,\ k_2' = K_2'/K_1, \text{ and } k_2 = K_2/K_1$$

From (2.19) and (2.20), the characteristic equation is derived by putting

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \qquad (2.22)$$

Neglecting the term of $O(\mu^{1/2})$ compared with 1, the dominant term in the mode coupling can be approximated by

$$a_{12}a_{21} \simeq (1 - D/D')(1 - r/r')(1 - X_1')(\alpha_1 + X_1)\alpha_2 \qquad (2.23)$$

If the coupling effect is completely neglected, a_{11} and a_{22} become the characteristic equations for the barotropic and baroclinic modes, respectively. It is noticed that a_{22} can be converted to a_{11} by changing the parameters:

$$(r/r') \to (D/D') \text{ and } (K_2, K_2') \to (K_1, K_1')$$

Thus, the mode characteristics are similar to each other. The main point in the present paper is the modification of these mode characteristics by the presence of coupling.

By the substitution of (2.17) into (2.21a), (2.21b) and (2.23), (2.22) may be written in an explicit form:

$$\begin{aligned} &\{(1 - X_1')(X_1 - T_1) - T_1P_1(X_1^2 - 1)\} \\ &\times \{(1 - X_2')(X_2 - T_2) - T_2P_2(X_2^2 - 1)\} \\ &\quad - (1 - P_1)(1 - P_2)(1 - X_1')(X_1^2 - 1) \\ &\times (X_2 - T_2)T_1 = 0 \end{aligned} \qquad (2.24)$$

where $P_1 = D/D'$, and $P_2 = r/r'$.

It is immediately found that one root of (2.24) is zero. This shows the generation of a stationary wave mode by the coupling of barotropic and baroclinic motions. On the other hand, if $(mc_2'/f)^2 \ll 1$, we have $k_2^2 > (k_2')^2 \gg 1 \gtrsim (k_1')^2$: $(k_2^2, (k_2')^2 \sim \mu^{-1})$ and the interaction term in (2.24) may be neglected within the error of $O(k_2^{-1})$ for a solution X_1 of $O(1)$. Thus, the mode characteristics for $X_1 \gtrsim O(1)$ can be approximated by those of the barotropic mode.

From (2.4a), the ratio γ of the surface displacement at the shelf break due to the baroclinic motion to that of the barotropic motion is

$$\gamma = \mu r^2 R_2/R_1 \quad \text{at} \quad x = 0$$

This can be transformed to

$$\begin{aligned} \gamma = &\{P_2/(1 - P_2)\}\{(X_1' - 1)(X_1 - T_1) \\ &+ T_1P_1(X_1^2 - 1)\}\{(X_1 - T_1)(X_1' - 1)\}^{-1} \\ &\text{for} \quad X_1 \neq T_1,\ X_1' \neq 1 \end{aligned} \qquad (2.25)$$

It is noticed that γ is $O(k_2^{-1})$ at most for the mode with $X \gtrsim O(1)$, so that the baroclinic motion may be neglected. However, the interface displacement near the shelf break may be governed by the baroclinic motion because $R_2/R_1 \sim O(\mu^{-1/2})$. For small values of X_1, γ is of $O(1)$ in general so that barotropic and baroclinic motions are equally important and $\gamma < 0$ for $X_1 > T_1$ and $\gamma > 0$ for $X_1 < T_1$.

3. Mode characteristics for a homogeneous fluid

Before going into discussions of mode characteristics for a two-layer fluid system, let us review the modes for a homogeneous fluid (see, for example, LARSEN, 1969). The characteristic equation is given by $a_{11}=0$ or from (2.24)

