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Periodic Solutions of the Differential Delay Equation $\dot{x}(t) = -f(x(t-1))$

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Abstract. For an odd function $f(x)$ defined only on a finite interval, this paper deals with the existence of periodic solutions and the number of simple periodic solutions of the differential delay equation (DDE) $\dot{x}(t) = -f(x(t-1))$. By use of the method of qualitative analysis combined with the constructing of special solutions a series of interesting results are obtained on these problems.

Keywords. Odd function, Periodic solution, Differential delay equation

1 Introduction

J. L. Kaplan and J. A. Yorke[1] made use of differential equations for the first time to discuss the conditions for the existence of periodic solutions of differential delay equations

$$\dot{x}(t) = -f(x(t-1)) \quad \text{and} \quad \dot{x}(t) = -f(x(t-1)) - f(x(t-2)),$$

where $f \in C^0(\mathbb{R}, \mathbb{R})$ being odd, $xf(x) > 0$ for $x \neq 0$. After then their results have been extended[2,3]. All those contributions are made under the condition that $f(x)$ is continuous on $\mathbb{R}$. This paper deals with the existence of periodic solutions and the number of simple periodic solutions of the differential delay equation

$$\dot{x}(t) = -f(x(t-1)), \tag{1.1}$$

where f satisfies

(H_0): $f \in C^0([-a, a], \mathbb{R}), f(-x) = -f(x)$ and $f(x) > 0$ for $x \in (0, a)$.

By the use of the method of qualitative analysis combined with constructing special solutions to Eq. (1.1) we give some new results in Section 2 about the existence. The number of simple periodic solutions is discussed in Section 3.

Definition 1.1. *Suppose both $x_1(t)$ and $x_2(t)$ are solutions of Eq.* (1.1). *Then $x_1(t)$ and $x_2(t)$ are said to be identical when there is a constant r such that $x_1(t) \equiv x_2(t+r)$.*

This definition is essential in counting the number of solutions.

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2 Existence of Periodic Solutions

We consider first the relation between Eq. (1.1) and the ordinary differential equations

$$\dot{x} = -f(y), \quad \dot{y} = f(x). \tag{2.1}$$

Lemma 2.1. *Suppose* (H_0) *holds. If Eqs.* (2.1) *has a nontrivial periodic solution* $(x(t), y(t))$ *with period* $4/(4l+1), l \geq 0$, *and with* $\|x\| < a$, *then* $x(t)$ *is a periodic solution of Eq.* (1.1). *Here* $\|x\| = \max|x(t)|$.

Proof. It is easy to see that all the trajectories of Eqs. (2.1) in the square $\{(x,y)||x| \leq a, |y| \leq a\}$ are determined by

$$F(x) + F(y) = c, \tag{2.2}$$

where $F(x) = \int_0^x f(\xi)d\xi$ for $|x| \leq a$ and $0 \leq c \leq 2F(a)$. When $0 < c \leq F(a)$ all the curves represented by Eq. (2.2) are closed and symmetric with respect to the x-axis and y-axis and each of them except for the case when $c = F(a)$ consists of a periodic solution of Eqs. (2.1). And $F(x) + F(y) = F(a)$ can represent a periodic solution of (2.1) only when the time that a moving point goes round the curve in a cycle is finite.

Since $(x(t), y(t))$ is a periodic solution of Eqs. (2.1) with period $4/(4l+1)$, we have

$$\dot{x}(t) = -f(y(t)), \quad \dot{y}(t) = f(x(t)). \tag{2.3}$$

Let $\Gamma = \{(x(t), y(t))|t \in \mathbb{R}\}$ and denote by A, B, C and D the four intersection points of Γ with the two axes in the counter-clockwise direction:

$$A = \left(x\left(t_0 + \frac{4k}{4l+1}\right), y\left(t_0 + \frac{4k}{4l+1}\right)\right) = (r, 0),$$
$$B = \left(x\left(t_1 + \frac{4k}{4l+1}\right), y\left(t_1 + \frac{4k}{4l+1}\right)\right) = (0, r),$$
$$C = \left(x\left(t_2 + \frac{4k}{4l+1}\right), y\left(t_2 + \frac{4k}{4l+1}\right)\right) = (-r, 0),$$
$$D = \left(x\left(t_3 + \frac{4k}{4l+1}\right), y\left(t_3 + \frac{4k}{4l+1}\right)\right) = (0, -r),$$

