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Energy and Angular Momentum of Charged Rotating Black Holes

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We show that the pseudotensors of Einstein, Tolman, Landau and Lifshitz, Papapetrou, and Weinberg essentially coincide for any Kerr–Schild metric if calculations are carried out in Kerr–Schild Cartesian coordinates. This generalizes a previous result by Gürses and Gürsey that dealt only with the pseudotensors of Einstein and Landau–Lifshitz. We compute exactly the energy and angular momentum distributions for the Kerr–Newman metric in Kerr–Schild Cartesian coordinates and compare the results with those obtained by using different definitions of quasi-local mass, which unlike pseudotensors do not agree for all Kerr–Schild metrics.

KEY WORDS : Energy pseudotensors ; Kerr–Schild black holes

1. INTRODUCTION

Following the energy-momentum pseudotensor of Einstein, a plethora of definitions for energy, momentum, and angular momentum of a general relativistic system has been proposed by many authors (see Ref. 1). To use the pseudotensors of Einstein, Tolman, and Landau and Lifshitz one is restricted to quasi-Minkowskian coordinates (Ref. 2, Ref. 3, p.227, Ref. 4, p.280). Møller [2], arguing that to single out a particular coordinate system is not satisfactory from the general relativistic point of view, constructed

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a new energy-momentum pseudotensor and claimed that with it one was not constrained to use asymptotically Minkowskian coordinates. However, three years later, Møller observed a serious drawback of his prescription [5], i.e., the total energy-momentum vector of a closed physical system is not a Lorentz four-vector. Thus, Møller's attempt to give a coordinate-independent definition for energy calculations failed and therefore we will not discuss Møller's pseudotensor any more in this paper. Møller's further effort was also not successful (see Ref. 6). Komar [7] and many others (see Ref. 1) proposed coordinate independent definitions of energy. Bergqvist [8] considered several different definitions of quasi-local mass and found that not any two of them give the same result for the Reissner–Nordström (RN) and Kerr spacetimes. Despite these problems there has been considerable interest in this subject in recent years (see Refs. 9,10, and references therein).

One of the present authors (Virbhadra, referred to as KSV hereafter) [11] showed that, up to the third order of the rotation parameter, the pseudotensors of Einstein, Tolman, and Landau–Lifshitz (ETLL) give the same and reasonable energy distribution in the Kerr–Newman (KN) field when calculations are carried out in Kerr–Schild (KS) Cartesian coordinates. Cooperstock and Richardson [12] extended the energy calculations up to the seventh order of the rotation parameter and found that the pseudotensors of ETLL give the same energy distribution for the KN metric. Moreover, their result supported the conjecture of KSV that there is no energy associated with the exterior of the Kerr black hole. Later on KSV [13] showed that the pseudotensors of ETLL yield the same energy and energy current density components for the Vaidya metric. Recently two of the present authors (Chamorro and KSV) [14] obtained the energy distribution in the Bonnor–Vaidya (BV) spacetime in the prescriptions of Einstein and LL. Both definitions give the same and reasonable result. Tod [15] calculated the Penrose quasi-local mass [16] for the BV metric and got the same result.

Only recently has been brought to our attention that in an interesting paper Gürses and Gürsey [17] showed that the pseudotensors of Einstein and LL coincide for all Kerr–Schild metrics. In this paper we extend that result by showing that the pseudotensors of Einstein and LL as well as those of Tolman (Ref. 3, p.227), Papapetrou [18] and Weinberg (Ref. 19, p.165) (PW) coincide for any Kerr–Schild metric in the precise sense described below and, in consequence, give the same energy and energy current density components for the KN as well as BV spacetimes. We also extend the results of KSV, Cooperstock and Richardson for the Kerr–Newman metric by performing a non-perturbative calculation. Since the pseudotensors of

Einstein and Tolman have mixed indices, when we say that they coincide with those of Landau–Lifshitz, Papapetrou and Weinberg (which have only upper indices) we mean that they are equal except for a trivial factor: a Minkowski metric used to raise the lower index.