$$aX_1^2+bX_1+c=0 \quad (3.1a)$$

where

$$a=k_1'+T_1P_1 \quad (3.1b)$$

$$b=-(1+T_1k_1') \quad (3.1c)$$

$$c=T_1(1-P_1) \quad (3.1d)$$

a. Very large shelf width; $K_1L \gg 1$ and $T_1 \simeq 1$

Two roots of (3.1a) are

$$X_1=\begin{cases}1 & (3.2a)\\ (1-P_1)/(k_1'+P_1) & (3.2b)\end{cases}$$

Now, for the very large long-shore wave length, $(c_1'm/f)^2 \ll 1$, we have $(k_1')^2=P_1$ and X_1 becomes the phase velocity c_p of the mode relative to c_1. Thus, we have

$$c_p/c_1=\begin{cases}1 & (3.3a)\\ P_1^{-1/2}(1-P_1^{1/2}) & (3.3b)\end{cases}$$

The former corresponds to the Kelvin wave on the shelf and the latter corresponds to the double Kelvin wave near the edge of the shelf (LONGUET-HIGGINS, 1968). The phase speed of the double Kelvin wave is faster or slower than the shallow-water Kelvin wave depending on the depth ratio P_1 smaller or larger than 1/4.

For the very small long-shore wave length, $(c_1m/f)^2 \gg 1$, we have $k_1' \simeq 1$, so that

$$\omega/f=\begin{cases}1 & (3.4a)\\ (1-P_1)/(1+P_1) & (3.4b)\end{cases}$$

The former is an apparent root which does not correspond to the reality unless the condition of Kelvin wave is satisfied. The latter corresponds to the non-divergent short wave length limit of the double Kelvin wave near the edge of the shelf (RHINES, 1969).

b. Small shelf width; $T_1 \sim K_1L \ll 1$

In this approximation, (3.1a) may be reduced to

$$X_1'^2-X_1'+K_1'L(1-P_1)=0 \quad (3.5)$$

and the two roots are approximated by

$$X_1'=\begin{cases}1-(1-P_1)K_1'L & (3.6a)\\ (1-P_1)K_1'L & (3.6b)\end{cases}$$

If $(mc_1'/f)^2 \ll 1$, the first root corresponds to the deep-water Kelvin wave mode, and the second root is the shelf-wave derived by LARSEN (1969). Phase velocities of these waves are approximately

$$c_p=\begin{cases}c_1' & (3.7a)\\ fL(1-P_1) & (3.7b)\end{cases}$$

If $(mc_1/f)^2 \gg 1$, the former root reduces to $\omega \simeq f$ but the latter is unchanged.

c. Very large wave length; $(mc_1'/f)^2 \ll 1$

In this case $k_1'^2=P_1$ and the variation of phase velocity of each mode with respect to K_1L can be studied by tracing the two roots of (3.1a) as shown in Figure 2. If $P_1<1/4$, the

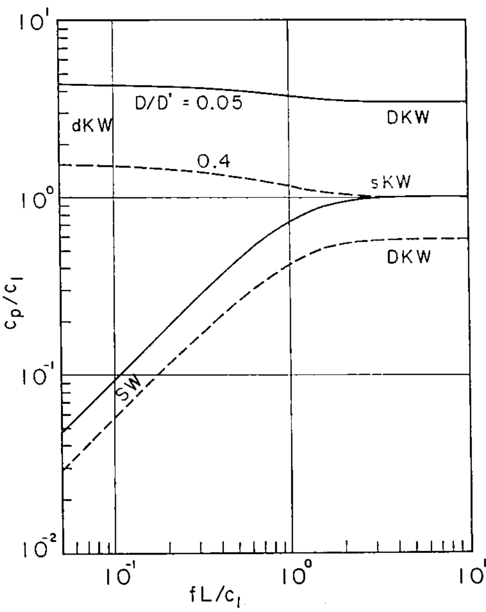

Fig. 2. Phase velocity c_p/c_1 as a function of the shelf width fL/c_1 for the case of a homogeneous fluid with $(mc_1'/f)^2 \ll 1$. sKW: shallow-water Kelvin wave, dKW: deep-water Kelvin wave, DKW: double Kelvin wave, SW: shelf wave.

deep-water Kelvin wave mode and the shelf-wave mode for small K_1L are transformed into double Kelvin wave mode and shallow-water Kelvin wave mode, respectively, as K_1L increases. If $P_1>1/4$, the opposite is true, because the phase velocity of the double Kelvin wave mode is smaller than that of the shallow-water Kelvin wave mode in this case.