where $0 \leq t_0 < t_1 < t_2 < t_3 \leq 4/(4l+1), k$ is an integer. Γ is now determined by $F(x)+F(y) = F(r)$. It follows from the symmetry of the trajectory that $t_1 = t_0 + \frac{1}{4l+1}, t_2 = t_0 + \frac{2}{4l+1}, t_3 = t_0 + \frac{3}{4l+1}$. It is easy to verify that $(y(t), -x(t))$ and $\left(x\left(t - \frac{1}{4l+1}\right), y\left(t - \frac{1}{4l+1}\right)\right)$ are both solutions of Eqs. (2.1) and

$$(y(t_0), -x(t_0)) = \left(x\left(t_0 - \frac{1}{4l+1}\right), y\left(t_0 - \frac{1}{4l+1}\right)\right) = (0, -r).$$

Since the uniquencess of solution of Eqs. (2.1) for any given initial conditions holds when $|x|, |y| < a$, we have

$$(y(t), -x(t)) = \left(x\left(t - \frac{1}{4l+1}\right), y\left(t - \frac{1}{4l+1}\right)\right).$$

It follows that

$$y(t) = x\left(t - \frac{1}{4l+1}\right) = -y\left(t - \frac{2}{4l+1}\right) = x\left(t - \frac{1}{4l+1} - \frac{4l}{4l+1}\right) = x(t-1),$$

and therefore

$$\dot{x}(t) = -f(y(t)) = -f(x(t-1)).$$

Lemma 2.2. *Suppose* (H_0) *holds and* $F(a) = 0$. *Let* $(p,q) \in \Gamma = \{(x,y)|F(x)+F(y) = F(a)\}, p,q > 0$ *and* $(x_0(t), y_0(t))$ *be a solution of Eqs.* (2.1) *with* $(x_0(t_0), y_0(t_0)) = (p,q)$. *Suppose* $s_1, s_2 > 0$ *are two constants such that* $x(t), y(t) > 0$ *for* $t \in (t_0 - s_1, t_0 + s_2)$ *and* $\lim_{t\to t_0-s_1}(x_0(t), y_0(t)) = (a, 0)$ *as well as* $\lim_{t\to t_0+s_2}(x_0(t), y_0(t)) = (0, a)$. *If* $s_1 + s_2 < \infty$, *then Eq.* (2.1) *has a* $4(s_1+s_2)$*-periodic solution* $(x(t), y(t))$ *satisfying*

1) $(x(t+s_1+s_2), y(t+s_1+s_2)) = (-y(t), x(t))$;

2) $\max_{t\in\mathbb{R}} |x(t)| = a$.

Proof. Since the trajectories of Eqs. (2.1) are determined by the family of curves (2.2), we have

$$F(x_0(t)) + F(y_0(t)) = F(a), \tag{2.4}$$

$$\dot{x}_0(t) = -f(y_0(t)), \quad \dot{y}_0(t) = f(x_0(t)) \tag{2.5}$$

for $t \in (t_0 - s_1, t_0 + s_2)$. Set $s = s_1 + s_2$. Define $(x(t), y(t))$ as

$$\begin{aligned}
(x(t), y(t)) &= (x_0(t-4ks), y(t-4ks)), \\
&\quad t \in [t_0 - s_1 + 4ks, t_0 + s_2 + 4ks); \\
(x(t), y(t)) &= (-y_0(t-(4k+1)s), x_0(t-(4k+1)s)), \\
&\quad t \in [t_0 - s_1 + (4k+1)s, t_0 + s_2 + (4k+1)s); \\
(x(t), y(t)) &= (-x_0(t-(4k+2)s, -y_0(t-(4k+2)s)), \\
&\quad t \in [t_0 - s_1 + (4k+2)s, t_0 + s_2 + (4k+2)s); \\
(x(t), y(t)) &= (y_0(t-(4k+3)s), -x_0(t-(4k+3)s)), \\
&\quad t \in [t_0 - s_1 + (4k+3)s, t_0 + s_2 + (4k+3)s),
\end{aligned}$$

where $k = 0, \pm 1, \cdots$. Clearly the vector function $(x(t), y(t))$ is $4s$-periodic and satisfies the requirements 1) and 2). We now prove that $(x(t), y(t))$ is a solution of Eqs. (2.1).

