VENKOV'S REDUCTION THEORY OF POSITIVE-QUADRATIC FORMS

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We prove the "absolute" finiteness of the number of faces (independent of the parameter $\varphi$ ) of Venkov's reduction domain $V(\varphi)$ (Izv. Akad. Nauk SSSR, Ser. Mat. 4, 37-52 (1940)) of $n$-ary positive quadratic forms. The case $n=3$ is given special consideration. We study the change of the reduction domain $V(\varphi)$ when $\varphi$ changes along a line segment in the space of coefficients.

The present paper is concerned with the reduction theory of positive quadratic forms according to Venkov [3]. The reader can get acquainted with the definitions and results of the reduction theory of positive quadratic forms through [1-11, 13-17]. Before we state the main result of this paper — Theorem 1 — we introduce some notation and definitions from [3, 9]. Let

$$f(x)=\sum_{i,j=1}^{n} a_{ij}x_i x_j,\ g(x)=\sum_{i,j=1}^{n} b_{ij}x_i x_j\ (a_{ij}=a_{ji}, b_{ij}=b_{ji}, i,j=1,\dots,n)$$

be two positive-definite quadratic forms; we write

$$(f,g)=\sum_{i,j=1}^{n} a_{ij}b_{ij}=a_{11}b_{11}+\dots+a_{nn}b_{nn}+2a_{12}b_{12}+\dots+2a_{n-1\,n}b_{n-1\,n} \tag{1}$$

The expression $(f,g)$ is called the Voronoi semiinvariant [5], sometimes also the scalar or inner product of $f(x)$ and $g(x)$. For

$$g(x)=\sum_{i=1}^{k}\mu_i(\lambda_{i1}x_1+\dots+\lambda_{in}x_n)^2$$

we obtain

$$(f,g)=\sum_{i=1}^{k}\mu_i f(\lambda_{i1},\dots,\lambda_{in}). \tag{2}$$

The form $F(x)=\sum_{i,j=1}^{n} A_{ij}x_i x_j=\bar{f}(x)$ is called adjoint to the form $f$, if the coefficients $A_{ij}$ are the cofactors of the elements $a_{ij}$ of the matrix $(a_{ij})$ of $f$. $fS$ denotes the quadratic form obtained from $f$ by the unimodular transformation $S$.

Definition. Let $\varphi$ be a given positive-definite quadratic form, $\bar{\varphi}$ its adjoint. We denote by $V(\varphi)$ the set of all positive quadratic forms $f$ satisfying the inequality

$$(f,\bar{\varphi})\leqslant(f,\bar{\varphi}S), \tag{3}$$

where $S$ runs through the set of all unimodular integral matrices. The form $\varphi$ is called the basic form of the Venkov reduction domain $V(\varphi)$, and the form $f$ is called $\varphi$-reduced or Venkov reduced with respect to the basic form $\varphi$.

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In [3] Venkov showed that the reduction domain $V(\varphi)$ is a simple gonohedron in the sense of the following definition of Ryshkov and Baranovskii [9]:

Definition. A quadratic form $f(x_1,\dots,x_n)$ of rank $r$ is called simple if it can be transformed by an integral unimodular transformation into the form $f(x_1,\dots,x_r)$. A bounded polyhedron $Q\subset \mathbb{K}$ ($\mathbb{K}$ is the positive cone) with finitely many faces is called simple if all its vertices correspond to simple quadratic forms. An unbounded pyramid with vertex $0$ the same as that of the positive cone $\mathbb{K}$ is called a positive gonohedron if its base is a simple polyhedron $Q\subset \mathbb{K}$.

THEOREM 1. Let $\Pi$ be a simple gonohedron. Then there exists a finite sequence of integral unimodular matrices $\mathfrak{B}(\Pi)=\{S_i\}_{i\in I}$ such that for every form $\overline{\varphi}\in\Pi$ all inequalities (3) which define the Venkov reduction domain $V(\varphi)$ are consequences of a finite number of inequalities

$$(f,\overline{\varphi})\leqslant(f,\overline{\varphi}S_i)\quad i\in\overline{I};\tag{4}$$

here $S_i$ runs through the set* $\mathfrak{B}(\Pi)$.