Completely analogous results are expected for the internal wave modes if the coupling of the barotropic and baroclinic motions is neglected.

4. Mode characteristics in a two-layer fluid with the thickness of the upper layer shallower than the shelf depth

Taking the mode characteristics for a barotropic fluid into consideration, let us examine the modes for a two-layer fluid given by (2.24). At first, let us consider the case of the very large shelf width. If $T_2\simeq 1$, $X_2=1$ is a root and, furthermore, if $T_1\simeq 1$, $X_1=1$ is also a root. These roots correspond to the baroclinic and barotropic Kelvin wave modes in shallower water if $(mc_2/f)^2\ll 1$ and $(mc_1/f)^2\ll 1$, respectively. In this case ($T_1\rightarrow 1$), the last root is given by

$$X_1=\frac{(1-P_1)(P_2k_2+k_2')+(1-P_2)(1+P_1k_1')}{(P_1+k_1')(P_2k_2+k_2')+(1-P_1)(1-P_2)k_1'} \tag{4.1}$$

which is reduced to (3.2b) when $(mc_2'/f)^2\ll 1$.

Since in general (2.24) is complicated, simplified equations are derived for two cases of the long-shore wave length; (a) $(mc_2/f)^2\gg 1$ and (b) $(mc_2/f)^2\ll 1$. The case (a) corresponds to the assumption of non-divergence of the baroclinic motion as well as the barotropic motion. The case (b) gives the strong divergence of the baroclinic motion, so that the coupling of the motions in the upper and lower layers is large.

(a) $(mc_2/f)^2\gg 1$

In this case, $k_1'=k_2'=k_2=1$ and $X_1'=X_2'=X_2=X_1=\omega/f$. Therefore, (2.24) can be simplified to yield

$$X_1(1-X_1)^2(X_1-W)=0 \tag{4.2}$$

with

$$W=\frac{T_2(1+T_1)-P_2(T_2-T_1)-T_1P_1P_2(1+T_2)}{(1+T_1)+P_2(T_2-T_1)+T_1P_1P_2(1+T_2)}$$

The roots of (4.2) are $X_1=0$, 1, and W among which $X_1=0$ corresponds to the stationary mode and $X_1=1$ is the inertial oscillation and only valid when the condition of Kelvin wave in shallower water is satisfied. The most interesting root is $X_1=W$.

If T_1, $T_2\rightarrow 1$, this root becomes

$$X_1=(1-P_1P_2)/(1+P_1P_2) \tag{4.3}$$

and the wave forms of both the barotropic and baroclinic motions are of the double Kelvin wave type ($\alpha_i=1$). Compared with (3.4b) for the purely barotropic case, it is seen that the frequency is increased by the presence of the stratification. Furthermore, the volume transport vector $\boldsymbol{Q}_1$ in the upper layer given by (2.5a) reduces to zero since $(1-r)\boldsymbol{U}_1=-r\boldsymbol{U}_2$ and the volume transport vector in the lower layer given by (2.5b) becomes $\boldsymbol{Q}_2=\boldsymbol{U}_1$. Thus, the wave is analogous to the bottom wave discussed by RHINES (1970). This is in contrast to the stationary mode ($X_1=0$), in which the motion is confined in the upper layer only in the present approximation.

If $T_1\ll 1$ and $T_2\simeq 1$, we have approximately

$$X_1=(1-P_2)/(1+P_2) \tag{4.4}$$

The wave forms of the barotropic and baroclinic motions are the Kelvin wave type and the double Kelvin wave type, respectively. Compared with (3.4b), it is found that P_1 is replaced by P_2, showing that the baroclinic motion is dominant.