In Section 2 we show that the five pseudotensors coincide if KS Cartesian coordinates can be used. Section 3 gives the results for the energy, momentum, and angular momentum distributions for the KN metric. In Section 4 it is pointed out that in the case of the RN spacetime the masses computed by means of the pseudotensors agree with the quasi-local masses of Hawking, Penrose, Ludvigsen–Vickers, Bergqvist–Ludvigsen, Dougan–Mason and Hayward, but not with that of Komar (see Refs. 8,20). In the case of the energy contents of the Kerr horizon, we find that the five pseudotensors give the same result as the quasi-local masses of Komar and Bergqvist–Ludvigsen [8].

Conventions: We use geometrized units in which the speed of light in vacuum c and the Newtonian gravitational constant G are taken to be equal to 1, the metric has signature $+---$, and Latin (Greek) indices take values $0\dots3$ $(1\dots3)$.

2. KERR–SCHILD METRICS AND ENERGY-MOMENTUM PSEUDOTENSORS

In the following we shall consider the algebraically special metrics of Kerr–Schild which are given by

$$g_{ik} = \eta_{ik} - 2Vl_il_k \tag{1}$$

($\eta_{ik} = \mathrm{diag}\,(1,-1,-1,-1)$) in terms of the scalar function V and the null vector l_i which satisfies the following properties:

$$g_{ik}l^il^k = \eta_{ik}l^il^k = 0, \qquad l^il_{k;i} = l^il_{k,i} = 0\,. \tag{2}$$

Gürses and Gürsey [17] pointed out that for these metrics the pseudotensors of Einstein and LL coincide and are proportional to the Einstein tensor.

We shall consider not only the pseudotensors of Einstein (see Ref. 2),

$$\Theta_i{}^k = \frac{1}{16\pi}\left\{\frac{g_{in}}{\sqrt{-g}}\,[-g(g^{kn}g^{lm} - g^{ln}g^{km})\,]_{,m}\right\}_{,l}, \tag{3}$$

and Landau and Lifshitz (Ref. 4, p.280),

$$L^{ik} = \frac{1}{16\pi}\,[-g(g^{ik}g^{lm} - g^{il}g^{km})\,]_{,lm}\,, \tag{4}$$

but also those of Tolman (Ref. 3, p.227),

$$\mathcal{T}_i{}^k = \frac{1}{8\pi}\left\{\sqrt{-g}\left[-g^{pk}V_{ip}{}^l + \frac{1}{2}\,g_i^k g^{pm}V_{pm}{}^l\right]\right\}_{,l}, \tag{5}$$

$$V_{jk}{}^i \equiv -\Gamma^i_{jk} + \frac{1}{2}\,g^i_j\Gamma^m_{mk} + \frac{1}{2}\,g^i_k\Gamma^m_{mj}, \tag{6}$$

Papapetrou [18],

$$\Sigma^{ik} = \frac{1}{16\pi}\,[\sqrt{-g}\,(g^{ik}\eta^{lm} - g^{il}\eta^{km} + g^{lm}\eta^{ik} - g^{lk}\eta^{im})\,]_{,lm}, \tag{7}$$

and Weinberg (Ref. 19, p.165),

$$W^{ik} = \frac{1}{16\pi}\left[\frac{\partial h^a_a}{\partial x_l}\eta^{ik} - \frac{\partial h^a_a}{\partial x_i}\eta^{lk} - \frac{\partial h^{al}}{\partial x^a}\eta^{ik} + \frac{\partial h^{ai}}{\partial x^a}\eta^{lk} + \frac{\partial h^{lk}}{\partial x_i} - \frac{\partial h^{ik}}{\partial x_l}\right]_{,l}. \tag{8}$$

(In the last definition $h_{ik} = g_{ik} - \eta_{ik}$ and indices on h_{ik} or $\partial/\partial x_i$ are raised or lowered with the help of η's.)