The proof of Theorem 1 will be given in Sec. 3. It is substantially different from the proof of the theorem about the finiteness of the number of independent inequalities defining the Venkov reduction domain $V(\varphi)$ (cf. [3, 16, 15]), where a positive-definite form $\varphi$ is fixed.

In Sec. 4 we consider the concrete situation where $n=3$, and choose for $\Pi$ the simple domain of Selling (Theorem 3). In the same section we evaluate (Theorem 4) the faces of the reduction domain $V(\varphi_0)$, where $\varphi_0=x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$, a form which Ryshkov [8] focused upon. Theorem 4 is a relatively simple consequence of Theorem 3.

In Sec. 5 we bring a result (Theorem 5) concerning the deformation of the Venkov reduction domain $V(\varphi)$, when $\varphi$ varies along a special line segment in the space of coefficients.

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## 2. PROOF OF AUXILIARY THEOREM

The proof of Theorem 1 rests on the following result.

THEOREM 2. Let $\Pi_1$ and $\Pi_2$ be simple gonohedra. Then there exists a finite set of unimodular integral matrices $\mathfrak{B}(\Pi_1,\Pi_2)=\{S_i\}_{i\in M}$ such that for all positive-definite quadratic forms $f_1\in\Pi_1$ and $f_2\in\Pi_2$ the set of all minima of

$$(f_1,\ f_2 S)\tag{5}$$

on the set of unimodular integral matrices $S\in GL(n,\mathbb{Z})$ equals $\mathfrak{B}(\Pi_1,\Pi_2)$.

First of all we note that for the proof of Theorem 2 we may assume, without loss of generality, that $\Pi_1$ and $\Pi_2$ are Minkowski reduction domains (multiple or single), since by Lemma 2.6 of [9] every simple gonohedron can be covered by a finite family of sets which are equivalent to Minkowski reduction domains. Every form from a Minkowski reduction domain has the form

*We stress that the set $\mathfrak{B}(\Pi)$ does not depend on the choice of the basic form $\varphi$.

$$f(x)=\sum_{i=1}^{m}\lambda_i\varphi_i(x_1,\dots,x_n),\ \lambda_1,\dots,\lambda_m\in\mathbb{R},\ \lambda_i\geqslant 0\ (i=1,\dots,m),$$

where $\varphi_1,\dots,\varphi_m$ are edge forms. If

$$f_1=\sum_{i=1}^{m}\lambda_i\varphi_i(x_1,\dots,x_n),\ f_2=\sum_{i=1}^{m}\mu_i\varphi_i(x_1,\dots,x_n),$$

then

$$(f_1,f_2S)=\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i\mu_j(\varphi_i,\varphi_jS);$$

each of the expressions $(\varphi_i,\varphi_jS)$ is a quadratic form in the elements of the matrix $S$. We have the following lemma which is a simple generalization of Lemma 1 of the paper by Crisalli [15].

LEMMA 1. Let $\varphi_i$ and $\varphi_j$ be edge forms of the Minkowski reduction domain, of rank $k$ and $\ell$, respectively. Then $(\varphi_i,\varphi_jS)$ is a positive-definite quadratic form in the elements of $S$ which are positioned at the intersection of rows number $n-\ell+1, n-\ell+2,\dots,n$ and columns number $n-k+1,\dots,n$.

LEMMA 2. Let $\mathfrak{A}$ be the class of all integral unimodular matrices whose elements in some given family of positions are fixed. Then the class $\mathfrak{A}$ contains a finite number of matrices $S$ such that $(f_1,f_2S)$ is a minimum for all positive-definite quadratic forms $f_1$ and $f_2$ which are Minkowski reduced.

The proof of this lemma is by reverse induction on the number of fixed elements in the matrices of the class. The lemma is obvious if all $n^2$ elements are fixed, i.e., if $\mathfrak{A}$ consists of one matrix $S$, which gives a minimum.

Assume now that our result holds for all classes of integral unimodular matrices in which no less than $k$ elements, $n^2\geqslant k\geqslant 1$, are fixed. Consider the class $\mathfrak{A}_1$ of integral unimodular matrices in which some $k-1$ elements are fixed.