(b) $(mc_2/f)^2\ll 1$

In this case, we may safely assume the inequality:

$$k_2^2,\ k_2'^2\gg 1\gtrsim k_1'^2,\ T_1,\ T_2$$

Therefore, the three roots besides zero can be determined approximately by the following characteristic equation within the error of $O(k_2^{-1})$:

$$aX_1^3+bX_1+\tilde{c}X_1+d=0 \tag{4.5}$$

where a and b are the same as (3.1b), (3.1c) and

$$\tilde{c}=c+P_2^{-1/2}k_2^{-1} \quad (4.6a)$$

$$d=-k_2^{-2}\{k_2T_1(1-P_1)G + T_2(1-P_2)(1+T_1k_1'P_1)H^{-1}\} \quad (4.6b)$$

with

$$G=(T_2+P_2^{1/2})(1+T_2P_2^{1/2})^{-1} \quad (4.6c)$$

$$H=P_2^{1/2}(1+T_2P_2^{1/2}) \quad (4.6d)$$

For the large shelf width ($K_1L\gg1$), we may put T_1, $T_2\simeq1$ and the relative orders of the coefficients in (4.5) are

$$a, b, \tilde{c}\sim O(1) \text{ and } d\sim O(k_2^{-1})$$

Therefore, the smallest root is approximated by

$$X_1\simeq k_2^{-1}$$

This corresponds to the baroclinic Kelvin wave mode in shallower water. The remaining two roots are approximately the same as for the homogeneous fluid (3.2a, b).

On the other hand, for the small shelf width ($K_1L\ll1$), we have

$$k_2T_1\simeq K_2L\lesssim1$$

so that

$$a, b\sim O(1),\ \tilde{c}\sim O(k_2^{-1}), \text{ and } d\sim O(k_2^{-2})$$

In this case, the largest root is approximately the same as for the homogeneous fluid (3.6a): namely the deep-water Kelvin wave mode if $(mc_1'/f)^2\ll1$. The remaining two roots can be determined approximately by

$$X_2^2-c'X_2+d'=0 \quad (4.7)$$

where

$$c'=(K_2L)(1-P_1)+P_2^{-1/2} \quad (4.8a)$$

$$d'=(K_2L)(1-P_1)G+T_2(1-P_2)H^{-1} \quad (4.8b)$$

In particular, if $K_2L\ll1$, two roots are approximated by

$$X_2=\begin{cases}P_2^{-1/2} & (4.9a)\\ (K_2L)(1-P_1P_2) & (4.9b)\end{cases}$$

The former corresponds to the baroclinic Kelvin wave mode in deep-water ($\alpha_2<0$) if $(mc_2'/f)^2\ll1$ and the latter to the shelf-wave ($0<\alpha_2<1$) which is different from the barotropic shelf-wave mode (3.6b).

The variation of phase velocity of each mode with K_2L is shown in Figure 3 by solving (4.5) numerically with $(mc_1'/f)^2\ll1$. Let us denote the modes given by three roots of (4.5) as the first, second and third modes in the decreasing order of the roots. The first mode is then found to be almost the same as the barotropic mode provided that the baroclinic motion is also significant near the shelf break. For $P_1>1/4$, the wave form of the barotropic motion varies from the Kelvin wave type to the double peak type and finally to the double Kelvin wave type as K_1L increases from zero to infinity. The accompanying wave form of

