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The Dimension of Leaf Closures of K-Contact Flows

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Abstract: For K-contact flows on $(2n+1)$-dimensional compact manifolds, we show that the dimension of any leaf closure is at most the smaller of $(n+1)$ and $(2n+1)$ minus the rank of the vector space of harmonic vector fields.

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1. Introduction

Weinstein Conjecture ([12]) states that any compact contact manifold should carry at least one closed characteristic. This conjecture has been proved in some special cases, among others, it is proved in [10] for compact hypersurfaces of contact type in $\mathbf{R}^{2n}$ and in [7] for compact K-contact manifolds.

Many properties of a flow can be derived from those of the algebra of foliate vector fields; for example the leaf closures of an isometric flow on a compact manifold are orbits of an abelian Lie algebra of Killing foliate vector fields ([6]). For K-contact flows, we point out the fact that the leaf closures can have dimension at most $n+1$ when the manifold is $(2n+1)$-dimensional.

In Section 4, a slightly more precise bound for the dimension of leaf closures of K-contact flows is obtained in terms of the first Betti numbers of the manifold.

2. The Contact Characteristic Vector Field

Let M be a $(2n+1)$-dimensional manifold with a contact form α on it. It is well known that there is a unique vector field ξ on M such that $\alpha(\xi) = 1$ and $i_\xi \mathrm{d}\alpha = 0$. The vector field ξ is called the *characteristic vector field* of the contact manifold M. The contact form α defines a $2n$-dimensional distribution D on M.

$$D = \ker \alpha.$$

The distribution D, which is left invariant by the vector field ξ ([11]), is called the *contact distribution.* Sections of the bundle D are called *horizontal* vector fields, whereas any vector field proportional to ξ is called a *vertical* vector field.

The 2-form $\mathrm{d}\alpha$ induces a symplectic structure on the distribution D; that is, D is a symplectic vector bundle over M.

Definition 1. A vector field X is said to be *foliate* if its Lie bracket with ξ is a vertical vector field. X will be called *transverse* if it is foliate and horizontal. In that case, X commutes with ξ, that is $[X, \xi] = 0$.

Definition 2. A differential form β is said to be *basic* if $i_\xi \beta = 0$ and $L_\xi \beta = 0$.

3. Killing Foliate Vector Fields

This section is devoted to a particular type of foliate vector fields, mainly those which are infinitesimal isometries. They arise naturally in the setting of isometric flows ([2]) where they describe the leaf closures.

Let M be a $(2n+1)$-dimensional compact K-contact manifold. Since ξ is Killing with respect to some K-contact metric ([7]), it is known that the closure of the characteristics are orbits of an abelian Lie algebra $\mathcal{G}$ of Killing vector fields ([5]).

Definition 3. A vector field X on M is said to be a *contact vector field* if the identity

$$L_X \alpha = h\alpha$$

holds for some function h on M. If $h \equiv 0$, that is $L_X \alpha = 0$, X is said to be *strictly contact.*

The function $\alpha(X)$ is usually called the *contact hamiltonian* function of the contact vector field X and is related to h by the formula

$$\mathrm{d}(\alpha(X))(\xi) = h \tag{1}$$

where ξ is the characteristic vector field of the contact form α.

Remark. Unlike the group of isometries of a compact riemannian manifold, little is known about the group of contact transformations in general. The following proposition, which will not be needed in proving our main result, implies that the group of contact transformations of a compact K-contact manifold contains a compact subgroup.

Proposition 1. *Let M be a K-contact manifold with K-contact metric g. Then any Killing foliate vector field on M is a strictly contact vector field with basic contact hamiltonian.*

Proof. Let g be a fixed K-contact metric and let X be a Killing foliate vector field on M. Since α is the metric dual of ξ ([7]), one has, for any horizontal vector field A:

$$\begin{aligned} L_X\alpha(A) &= X\alpha(A) - \alpha([X,A]) = Xg(\xi, A) - g(\xi, [X,A]) \\ &= Xg(\xi, A) - g(\xi, [X,A]) - g([X,\xi], A) \\ &= L_X g(\xi, A) = 0. \end{aligned}$$