Consider some matrix $S_1\in\mathfrak{A}_1$. Then we have for every $S\in\mathfrak{A}_1$ such that $(f_1,f_2S)$ is a minimum:

$$(f_1,f_2S)=\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i\mu_j(\varphi_i,\varphi_jS)\leqslant(f_1,f_2S_1)=\sum_{i=1}^{m}\sum_{j=1}^{m}\lambda_i\mu_j(\varphi_i,\varphi_jS_1). \tag{6}$$

The expressions $(\varphi_i,\varphi_jS)$ and $(\varphi_i,\varphi_jS_1)$ are equal if they depend only on the fixed elements of the matrices in the class $\mathfrak{A}$. Since $f_1$ and $f_2$ are positive-definite quadratic forms it follows from inequality (6) that there exists at least one set of indices $(i,j)$ with $\lambda_i>0, \mu_j>0$ such that

$$(\varphi_i,\varphi_jS)\leqslant(\varphi_i,\varphi_jS_1), \tag{7}$$

where both sides of the inequality depend also on nonfixed elements of matrices in $\mathfrak{A}_1$. By Lemma 1 all nonfixed elements of the matrix $S\in\mathfrak{A}_1$, on which $(\varphi_i,\varphi_jS)$ depends, must belong to a finite set. Since the number of edge forms of a Minkowski reduction domain is finite, it follows that all matrices $S$ in $\mathfrak{A}_1$ on which $(f_1,f_2S)$ takes a minimum belong to at least one of the finite number of classes in which no less than $k$ elements are fixed. Using the inductive hypothesis, we obtain Lemma 2.

Proof of Theorem 2. Since $GL(n,\mathbb{Z})$ is the class of all integral unimodular matrices in which the elements are fixed on an empty set of positions, it follows from Lemma 2 that there exists a finite set of matrices $S$ on which $(f_1, f_2 S)$ takes a minimum for all positive-definite quadratic forms $f_1, f_2$ which are Minkowski reduced. Theorem 2 is established.

## 3. PROOF OF THE MAIN THEOREM

Let $\mathfrak{M}$ be a Minkowski reduction domain, $\Pi$ a simple gonohedron and $\mathfrak{B}(\mathfrak{M},\Pi)$ a set of integral unimodular matrices such that for every $S\in\mathfrak{B}(\mathfrak{M},\Pi)$ there exist at least one positive-definite quadratic form $f\in\mathfrak{M}$ and one positive-definite quadratic form $\bar{\varphi}\in\Pi$, for which the value of $(f,\bar{\varphi}S)$ on the set of all integral unimodular matrices takes its minimum on $S$. By Theorem 2 $\mathfrak{B}(\mathfrak{M},\Pi)$ is finite. This means that all Venkov reduction domains $V(\varphi)$ with $\bar{\varphi}\in\Pi$ can be covered by a finite set of Minkowski domains which intersect in nondegenerate forms of the Venkov domain $V(\varphi)$ for all $\bar{\varphi}\in\Pi$.

It follows now from the definition of the Venkov reduction domain $V(\varphi)$ that every face of highest dimension of $V(\varphi)$ is defined by an equation of the form

$$(f,\bar{\varphi})=(f,\bar{\varphi}S), \tag{8}$$

where $S$ is an integral unimodular matrix. Since

$$(f,\bar{\varphi}S)=(fS^T,\bar{\varphi}),$$

we find that $S^T$ transforms the forms from face (8) of the domain $V(\varphi)$ into some other face of $V(\varphi)$.

For the proof of the main theorem it suffices to prove that the number of transformations of $\varphi$-reduced positive-definite quadratic forms into $\varphi$-reduced forms (i.e., into $V(\varphi)$) is finite for all $\bar{\varphi}\in\Pi$ and that they all belong to one finite set.