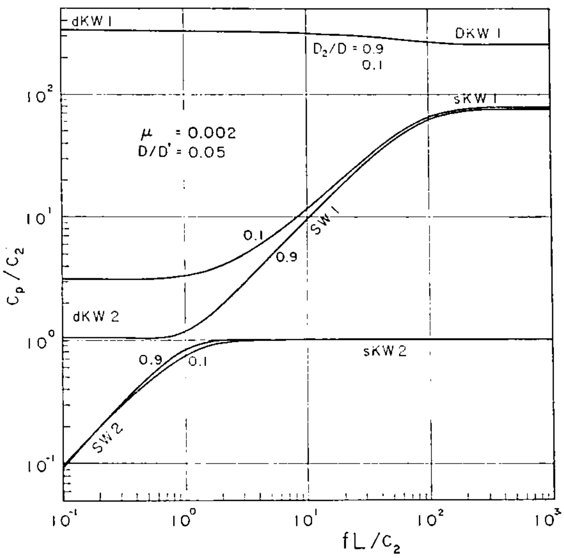

Fig. 3. Phase velocity c_p/c_2 as a function of the shelf width fL/c_2 for the case of a two-layer fluid with $(mc_1'/f)^2\ll1$. Thickness of the upper layer is smaller than the shelf depth. Illustrations in the figure are the same as in Fig. 2, provided that 1 and 2 denote predominantly barotropic and baroclinic characters, respectively. The difference of phase velocity of the second mode for large fL/c_2 is an error introduced by the approximation in (4.5). True value should approach $c_p/c_2=74.53$ for the cases of $D_2/D=$ 0.1 and 0.9.

the baroclinic motion is of the double Kelvin wave type for $K_2L \gg 1$. Near the shelf break, the interface disturbance is larger than the surface disturbance by $O(k_2)$.

The second mode with the intermediate phase velocity is the most interesting. The barotropic wave-form changes from the Kelvin wave type to the shelf-wave type at $X_1 = T_1$ as K_1L increases from zero and approaches to the Kelvin wave in shallower water as $K_1L \gg 1$. On the other hand, the wave-form of the baroclinic motion changes from the Kelvin wave type to the double peak type at $k_2X_1T_2 = 1$ as K_2L increases and finally approaches the double Kelvin wave type for $K_2L \gg 1$. For a moderate value of K_1L when the phase velocity of the second mode is close to that of the barotropic shelf-wave mode, the baroclinic motions are still important near the edge of the shelf. It may be noticed that the characteristics of the second mode shows an apparent effect of increasing phase velocity of the barotropic shelf-wave due to the presence of stratification.

The third mode with the smallest phase velocity ($X_2 < 1$) is predominantly of the baroclinic nature with significant contribution from barotropic motions. The wave-forms of both the barotropic and baroclinic motions are of the shelf-wave type. With $K_2L \to \infty$, however, the baroclinic motion is confined near the coastal boundary as a Kelvin wave and the barotropic motion disappears.

(c) Stationary wave mode: ($X_1 = 0$)

If we take the solution of the form (2.12) together with (2.15) and (2.16), it is straightforward to derive the stream functions Ψ_1 and Ψ_2 in the upper and lower layers satisfying (2.6) and (2.7) in the form:

$$\Psi_1 = (1-r)\phi_1 + r\phi_2 \qquad (4.10a)$$

$$\Psi_2 = r(\phi_1 - \phi_2) \qquad (4.10b)$$

with

$$\phi_i/C = \{(\sinh K_i x)/T_i + \cosh K_i x\} \times \exp[imy] \qquad (4.11a)$$

where C is a constant. In deep water, all quantities in (4.10a) and (4.10b) are dashed and

$$\phi_i'/C = \exp[-K_i'x + imy] \qquad (4.11b)$$

It is easily seen that, if $(mc_2/f)^2 \gg 1$, namely when the divergence of the baroclinic motion is negligible, we have $K_1 = K_2 = m$. Therefore, under the condition of $mL \gg 1$, we have $T_i = 1$, $\phi_1 = \phi_2$, and from (4.10b), the motion in the lower layer vanishes. Thus, we may call this mode a surface mode in contrast to a bottom made discussed in (a) of this section.

It is mentioned that, since $\omega = 0$, the flow is purely geostrophic and the solution is not ristricted to the form (4.11a, b). More general form of a stationary vortex can satisfy the boundary conditions (2.6) to (2.11).