By using the properties of Kerr–Schild metrics it is not difficult to prove that for KS metrics the five pseudotensors of ETLLPW essentially coincide, as one always has in KS Cartesian coordinates

$$\Theta_i{}^k = \mathcal{T}_i{}^k = \eta_{ij}L^{jk}, \tag{9}$$

$$L^{ik} = \Sigma^{ik} = W^{ik} = \frac{1}{16\pi}\,\Lambda^{iklm}{}_{,lm}, \tag{10}$$

$$\Lambda^{ikpq} \equiv 2V(\eta^{ik}l^pl^q + \eta^{pq}l^il^k - \eta^{ip}l^kl^q - \eta^{kq}l^il^p). \tag{11}$$

The demonstration of eqs. (9) and (10) is rather long but straightforward. One only has to substitute eq. (1) into the definitions (3)–(8) and use the fact that for any KS metric eq. (2) and the following properties hold:

$$g = -1, \qquad g^{ik} = \eta^{ik} + 2Vl^il^k, \tag{12}$$

$$l^i = g^{ik}l_k = \eta^{ik}l_k, \qquad l_ig^{ki}{}_{,m} = 0, \tag{13}$$

$$g^{ik}g^{lm} - g^{il}g^{km} = \eta^{ik}\eta^{lm} - \eta^{il}\eta^{km} + \Lambda^{iklm}, \tag{14}$$

$$l^il^k_{,i} = 0, \qquad l_il^i_{,k} = 0, \tag{15}$$

$$h^{ik} = -2Vl^il^k = \eta^{ik} - g^{ik}, \qquad h^i_i = 0, \tag{16}$$

$$\Gamma^m_{mk} = 0, \tag{17}$$

$$V_{jk}{}^i = -\Gamma^i_{jk}, \qquad g^{pm}V_{pm}{}^i = g^{im}{}_{,m}. \tag{18}$$

In consequence, the energy and momentum are

$$P^i = \frac{1}{16\pi} \iint \Lambda^{i0\alpha m}{}_{,m} n_\alpha \, dS, \tag{19}$$

and the spatial components of J^{ik} are

$$J^{\alpha\beta} = \frac{1}{16\pi} \iint (x^\alpha \Lambda^{\beta 0\sigma m}{}_{,m} - x^\beta \Lambda^{\alpha 0\sigma m}{}_{,m} + \Lambda^{\alpha 0\sigma\beta}) \, n_\sigma \, dS, \tag{20}$$

where n_α is the outward unit normal vector and dS is the infinitesimal surface element.

3. THE KERR–NEWMAN METRIC

The KN spacetime is given in KS Cartesian coordinates by the line element in eq. (1) with the following choices for V and l_i [21]:

$$V = \frac{2M\rho^3 - Q^2\rho^2}{2(\rho^4 + a^2 z^2)}, \tag{21}$$

$$\begin{aligned} l_i \, dx^i &= dt + \frac{z}{\rho} \, dz + \frac{\rho}{\rho^2 + a^2} (x \, dx + y \, dy) \\ &\quad - \frac{a}{\rho^2 + a^2} (x \, dy - y \, dx), \end{aligned} \tag{22}$$

where ρ is defined by the positive root of

$$\frac{x^2 + y^2}{\rho^2 + a^2} + \frac{z^2}{\rho^2} = 1. \tag{23}$$

By using the results of the previous section we calculate the energy, momentum, and angular momentum for the KN metric in KS Cartesian coordinates. The intermediate mathematical expressions are very lengthy and therefore we give only the final results, which have been obtained and checked by means of two different computer algebra systems. The energy and momentum inside a surface with constant ρ in all the prescriptions of ETLLPW are

$$\begin{aligned} E(\rho) &= M - \frac{Q^2}{4\rho} \left[1 + \frac{(a^2 + \rho^2)}{a\rho} \arctan\left(\frac{a}{\rho} \right) \right], \\ P_1(\rho) &= P_2(\rho) = P_3(\rho) = 0\,. \end{aligned} \tag{24}$$