Let $f$ be any positive-definite form from the Minkowski reduction domain and let $S$ be any one of the transformations of $f$ of $\varphi$-reduced form. By definition of $\mathfrak{B}(\mathfrak{M},\Pi)$ and of the domain $V(\varphi)$ (for $\bar{\varphi}\in\Pi$) we obtain that $S^T\in\mathfrak{B}(\mathfrak{M},\Pi)$. Therefore, all transformations of $\varphi$-reduced forms into $\varphi$-reduced forms are among the transformations

$$\{(S_i^T)^{-1}S_j^T\}, \tag{9}$$

where $S_i, S_j\in\mathfrak{B}(\mathfrak{M},\Pi)$. Since $\mathfrak{B}(\mathfrak{M},\Pi)$ is finite we conclude that set (9) is finite. Since $\mathfrak{B}(\Pi)$ is contained in (9) it, too, is finite. This concludes the proof of Theorem 1.

## 4. CASE $n=3$ OF THE MAIN THEOREM FOR A SIMPLE SELLING REDUCTION DOMAIN

Quadratic forms in the Selling reduction domain as principal perfect domain have the following representation:

$$g=\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2+\lambda_4(x_1-x_2)^2+\lambda_5(x_1-x_3)^2+\lambda_6(x_2-x_3)^2 \tag{10}$$

where $\lambda_1,\lambda_2,\dots,\lambda_6\geqslant 0$. Let $S=(s_1,s_2,s_3)$ be an integral unimodular matrix with columns $s_1,s_2,s_3$. Then

$$(fS,g)=\lambda_1f(s_1)+\lambda_2f(s_2)+\lambda_3f(s_3)+\lambda_4f(s_1-s_2)+\lambda_5f(s_1-s_3)+\lambda_6f(s_2-s_3), \tag{11}$$

where the form $g$ has form (10). With every matrix $S$ we associate the augmented matrix

$$\tilde{S}=(S_1, S_2, S_3, S_1-S_2, S_1-S_3, S_2-S_3). \tag{12}$$

Note that the group $G_0$ of integral unimodular transformations of the squares $x_1^2, x_2^2, x_3^2$, $(x_1-x_2)^2, (x_1-x_3)^2, (x_2-x_3)^2$ transforms every Selling reduced form into another Selling reduced form. To the group $G_0$ correspond permutations (possibly with sign changes) of the columns in the augmented matrix (12).

A simple Selling reduction domain [13] contains forms of type (10) for which $0\le\lambda_4\le\lambda_5\le\lambda_6$, $\lambda_4\le\lambda_3, \lambda_5\le\lambda_1, \lambda_5\le\lambda_2$ .

THEOREM 3. The faces of highest dimension of the Venkov reduction domain $V(\varphi)$ for $\bar{\varphi}$ in the simple Selling domain are defined by integral unimodular matrices $S$ for which $|s_{ij}|\le 2$ .

The proof of this theorem is based on the proofs of Theorems 1 and 2 (for $\Pi_1$ and $\Pi_2$ simple Selling reduction domains) and the following Lemma 3.

LEMMA 3. Assume that $f$ does not lie on a face of the Selling reduction domain with dimension less than $N-1=\frac{n(n+1)}{2}-1=5$ . Then the augmented matrix $\tilde{S}$ of the matrix $S$ for which $(fS, \bar{\varphi})$ takes a minimum contains three linearly independent proximity vectors which are Selling reduced.

We apply Theorem 3 and Lemma 3 to the description of $V(\varphi_0)$ (cf. [8]) where $\varphi_0=x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$ is a principal perfect form.

THEOREM 4. The reduction domain $V(\varphi_0)$ is defined by the following 42 independent inequalities:

$$\begin{gathered}a_{ij}+a_{ik}\ge 0;\ 2a_{ii}-a_{ij}-a_{ik}\ge 0;\ 2a_{ii}-2a_{ij}-a_{ik}+a_{jk}\ge 0;\\ a_{ii}+6a_{ij}-2a_{ik}\ge 0;\ 5a_{ii}+2a_{ij}-6a_{ik}\ge 0;\\ 5a_{ii}+a_{jj}-6a_{ij}+2a_{ik}-2a_{jk}\ge 0;\ 5a_{ii}+a_{jj}-6a_{ij}-6a_{ik}+6a_{jk}\ge 0;\\ a_{ii}-2a_{ij}-2a_{ik}+4a_{jk}\ge 0;\ a_{ii}+a_{jj}+2a_{ij}-2a_{ik}-2a_{jk}\ge 0,\end{gathered}$$

where $i, j, k$ run through all permutations of the numbers 1, 2, 3.