When $t \in [t_0 - s_1 + (4k+1)s, t_0 + s_2 + (4k+1)s)$, let $r = t - (4k+1)s$. Then we have $r \in [t_0 - s_1, t_0 + s_2)$ and

$$\dot{x}(t) = -\dot{y}_0(r) = -f(x_0(r)) = -f(y(t)),$$

$$\dot{y}(t) = \dot{x}_0(r) = -f(-y_0(r)) = -f(-x(t)) = f(x(t)).$$

Obviously $(x(t), y(t))$ satisfies Eqs. (2.1). The other three cases can be proved in a similar way. Besides, $(x(t), y(t))$ is continuous and differentiable at each of the connection points. Therefore $(x(t), y(t))$ is a $4s$-periodic solution of Eqs. (2.1).

We now consider a segment of curve $F(x) + F(y) = F(a), \quad x, y \geq 0$.

When (H_0) holds, $F : [0,a] \rightharpoonup [0,F(a)] \subset \mathbb{R}$ is a strictly monotonic function and hence F^{-1} exists on $[0,F(a)]$. Since $F(a) - F(x) \in [0,F(a)]$ for $x \in [0,a]$, we define

$$H(x) := f[F^{-1}(F(a) - F(x))] \quad \text{for} \quad x \in [0,a].$$

Obviously $H : [0,a] \rightharpoonup [0,a]$ is a continuous mapping.

Theorem 2.1. *Suppose (H_0) holds. If* $\lim\limits_{x\to 0} f(x)/x = \alpha > \frac{\pi}{2}(4k+1)$ *and* $\int_0^a \frac{dx}{H(x)} > \frac{1}{4k+1}$ *for an integer* $k \geq 0$, *then Eq.* (1.1) *has at least one* $4/(4k+1)$*-periodic solution* $x(t) : |x(t)| < a$.

Proof. Take $(x_0,y_0) \in \Gamma = \{(x,y)|F(x)+F(y) = f(a)\}$, $x_0,y_0 > 0$. Denote by $(x_1(t),y_1(t))$ the solution of Eqs. (2.1) satisfying $(x_1(t_0),y_1(t_0)) = (x_0,y_0)$. Then there are $s_1,s_2 > 0$ such that $(x_1(t_0),y_1(t_0)) = (x_0,y_0)$ and $(x_1(t_0-s_1),y_1(t_0-s_1)) = (a,0),(x_1(t_0+s_2),y_1(t_0+s_2)) = (0,a)$. We have $f(y_1(t)) = f[F^{-1}(F(a) - F(x_1(t))] = H(x_1(t))$ and then

$$dt = -dx_1(t)/f(y_1(t)) = -dx_1(t)/H(x_1(t)).$$

Therefore

$$s_1 + s_2 = -\int_a^0 \frac{dx}{H(x)} = \int_0^a \frac{dx}{H(x)} > 1/(4k+1).$$

That's to say, the time that a moving point goes along the trajectory from $(a,0)$ to $(0,a)$ is greater than $1/(4k+1)$. It follows from the continuous dependence of solutions upon the initial conditions that there is $b < a$, sufficiently near a, such that the time that a point goes along the trajectory from $(b,0)$ to $(0,b)$ is greater than $\frac{1}{4k+1}$, too. The trajectory r_b determined by

$$F(x) + F(y) = F(b) \tag{2.6}$$

is a closed curve. It follows from the symmetry of the trajectories of Eqs. (2.1) that the period T_b of the closed trajectory r_b is greater than $4/(4k+1)$.

Let $c = F(s), s \in (0,a)$ in Eq. (2.2). For a trajectory $r_s := \{(x_s(t),y_s(t))|t \in \mathbb{R}\}$ determined by $F(x) + F(y) = F(s)$, let $x = \rho\cos\theta, y = \rho\sin\theta$. Then we have the equivalent equations

$$\begin{cases} \dot\rho = -f(\rho\sin\theta)\cos\theta + f(\rho\cos\theta)\sin\theta, \\ \dot\theta = \frac{1}{\rho}[f(\rho\cos\theta)\cos\theta + f(\rho\sin\theta)\sin\theta]. \end{cases} \tag{2.7}$$