5. Mode characteristics in a fluid with two layers in deep water only

Parallel arguments to Section 2 lead to the characteristic equation for a model with the two-layer in deeper water only:

$$(X_1 - T_1)(1 - X_1k_1')(1 - X_1k_2') - T_1(X_1^2 - 1)(D/D') \times \{(D_2'/D_1')(1 - X_1k_1') + (1 - X_1k_2')\} = 0 \qquad (5.1)$$

(a) $(mc_2'/f)^2 \gg 1$

In this case, we have k_1', $k_2' = 1$ and (5.1) reduces to

$$(1 - X_1)^2\{X_1(1 + T_1P_3) - T_1(1 - P_3)\} = 0 \qquad (5.2)$$

where

$$P_3 = D/D_1' \leq 1$$

The root $X_1 = 1$ is trivial and the important root is

$$X_1 = T_1(1 - P_3)/(1 + T_1P_3) \qquad (5.3)$$

Since $X_1 < T_1$, the motion on the shelf is of the shelf-wave type. For $T_1 \to 1$, we arrive at the double Kelvin wave. The difference of this wave from the barotropic case (3.4b) is the replacement of P_1 by P_3: namely the actual depth of deeper water is replaced by the thickness of the upper layer.

(b) $(mc_2'/f)^2 \ll 1$

In this case, we have $k_2'^2 \gg 1 \gtrsim k_1'^2$, T_1, and

(5.1) may be transformed to

$$aX_1^3+bX_1^2+c^*X_1+d^*=0 \qquad (5.4)$$

within the error of $O(k_2'^{-1})$. Here, a and b are the same as (3.1b) and (3.1c), and

$$c^*=c+(k_2')^{-1} \qquad (5.5a)$$

$$d^*=-T_1(1-P_3)/k_2' \qquad (5.5b)$$

For a very large shelf when $K_1L\gg1$, we have $T_1=1$ and the larger two roots can be approximated by the roots for the homogeneous fluid, (3.3a, b). The smallest root is approximated by

$$X_2'=(1-P_3)/(1-P_1) \qquad (5.6)$$

This root corresponds to the double Kelvin wave mode ($\alpha_1=1$) with the internal Kelvin wave type in deeper water.

For a small shelf width when $K_1L\ll1$, we have $T_1\ll1$ and the largest root can be approximated by the corresponding root for the homogeneous fluid (3.6a). The smaller two roots are given by

$$X_2'^2-\{1+K_2'L(1-P_1)\}X_2'+K_2'L(1-P_3)=0 \qquad (5.7)$$

If $K_2'L\ll1$, we have

$$X_2'=\begin{cases}1-(P_1-P_3)K_2'L & (5.8a)\\ (1-P_3)K_2'L & (5.8b)\end{cases}$$

(5.8a, b) is similar to (3.6a, b) and the larger root corresponds to the deep-water baroclinic Kelvin wave mode and the smaller root is the shelf-wave mode with the baroclinic character dominating in deep water.

If we put formally $K_2'L\gg1$, the larger root of (5.7) becomes

$$X_2'=(1-P_1)K_2'L \qquad (5.9)$$

Namely, the mode corresponding to the larger root resembles the barotropic shelf-wave mode (3.6b). The increase of the phase velocity of a shelf-wave due to the effect of stratification discussed by MYSAK (1967) is thus explained by the transition of the predominant motion of this mode from the barotropic shelf-wave mode to the baroclinic Kelvin wave mode in deeper water as the shelf width decreases ($K_2'L\lesssim1$). He apparently missed the baroclinic shelf-wave mode of much slower phase velocity (5.8b).

It should be noticed that, as $P_3\to1$, the smallest root approaches zero; in other words the stationary mode is a limiting case of the mode corresponding to the smallest root in the present model as $D/D_1'\to1$.