The spatial components of the angular momentum are

$$J^{12}(\rho) = a\left\{M - \frac{Q^2}{4\rho}\left[1 - \frac{\rho^2}{a^2} + \frac{(a^2+\rho^2)^2}{a^3\rho}\arctan\left(\frac{a}{\rho}\right)\right]\right\},$$
$$J^{23}(\rho) = J^{31}(\rho) = 0\,. \tag{25}$$

The total energy, momentum, and angular momentum ($\rho \to \infty$ in the above expressions) are $E = M$, $P_1 = P_2 = P_3 = 0$, $J^{12} = Ma$ and $J^{23} = J^{31} = 0$. The energy and energy current density components for the KN metric are

$$(L^{00}, L^{10}, L^{20}, L^{30}) = \frac{Q^2\rho^4}{8\pi(\rho^4 + a^2z^2)^3}\,(\rho^4 + 2a^2\rho^2 - a^2z^2, -2ay\rho^2, 2ax\rho^2, 0). \tag{26}$$

Notice that the mass parameter M does not appear in the densities in (26), but it does appear in (24) and (25). The reason is that the results in (24) and (25) are not computed by integrating the density components (which are given in (26) for the exterior of the black hole), but are obtained by using Gauss's theorem and integrating over a surface (ρ = constant), according to (19) and (20). This takes into account the contribution from the interior of the black hole, which is not described by the Kerr–Newman metric.

4. DISCUSSION

We have shown that for any KS metric the five pseudotensors of ETLLPW coincide [in the sense of eqs. (9) and (10)] if calculations are carried out in KS Cartesian coordinates. We have obtained energy and angular momentum distributions for the KN metric for arbitrary values of the mass, charge, and rotation parameters. For the Kerr black hole the energy is confined to its interior, because by taking $Q = 0$ in eqs. (24) and (26) one sees that the energy and energy current density components vanish outside the black hole and that the energy distribution [given by eq. (24)] is independent of ρ. This proves a previous conjecture of KSV [11] and is compatible with Cooperstock's conjecture [10]. It is clear from eqs. (24) and (25) that the energy distribution for the KN metric is independent of the sign on the charge as well as rotation parameters whereas the direction of the angular momentum depends on the sign of the rotation parameter and is independent of the sign on the charge parameter. This is obviously a reasonable result.

For the RN metric ($a = 0$), one gets from eq. (24) $E = M - Q^2/2r$. The quasi-local masses of Hawking, Penrose, Ludvigsen–Vickers, Bergqvist–Ludvigsen, Dougan–Mason, and Hayward give the same result, while that of Komar leads to to $E = M - Q^2/r$ (see Refs. 8,20). From eq. (24) one gets $E = M$ for the Kerr metric ($Q = 0$), in accordance with the energy content of the event horizon as computed by using the quasi-local masses of Komar and Bergqvist–Ludvigsen. However, the quasi-local masses of Hawking, Penrose, and Dougan–Mason give results that differ from each other [8] as well as from the definitions of ETLLPW discussed above.

For the BV metric two of the present authors (Chamorro and KSV) [14] found that the pseudotensors of Einstein and LL give the same result for the energy as well as for the energy current densities. In the light of Gürses and Gürsey's result [17] one sees the reason for the coincidences, which will in fact happen for the five prescriptions of ETLLPW. The energy distribution is $E = M(u) - Q(u)^2/2r$, where u is the retarded time coordinate. Tod [15] found the same result for the BV metric in Penrose prescription.

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Editor's note: For further results on Møller's energy-momentum complex the reader may consult also G. Lessner, *Gen. Rel. Grav.* **28** (1996), 527.

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