Therefore

$$L_X \alpha = h\alpha \tag{2}$$

for some function h, where $h = \mathrm{d}(\alpha(X))(\xi) = i_\xi \mathrm{d} i_X \alpha$ by formula (1). Note that

$$\alpha([X,\xi]) = g(\xi, [X,\xi]) = Xg(\xi,\xi) - L_X g(\xi,\xi) - g(\xi,[X,\xi]) = -\alpha([X,\xi]);$$

hence $\alpha([X,\xi]) = 0$, which implies that

$$[X,\xi] = 0, \tag{3}$$

since $[X,\xi]$ is a vertical vector field. Using the identities (2) and (3), we see that

$$h = L_X\alpha(\xi) = X\alpha(\xi) - \alpha([X,\xi]) = 0.$$

□

We shall denote by $\mathcal{L}$ the Lie algebra of Killing foliate vector fields. Any element X in $\mathcal{L}$ decomposes uniquely as

$$X = \bar{X} + \alpha(X)\xi.$$

The fact that $\alpha(X)$ is basic, implies that $\bar{X}$ commutes with ξ, that is $[\bar{X},\xi] = 0$. Also the identity

$$i_{\bar{X}}\mathrm{d}\alpha = -\mathrm{d}i_X\alpha \tag{4}$$

holds.

Remark. On a compact manifold M with an isometric flow, the Lie algebra $\mathcal{G}$ which describes the leaf closures is a subalgebra of $\mathcal{L}$.

Lemma 1. *Let M be a K-contact compact manifold with Lie algebra of Killing foliate vector fields $\mathcal{L}$. If X, $Y \in \mathcal{L}$ and $[X,Y] = 0$, then $[\bar{X},\bar{Y}] = 0$.*

Proof. The vanishing of $[X,Y]$ implies that

$$0 = [\bar{X} + \alpha(X)\xi, \bar{Y} + \alpha(Y)\xi] = [\bar{X},\bar{Y}] + [\bar{X}(\alpha(Y)) - \bar{Y}(\alpha(X))]\xi.$$

So $[\bar{X},\bar{Y}]$ is a vertical vector field and hence, applying identity (4),

$$\begin{aligned} 0 = i_{[\bar{X},\bar{Y}]}\mathrm{d}\alpha &= L_{\bar{X}}i_{\bar{Y}}\mathrm{d}\alpha - i_{\bar{Y}}L_{\bar{X}}\mathrm{d}\alpha \\ &= \mathrm{d}i_{\bar{X}}i_{\bar{Y}}\mathrm{d}\alpha + i_{\bar{X}}\mathrm{d}i_{\bar{Y}}\mathrm{d}\alpha - i_{\bar{Y}}i_{\bar{X}}\mathrm{d}^2\alpha - i_{\bar{Y}}\mathrm{d}i_{\bar{X}}\mathrm{d}\alpha \\ &= \mathrm{d}(\mathrm{d}\alpha(\bar{Y},\bar{X})), \end{aligned}$$

thus $\mathrm{d}\alpha(\bar{X},\bar{Y})$ is a constant function. But since $\mathrm{d}\alpha(\bar{X},\bar{Y}) = -\mathrm{d}(\alpha(X))(\bar{Y})$ must vanish at some point on the compact manifold M, we deduce that $\mathrm{d}\alpha(\bar{X},\bar{Y}) = 0$, which implies that

$$\alpha([\bar{X},\bar{Y}]) = -\mathrm{d}\alpha(\bar{X},\bar{Y}) = 0,$$

and therefore $[\bar{X},\bar{Y}] = 0$ since $[\bar{X},\bar{Y}]$ is a vertical vector field. □

Before we state our next proposition, we define the rank of a set of vector fields on a given manifold to be the maximum dimension of tangent planes spanned by elements of the set at any point on the manifold.

Proposition 2. *Let M be a K-contact compact $(2n+1)$-dimensional manifold and $\mathcal{L}_0$ a maximal abelian Lie subalgebra of Killing vector fields containing the characteristic vector field. Then the rank of $\mathcal{L}_0$ is at most equal to $n+1$.*

Proof. Each set of linearly independent local commuting Killing foliate vector fields $\{X_1,\ldots,X_k\}$ determines a local $(k-1)$-dimensional horizontal integrable distribution by Lemma 1, and it is well known that the maximum dimension of an integral submanifold of the contact distribution is n, so $k-1 \leq n$ and hence $k \leq n+1$. □

Corollary 1. *A leaf closure of a K-contact $(2n+1)$-dimensional compact manifold has at most dimension $n+1$.*

Proof. Corollary 1 is an immediate consequence of the fact that the leaf closures are orbits of $\mathcal{G}$, a subalgebra of some $\mathcal{L}_0$. □

4. Cohomological Rank of K-Contact Manifolds

Starting with a contact metric structure on M with structure tensors J and g, one defines an almost complex operator $\tilde{J}$ on $M \times \mathbf{R}$ by

$$\tilde{J}(X + f\frac{\mathrm{d}}{\mathrm{d}t}) = JX - \alpha(X)\frac{\mathrm{d}}{\mathrm{d}t} + f\xi.$$

If the almost complex structure defined by $\tilde{J}$ is integrable ([1]), then the original contact metric structure is called *Sasakian.* We refer to [1] for more properties of these structures. Let us point out, however, that we have the following inclusion of classes of flows:

Sasakian $\hookrightarrow$ *K-contact.*

On 3-dimensional compact manifolds, the inclusion is an identity ([8]).