The variation of phase velocity of each mode with $K_2'L$ is shown in Figure 4, by solving (5.4) numerically. It is seen that the mode with the smallest phase velocity resembles the shelf-wave mode for small $K_2'L$ and the baroclinic double Kelvin wave mode as $K_2'L$ increases. On the other hand, the mode corresponding to the deep-water baroclinic Kelvin wave mode for small $K_2'L$ is transformed to the mode corresponding to the barotropic shelf-wave mode and finally to the shallow-water Kelvin wave mode (or barotropic double Kelvin

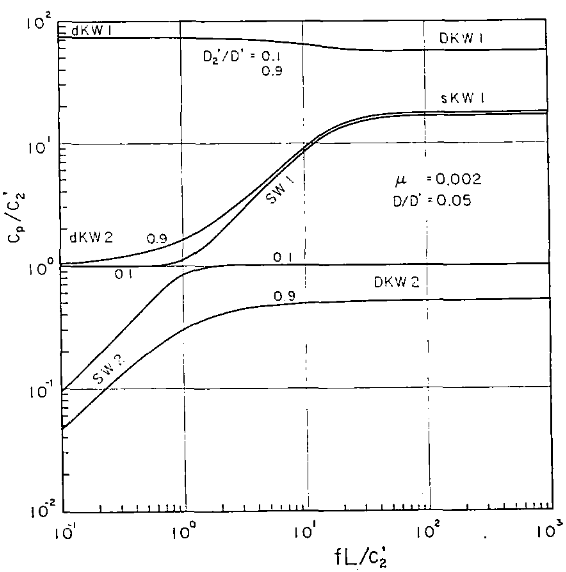

Fig. 4. Phase velocity c_p/c_2' as a function of the shelf width fL/c_2' for the case of a two-layer fluid with $(mc_1'/f)^2\ll1$. Thickness of the upper layer is larger than the shelf depth. Illustrations in the figure are the same as in Figs. 2 and 3. The difference of phase velocity of the second mode for large fL/c_2' is an error introduced by the approximation in (5.4). True value should approach $c_p/c_2'=16.67$ for the cases of $D_2'/D'=0.1$ and 0.9.

mode if $D/D'<1/4$) as $K_2'L$ increases.

6. Conclusions

The characteristics of the trapped modes in a two-layer shelf model are examined. The relative order of various scales: the long-shore wave length, the Rossby's deformation radius and the shelf width, are crucial to determine the characteristics of various modes. The variation of the phase velocity of each mode with the increase of the shelf width is shown in Figures 3 and 4, for a very large long-shore wave-length. The shelf-wave mode splits into two parts near $fL/c_2\sim1$ or 2 by the presence of stratification. With the decrease of the shelf width the barotropic shelf-wave mode is transformed into the deep-water baroclinic Kelvin wave mode and the shelf-wave mode formed by the combination of the barotropic and baroclinic motions with much smaller phase velocity appears.

If the interface of the two fluid layers is shallower than the shelf depth, a stationary mode is possible, and for large shelf width the double Kelvin wave mode of baroclinic nature does not exist. If $(mc_2/f)^2\gg1$ and $(mL)^2\gg1$, the stationary wave mode and the shelf-wave mode become the surface mode and the bottom mode, respectively, where the motion is confined either in the surface or bottom layer only.

If the interface is deeper than the shelf depth, the shelf-wave of the baroclinic character in deeper water exists for small shelf width. In contrast, the barotropic shelf-wave mode is transformed with the decrease of the shelf width into deep water baroclinic Kelvin wave mode near $fL/c_2'\sim1$ or 2, so that the phase velocity of this mode apparently increases by the presence of stratification. For large shelf width, two double Kelvin wave modes appear with barotropic or baroclinic characters in deeper water, respectively.

The present model is admittedly too simple to understand fully the mode characteristics of trapped waves in a real ocean. The inclusion of the depth variation on the shelf would complicate the coupling of the barotropic and baroclinic motions, because of the presence of shelf-waves of higher modes. However, if the slope on the shelf is small, the coupling of two kinds of motions on the shelf may be safely neglected if the Rossby's internal deformation radius is much smaller than the horizontal scale of the wave. The existence of the rapid variation of the depth at the shelf break is considered to be the essential factor causing the coupling, so that the present model may be justfied with respect to the qualitative nature of mode characteristics.