The proof of the necessity of this system of inequalities follows from the theorem of Minkowski—Farkas [12, Lemma 2.4]. The totality of all these inequalities is obtained by means of Theorem 3 from the set of matrices (as subset of the matrices of Lemma 3) for which $(fS, \bar{\varphi}_0)$ takes a minimum under the condition that $f$ does not lie in a face of the simple Selling reduction domain of dimension less than $N-1=\frac{n(n+1)}{2}-1=5$ .

## 5. A CLASS OF VENKOV REDUCTION DOMAINS

In this section we put $n=3$ and determine the faces of highest dimension of the reduction domain $V(\varphi_t)$ , where $t$ runs through all real values from $0$ to $\frac{1}{2}$ , and

$$\bar{\varphi}_t=x_1^2+x_2^2+x_3^2+2t(x_1x_2+x_1x_3+x_2x_3).$$

For $t=0$ we obtain the symmetrized Minkowski reduction domain $\mathfrak{M}^c$ and for $t=\frac{1}{2}$ the Selling reduction domain of $\varphi$ .

Consider the following four sets of matrices:

$$\mathrm{I}\quad \pm\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix};\ \pm\begin{pmatrix}1&1&1\\0&-1&0\\0&0&-1\end{pmatrix};\ \pm\begin{pmatrix}1&1&1\\-1&0&0\\0&0&-1\end{pmatrix};\ \pm\begin{pmatrix}1&1&1\\-1&0&0\\0&-1&0\end{pmatrix};$$

$$\pm\begin{pmatrix}-1&0&0\\0&-1&0\\0&1&1\end{pmatrix};\ \pm\begin{pmatrix}-1&0&0\\0&0&-1\\0&1&1\end{pmatrix};\ \pm\begin{pmatrix}0&-1&0\\-1&0&0\\1&0&1\end{pmatrix};\ \pm\begin{pmatrix}0&-1&0\\0&0&-1\\1&0&1\end{pmatrix};\ \pm\begin{pmatrix}0&0&-1\\-1&0&0\\1&1&0\end{pmatrix};\ \pm\begin{pmatrix}0&0&-1\\0&-1&0\\1&1&0\end{pmatrix};$$

$$\mathrm{II}\quad \pm\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix};\ \pm\begin{pmatrix}1&1&0\\0&-1&0\\0&1&1\end{pmatrix};\ \pm\begin{pmatrix}1&0&1\\0&1&1\\0&0&-1\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\0&-1&0\\0&1&1\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\0&0&-1\\0&1&1\end{pmatrix};\ \pm\begin{pmatrix}0&1&0\\-1&0&0\\1&0&1\end{pmatrix};$$

$$\pm\begin{pmatrix}0&0&1\\-1&0&0\\1&1&0\end{pmatrix};\ \pm\begin{pmatrix}0&0&1\\0&-1&0\\1&1&0\end{pmatrix};\ \pm\begin{pmatrix}0&1&0\\0&0&-1\\1&0&1\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\-1&-1&0\\0&1&1\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\-1&0&-1\\0&1&1\end{pmatrix};\ \pm\begin{pmatrix}0&1&0\\-1&-1&0\\1&0&1\end{pmatrix};$$

$$\pm\begin{pmatrix}0&1&0\\0&-1&-1\\1&0&1\end{pmatrix};\ \pm\begin{pmatrix}0&0&1\\-1&0&-1\\1&1&0\end{pmatrix};\ \pm\begin{pmatrix}0&0&1\\0&-1&-1\\1&1&0\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\-1&0&-1\\1&1&1\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\-1&-1&0\\1&1&1\end{pmatrix};\ \pm\begin{pmatrix}0&1&0\\-1&-1&0\\1&1&1\end{pmatrix};$$

$$\pm\begin{pmatrix}0&1&0\\0&-1&-1\\1&1&1\end{pmatrix};\ \pm\begin{pmatrix}0&0&1\\-1&0&-1\\1&1&1\end{pmatrix};\ \pm\begin{pmatrix}0&0&1\\0&-1&-1\\1&1&1\end{pmatrix};$$