Clearly $\lim\limits_{s\to 0} F(s) = 0$ implies $\lim\limits_{s\to 0} \rho = 0$. Therefore $\forall\varepsilon \in (0, \alpha - \frac{\pi}{2}(4k+1))$, there is an s small enough such that

$$\dot\theta = \frac{f(\rho\cos\theta)}{\rho\cos\theta}\cos^2\theta + \frac{f(\rho\sin\theta)}{\rho\sin\theta}\sin^2\theta > \alpha - \varepsilon > \frac{\pi}{2}(4k+1).$$

So the period T_s of the closed trajectory $r_s : F(x) + F(y) = F(s)$ is less than $4/(4k+1)$. Since the periods of the closed trajectories change continuously, there is a closed trajectory $r_d, \{(x(t),y(t))\}$, of Eqs. (2.1), $d \in (0,a)$, whose period equals $4/(4k+1)$. Then Lemma 2.1 implies that $x(t)$ is a periodic solution with period $4/(4k+1)$. It is clear that $|x(t)| < a$.

This theorem is now proved.

Corollary 2.1. *Suppose* (H_0) *holds,* $\underline{\lim}_{x\to 0} f(x)/x = \alpha > \frac{\pi}{2}(4k+1)$ *and* $\int_0^a \frac{dx}{H(x)} > 1/(4l+1)$, *where* $k \geq 1 \geq 0$. *Then Eq.* (1.1) *has at least* $k - l + 1$ *different nontrivial periodic solutions* $x_i(t)$ *with periods* $4/(4i+1)$, *respectively, and* $|x_i(t)| < a, i = l, l+1, \cdots, k$.

This can be directly deduced from Theorem 2.1.

Corollary 2.2. *Suppose* (H_0) *holds. If* $\lim_{x\to 0} f(x)/x = \infty$, *then Eq.* (1.1) *has infinite different periodic solutions.*

Proof. Since $\int_0^a \frac{dx}{H(x)}$ is a positive number, there is an integer $k_0 > 0$ such that

$$\int_0^a \frac{dx}{H(x)} > 1/(4k+1),$$

for $k \geq k_0$. At the same time $\lim_{x\to 0} f(x)/x > \frac{\pi}{2}(4k+1)$. It follows from Theorem 2.1 that Eq. (1.1) has at least one $4/(4k+1)$-periodic solution $x_k(t)$ for every $k \geq k_0$. The corollary is now proved.

Theorem 2.2. *Suppose* (H_0) *holds and* $f(a) = 0$. *If* $\int_0^a \frac{dx}{H(x)} \leq 1/(4k+1)$ *for an integer* $k \geq 0$, *then Eq.* (1.1) *has at least a* $4/(4k+1)$*-periodic solution* $x_k(t)$ *with* $\max\{x_k(t)\} = a$.

Proof. Let $\int_0^a \frac{dx}{H(x)} = s$ and take $(x_0, y_0) \in \{(x,y) | F(x) + F(y) = F(a)\}, x_0, y_0 > 0$. Suppose $(x_0(t), y_0(t))$ is the solution of Eqs. (2.1) satisfying $(x_0(t_0), y_0(t_0)) = (x_0, y_0), (x_0(t_0 - s_1), y_0(t_0 - s_1)) = (a, 0), (x_0(t_0 + s_2), y_0(t_0 + s_2)) = (0, a)$ and $x_0(t), y_0(t) > 0$ for $t \in (t_0 - s_1, t_0 + s_2)$. It follows from $F(x_0(t)) + F(y_0(t)) = F(a)$ and the first equation of Eqs. (2.1) that

$$s_1 + s_2 = -\int_{t_0 - s_1}^{t_0 + s_2} \frac{\dot{x}_0(t)dt}{f[F^{-1}(F(a) - F(x_0(t))]} = -\int_a^0 \frac{dx}{H(x)} = s.$$

Then Eqs. (2.1) has a $4s$-periodic solution $(x^*(t), y^*(t))$ satisfying

1) $(x^*(t+s), y^*(t+s)) = (-y^*(t), x^*(t))$,

2) $\max |x(t)| = a$,

since Lemma 2.2 holds. Property 1) implies $x^*(t+s) = -y^*(t) = -x^*(t-s)$ or $x^*(t) = -y^*(t-s) = -x^*(t-2s)$. Then

$$\dot{x}^*(t) = -f(y^*(t)), \quad \dot{y}^*(t) = f(x^*(t)).$$

Let $t_1 \in [0, 4s], (x^*(t_1), y^*(t_1)) = (a, 0)$. Then

$$(x^*(t_1 + s), y^*(t_1 + s)) = (-y^*(t_1), x^*(t_1)) = (0, a),$$

$$(x^*(t_1 + 2s), y^*(t_1 + 2s)) = (-y^*(t_1 + s), x^*(t_1 + s)) = (-a, 0)$$

and $(x^*(t_1 + 3s), y^*(t_1 + 3s)) = (0, -a)$.