Acknowledgment

The author is grateful to Miss KATO for her help in numerical computation.

References

ENDOH, M. (1973): A numerical experiment on the variations of western boundary currents, Part II Response to a moving typhoon. J. Oceanog. Soc. Japan, **29**, 28–43.

ISOZAKI, I. (1972): Unusually high mean sea level in September 1971 along the south coast of Japan —1. Some aspects of high sea level with time scale more than one week—. Pap. Meteorol. and Geophys., **23**, 243–257.

KAJIURA, K. (1962): A note on the generation of boundary waves of Kelvin type. J. Oceanog. Soc. Japan, **18**, 49–58.

LARSEN, J.C. (1969): Long waves along a single-step topography in a semi-infinite uniformly rotating ocean. J. Marine Res., **27**, 1–6.

LONGUET-HIGGINS, M.S. (1968): On the trapping of waves along a discontinuity of depth in a rotating ocean. J. Fluid Mech., **31**, 417–434.

MYSAK, L.A. (1967): On the theory of continental shelf waves. J. Marine Res., **25**, 205–227.

RHINES, P.B. (1969): Slow oscillations in an ocean of varying depth, Part 1. Abrupt topography. J. Fluid Mech., **37**, 161–189.

RHINES, P.B. (1970): Edge-, bottom-, and Rossby waves in a rotating stratified fluid. Geophys. Fluid Dynamics, **1**, 273–302.

ROBINSON, A.R. (1964): Continental shelf waves and the response of the sea surface to weather system. J. Geophys. Res., **69**, 367–368.

SHOJI, D. (1961): On the variation of the daily mean sea levels along the Japanese Islands. J. Oceanog. Soc. Japan, **17**, 141–152.

SUGINOHARA, N. (1973): Response of a two-layer ocean to typhoon passage in the western boundary region. J. Oceanog. Soc. Japan, **29**, 236–250.

YOSHIDA, K. (1960): The oceanic waves of days to months periods. Rec. Oceanog. Wks. Japan, **5**, 11-24.
YOSHIDA, K., D. SHOJI and J. MASUZAWA (1972): A possible interaction between the storm tides and the Kuroshio—a speculation on the recent floods. Rec. Oceanog. Wks. Japan, **17**, 47-51.

陸棚上の超長周期波に対する成層の効果

梶 浦 欣 二 郎

要旨: 二層モデルによって，一様水深の真直な陸棚上に生ずる超長周期波の特性を調べ，陸棚幅，岸沿いの波の波長等の違いによって，特性がどうなるかを明らかにした.

このモデルでは四つのモード波が出来るが，そのうち位相速度が最大のモード波は順圧モードで近似出来る．順圧の陸棚波に対応する二番目のモード波は傾圧的な海水運動によってかなりの影響をうけ，陸棚幅が狭い極限では外海側の内部ケルビン波に対応するモードに移行する(岸沿いの波の波長が大きい場合)．従って，この場合はみかけ上陸棚波速度が成層の影響によって増加するようにみえる．残りの二つのモード波は内部波的であるが，順圧的な海水運動も重要な効果を及ぼしている．そのうち，一つは陸棚幅が狭い場合には陸棚波的な振舞をし，陸棚幅が大きい場合には陸棚上の内部ケルビン波になる．もう一つは定常波モードである．岸沿いの波の波長が内部ロスビー変形半径より短いときには上，下両層の運動は無関係となり，表面モードと海底モードとが出来る．

もし，二層の境界面の深さが陸棚水深よりも大きいときには定常波モードはなくなり，第三番目のモード波の特性は陸棚幅が狭い場合には陸棚波的で，陸棚幅が広い場合には内部波の特性をもった二重ケルビン波に近づく．位相速度の大きい初めの二つのモード波の特性は境界面の深さが陸棚水深より小さい場合と本質的に変わりはない．