$$\mathrm{III}\quad \pm\begin{pmatrix}1&1&0\\1&0&1\\-1&-1&-1\end{pmatrix};\ \pm\begin{pmatrix}1&1&0\\0&1&1\\-1&-1&-1\end{pmatrix};\ \pm\begin{pmatrix}1&0&1\\0&1&1\\-1&-1&-1\end{pmatrix};$$

$$\mathrm{IV}\quad \pm\begin{pmatrix}1&\alpha&\beta\\0&\gamma&0\\0&0&\delta\end{pmatrix};\ \pm\begin{pmatrix}\alpha&1&\beta\\\gamma&0&0\\0&0&\delta\end{pmatrix};\ \pm\begin{pmatrix}\alpha&\beta&1\\\gamma&0&0\\0&\delta&0\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\0&\alpha&0\\0&\beta&\gamma\end{pmatrix};\ \pm\begin{pmatrix}1&0&0\\0&0&\alpha\\0&\beta&\gamma\end{pmatrix};$$

$$\pm\begin{pmatrix}0&1&0\\\alpha&0&0\\\beta&0&\gamma\end{pmatrix};\ \pm\begin{pmatrix}0&1&0\\0&0&\alpha\\\beta&0&\gamma\end{pmatrix};\ \pm\begin{pmatrix}0&0&1\\0&\alpha&0\\\beta&\gamma&0\end{pmatrix};\ \pm\begin{pmatrix}0&0&1\\\alpha&0&0\\\beta&\gamma&0\end{pmatrix};$$

here $\alpha, \beta, \gamma, \delta$ take independently the values $+1$ and $-1$. With each of the matrix families I-IV we associate also the matrices obtained from the given matrices by permuting the columns.

THEOREM 5. The Venkov reduction domain $V(\varphi_t)$, for $\bar{\varphi}_t = x_1^2 + x_2^2 + x_3^2 + 2t(x_1x_2 + x_1x_3 + x_2x_3)$, is defined by the independent inequalities

$$(f, \bar{\varphi}_t) \leqslant (f, \bar{\varphi}_t S),$$

where

1) for $0 < t < \frac{1}{2}$ the matrix $S$ runs through the sets I and II;

2) for $t = \frac{1}{2}$ the matrix $S$ runs through the sets I and III;

3) for $t = 0$ the matrix $S$ runs through the set IV.

If $S_1$ and $S_2$ are any two matrices from the sets I–IV, which differ only by a permutation of the columns, they yield a single independent inequality.

The theorem was proved earlier for $t = 0$ and $t = \frac{1}{2}$. We give here only the statement of a lemma on which the proof of the theorem rests.

LEMMA 4. Assume that the form $f$ satisfies the conditions of the theorem, with all inequalities strict. Then the coordinates of the proximity vectors of the form $f$ have absolute value not exceeding 1.

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# REPRESENTATION OF INTEGERS BY POSITIVE-TERNARY QUADRATIC FORMS

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The discrete ergodic method is applied to the problem mentioned in the title. We obtain estimates (of exact order) of the number of primitive representations in a given cone and a given residue class. The paper is an extension of the work of Peters (M. Peters, Acta Arithm., 34, 57-80 (1977)).

## 1. INTRODUCTION

1. Content of This Paper. The present paper is a continuation of the investigations [4, 5, 15] which deal with the application of the discrete ergodic method to the problem of representations of integers by positive ternary quadratic forms.

The introductory Sec. 1 contains basic definitions and notations and the statement of the main result of the paper: Theorem 1.1. At the end of this section we make a number of remarks on this result. The proof of Theorem 1.1 is contained in point 3 of Sec. 4.

The proof of Theorem 1.1 is based on the "key lemma" of the discrete ergodic method for generalized quaternions (Sec. 4, Proposition 4.3). Proposition 4.3 generalizes and unifies Lemmas 1 and 2 of Sec. 3 in Chap. V of [5]. With this generalization we can rectify some incorrect reasoning in [5, Chap. V, Secs. 4, 5].

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