Let $\sigma = \frac{1}{4k+1} - s$. We define $(x(t), y(t))$ as follows:

$$(x(t), y(t)) : \; = (a, 0), \qquad t \in \left[t_1 + \frac{4i}{4k+1}, t_1 + \frac{4i}{4k+1} + \sigma\right);$$

$$(x(t), y(t)) : \; = \left(x^*\left(t - \frac{4i}{4k+1} - \sigma\right), y^*\left(t - \frac{4i}{4k+1} - \sigma\right)\right),$$

$$t \in \left[t_1 + \frac{4i}{4k+1} + \sigma, t_1 + \frac{4i+1}{4k+1}\right);$$

$$(x(t), y(t)) : \; = (0, a), \qquad t \in \left[t_1 + \frac{4i+1}{4k+1}, t_1 + \frac{4i+1}{4k+1} + \sigma\right);$$

$$(x(t),y(t)):\ =\left(x^*\left(t-\frac{4i}{4k+1}-2\sigma\right),y^*\left(t-\frac{4i}{4k+1}-2\sigma\right)\right),\quad t\in\left[t_1+\frac{4i+1}{4k+1}+\sigma,t_1+\frac{4i+2}{4k+1}\right);$$

$$(x(t),y(t)):\ =(-a,0),\qquad t\in\left[t_1+\frac{4i+2}{4k+1},t_1+\frac{4i+2}{4k+1}+\sigma\right);$$

$$(x(t),y(t)):\ =\left(x^*\left(t-\frac{4i}{4k+1}-3\sigma\right),y^*\left(t-\frac{4i}{4k+1}-3\sigma\right)\right),\quad t\in\left[t_1+\frac{4i+2}{4k+1}+\sigma,t_1+\frac{4i+3}{4k+1}\right);$$

$$(x(t),y(t)):\ =(0,-a),\qquad t\in\left[t_1+\frac{4i+3}{4k+1},t_1+\frac{4i+3}{4k+1}+\sigma\right);$$

$$(x(t),y(t)):\ =\left(x^*\left(t-\frac{4i}{4k+1}-4\sigma\right),y^*\left(t-\frac{4i}{4k+1}-4\sigma\right)\right),\quad t\in\left[t_1+\frac{4i+3}{4k+1}+\sigma,t_1+\frac{4i+4}{4k+1}\right).$$

Obviously $(x(t),y(t))$ is a continuous and differentiable periodic vector function with period $4/(4k+1)$. We now prove that $(x(t),y(t))$ is a solution of Eqs. (2.1).

When $t\in\left[t_1+\frac{4i}{4k+1},t_1+\frac{4i}{4k+1}+\sigma\right)$, we have $f(x(t))=f(y(t))=0$ since $x(t)=a$, $y(t)=0$. Then

$$\dot{x}(t)=-f(y(t)),\quad \dot{y}(t)=f(x(t)) \tag{2.8}$$

holds. When $t\in\left[t_1+\frac{4i}{4k+1}+\sigma,t_1+\frac{4i+1}{4k+1}\right)$, we have

$$\dot{x}(t)=\dot{x}^*\left(t-\frac{4i}{4k+1}-\sigma\right)=-f\left[y^*\left(t-\frac{4i}{4k+1}-\sigma\right)\right]=-f(y(t)),$$

$$\dot{y}(t)=y^*\left(t-\frac{4i}{4k+1}-\sigma\right)=f\left[x^*\left(t-\frac{4i}{4k+1}-\sigma\right)\right]=f(x(t)).$$

The other cases can be proved in a similar way.

We now prove that $(x(t),y(t))$ satisfies

1) $\left(x\left(t+\frac{1}{4k+1}\right),y\left(t+\frac{1}{4k+1}\right)\right)=(-y(t),x(t))$ and

2) $\max|x(t)|=a$.