In [7], it was shown that the basic first cohomology of a K-contact flow on a compact manifold is isomorphic to its first de Rham cohomology. It is therefore expected that the de Rham cohomology will affect the topology of those flows. We will show how the first Betti number of the manifold puts some restrictions on the maximum dimension of leaf closures.

In [8], examples of non-almost regular K-contact flows on the 3-dimensional sphere $\mathbf{S}^3$ were provided. As a consequence of our next theorem, such non-almost regular K-contact flows cannot be found on a 3-dimensional compact manifold with nonvanishing first Betti number.

We first prove the following elementary lemma.

Lemma 2. *Let Z be a harmonic vector field on a $(2n+1)$-dimensional K-contact manifold M with K-contact metric g. Then Z is a transverse vector field.*

Proof. Since $\beta = g(Z,.)$ is a harmonic 1-form, it is basic ([7]); in particular the identity $L_\xi\beta = 0$ holds. Therefore, for any vector field Y on M, one has

$$\begin{aligned} g([\xi, Z], Y) &= \xi g(Z,Y) - L_\xi g(Z,Y) - g(Z,[\xi,Y]) = \xi\beta(Y) - \beta([\xi,Y]) \\ &= L_\xi\beta(Y) = 0. \end{aligned}$$

Thus

$$[\xi, Z] = 0. \tag{5}$$

□

By the rank of a set of vector fields we mean the maximum dimension of tangent planes spanned by elements in the set at any point of the manifold.

Definition 4. We call the rank of the vector space of harmonic vector fields on a manifold M the *first cohomological rank* of M and denote it by $r_1(M)$.

It follows from [9] that on a compact Sasakian manifold M, $r_1(M)$ is zero or even. Also, observe that if $r_1(M)$ is attained at a point $p \in M$, then it is attained on an open neighborhood of p.

Theorem 1. *Let M be a compact K-contact $(2n+1)$-dimensional manifold with K-contact metric g and first cohomological rank $r_1(M)$. Then a leaf closure of the K-contact flow has dimension at most* $\inf\{n+1,\ 2n+1-r_1(M)\}$.

Proof. Let $\{Z_k\}$, $k \in \{1, 2, \ldots, r_1(M)\}$, be $r_1(M)$ linearly independent harmonic vector fields whose rank is $r_1(M)$ at some point $p \in M$, hence on an open set O containing p. Recall that orbit closures on K-contact flows on a compact manifold are orbits of a compact abelian group G of isometries, hence a torus. More precisely, denoting by φ_t the 1-parameter group of isometries generated by ξ and by $\mathcal{I}(M)$ the group of isometries of M, then G is the closure of φ_t in $\mathcal{I}(M)$. Keeping the same notation as in the previous section, let $\mathcal{G}$ denote the Lie algebra of G, so that elements in $\mathcal{G}$ are commuting Killing vector fields on M. Let $V \in \mathcal{G}$, then V is an infinitesimal automorphism of the flow. On the one hand, the flow has at least 2 closed characteristics ([7]) and V is proportional to ξ along each of them, hence $g(V, Z_k) = 0$ along those closed orbits. On the other hand, due to the fact that V is Killing and the Z_k are harmonic, the functions $g(V, Z_k)$ are constant. It follows that one has

$$g(V, Z_k) = 0$$

everywhere, in particular on the open set O where $r_1(M)$ is attained. Consequently, in O, the dimension of orbits of $\mathcal{G}$ is at most $2n + 1 - r_1(M)$. By Lemma 2, all Z_k commute with ξ. As a consequence of this, the open set O contains all orbits of its points. Therefore, thanks to the theory of compact transformation groups ([3], p. 43), the open set O contains an orbit whose closure has maximum dimension; thus the maximum dimension of a leaf closure is $2n + 1 - r_1(M)$. This combined with Corollary 1 completes the proof of Theorem 1. □

Corollary 2. *Let M be a K-contact compact 3-dimensional manifold with nonzero first Betti number. Then the contact flow on M is almost regular.*

Proof. Since in dimension 3, K-contact compact manifolds are Sasakian ([8]), it follows that if there is a nonzero harmonic 1-form on M, then $r_1(M) = 2$ and by Theorem 1, a leaf closure has dimension at most equal to $\inf\{2,\ 1\} = 1$. Hence, all characteristics are closed. □

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