It is obvious that $\max|x(t)|=a$.

When $t\in\left[t_1+\frac{4i}{4k+1},t_1+\frac{4i}{4k+1}+\sigma\right)$, then $(x(t),y(t))=(a,0)$ and $\left(x\left(t_1+\frac{1}{4k+1}\right),\right.$ $\left.y\left(t_1+\frac{1}{4k+1}\right)\right)=(0,a)$ imply that the relation in 1) holds.

When $t\in\left[t_1+\frac{4i}{4k+1}+\sigma,t_1+\frac{4i+1}{4k+1}\right)$, then $t+\frac{1}{4k+1}\in\left[t_1+\frac{4i+1}{4k+1}+\sigma,t_1+\frac{4i+2}{4k+1}\right)$. So

$$\begin{aligned}\left(x\left(t+\frac{1}{4k+1}\right),y\left(t+\frac{1}{4k+1}\right)\right)&=\left(x^*\left(t+\frac{1-4i}{4k+1}-2\sigma\right),y^*\left(t+\frac{1-4i}{4k+1}-2\sigma\right)\right)\\&=\left(x^*\left(t-\frac{4i}{4k+1}-\sigma+s\right),y^*\left(t-\frac{4i}{4k+1}-\sigma+s\right)\right)\\&=\left(-y^*\left(t-\frac{4i}{4k+1}-\sigma\right),x^*\left(t-\frac{4i}{4k+1}-\sigma\right)\right)\\&=(-y(t),x(t)).\end{aligned}$$

The relation in 1) also holds. The other cases can be proved in a similar way.

Since $(x(t), y(t))$ is a solution of Eqs. (2.1) satisfying $\left(x\left(t+\frac{1}{4k+1}\right), y\left(t+\frac{1}{4k+1}\right)\right) = (-y(t), x(t))$, i.e., $(x(t), y(t)) = \left(-y\left(t-\frac{1}{4k+1}\right), x\left(t-\frac{1}{4k+1}\right)\right)$, it follows from the first equation in (2.1) that

$$\dot{x}(t) = -f(y(t)) = -f\left(x\left(t-\frac{1}{4k+1}\right)\right) = -f\left(x\left(t-\frac{1}{4k+1}-\frac{4k}{4k+1}\right)\right) = -f(x(t-1)).$$

Therefore $x(t)$ is a $4/(4k+1)$-periodic solution of Eq. (1.1). The proof is now completed.

Corollary 2.3. *Suppose* (H_0) *holds. If* $f(a) = 0$ *and* $\int_0^a \frac{dx}{H(x)} \le \frac{1}{4k+1}$ *for an integer* $k \ge 0$, *then Eq.* (1.1) *has at least* $k+1$ *different periodic solutions,* $x_l(t)$, *with period* $4/(4l+1), l = 0, 1, \cdots, k$.

Corollary 2.4. *Suppose* (H_0) *holds and* $f(x) = f(a-x)$ *for* $x \in [0, a]$. *If* $\int_0^a \frac{dx}{f(x)} \le \frac{1}{4k+1}$, *then Eq.* (1.1) *has at least* $k+1$ *different periodic solutions* $x_l(t) : \max|x_l(t)| = a$, *with periods* $4/(4k+1), l = 0, 1, \cdots, k$. *Here* $k \ge 0$ *is an integer.*

Proof. Clearly $f(0) = f(a) = 0$. Therefore it suffices to prove

$$H(x) = f(x). \tag{2.9}$$

In fact, the condition $f(x) = f(a-x)$ for $x \in [0, a]$ implies

$$F(a) - F(x) = \int_x^a f(\xi)d\xi \overset{u=a-\xi}{=} \int_0^{a-x} f(a-u)du = \int_0^{a-x} f(u)du = F(a-x),$$

$$H(x) = f[F^{-1}(F(a) - F(x))] = f[F^{-1}(F(a-x))] = f(a-x) = f(x).$$

So this corollary is a direct deduction drawn from Theorem 2.2.

Theorem 2.3. *Suppose* (H_0) *holds and* $f(a) = 0$. *If* $\lim_{x\to 0} f(x)/x = \alpha > \frac{\pi}{2}(4k+1)$ *for an integer* $k \ge 0$, *then Eq.* (1.1) *has at least* $k+1$ *different periodic solutions with periods* $4/(4l+1), l = 0, 1, \cdots, k$.

Proof. These periodic solutions are confirmed by Corollary 2.1 when $\int_0^x \frac{dx}{H(x)} > 1$ and by Corollary 2.3 when $\int_0^a \frac{dx}{H(x)} \le \frac{1}{4k+1}$. As for the case where $\int_0^x \frac{dx}{H(x)} = \beta : \frac{1}{4l+1} < \beta \le \frac{1}{4l-3}, 1 \le l \le k$, the $k-l+1$ periodic solutions are ensured by Corollary 2.1 and the other l ones by Corollary 2.3.

Remark. When $f(x)$ has finite discontinuous points of the first type, i.e., for any one of its discontinuity points, say , x_0, both $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ exist and are finite, all the above theorems remain valid.

Example 2.1. Consider a differential delay equation

$$\dot{x}(t) = -\alpha f(x(t-1)), \tag{2.10}$$

where $\alpha > 0$ is a parameter and

$$f(x) = \begin{cases} -\sqrt{x(1+x)}, & x \in [-1, 0], \\ \sqrt{x(1-x)}, & x \in [0, 1]. \end{cases}$$

Obviously $f(x)$ satisfies all requirements of Corollary 2.4. We know from $\lim_{x\to 0} \alpha f(x)/x = +\infty$ that $\forall\alpha > 0$ Eq. (2.10) has infinitely many oscillating periodic solutions $x_l(t)$ with periods $4/(4l+1), l = 0,1,2,\cdots$. Furthermore $\int_0^a \frac{dx}{f(x)} = \pi$ implies that when $\alpha \geq \pi(4l+1)$ for an integer $k \geq 0$, among these periodic solutions there are at least $k+1$ ones with amplitudes being 1.

3 Number of Simple Periodic Solutions

Definition 3.1. *An ω-periodic solution $x(t)$ of Eq. (1.1) is said to be simple periodic when the trajectory $\{(x(t), \dot{x}(t)) | t \in \mathbb{R}\}$ is a simple closed curve in the $x, \dot{x}$-plane.*

This definition is consistent with that in [5] when $f(x)$ is continuous on $\mathbb{R}$ and $xf(x) > 0$ for $x \neq 0$.

We have given in [5] a result as follows (see [5], Theorem 2.1).

Theorem 3.1. *Suppose $f \in C^0(\mathbb{R}, \mathbb{R})$ being odd, $f'(x) \geq 0$ and there is a constant $\varepsilon > 0$ such that $f'(x)$ is monotonically decreasing on $(0, \varepsilon)$ and not increasing on (ε, ∞). If there are two integers $k, n \geq 0$ such that*

$$\frac{\pi}{2}\max\{0, 4k-3\} \leq f'(\infty) < \frac{\pi}{2}(4k+1) \leq \frac{\pi}{2}[4(k+n)+1] < f'(0) \leq \frac{\pi}{2}[4(k+n)+5],$$

then Eq. (1.1) has exactly $n+1$ simple periodic solutions with periods $4/[4(k+l)+1], l = 0, 1, \cdots, n$.

When $f(x)$ is defined only on a finite interval $[-a, a]$, we give the following theorem.

Theorem 3.2. *Suppose (H_0) holds, $f \in C^1([-a,a],[0,\infty))$ and there is $\varepsilon \in (0,a)$ such that $f'(x)$ is monotonically decreasing on $(0,\varepsilon)$ and not increasing on (ε, a). If*

$$f(a) \leq a, \quad \frac{\pi}{2}[4n+1] < f'(0) \leq \frac{\pi}{2}[4n+5],$$

then Eq. (1.1) has exactly $n+1$ simple periodic solutions, $x_l(t)$, with periods $4/(4l+1), l = 0, 1, \cdots, n$.

Proof. We define a function as follows:

$$g(x) = \begin{cases} f(x), & x \in [-a,a] \\ f(a) + f'(a)(x-a), & x > a, \\ -f(a) + f'(a)(x+a), & x < -a. \end{cases}$$

Obviously $g(x)$ satisfies the requirements of Theorem 3.1. Since

$$\frac{f(a)}{a} = \frac{f(a)-f(0)}{a-0} < 1,$$

there is $\xi \in (0,a)$ such that $f'(\xi) \leq 1$ and therefore $f'(a) \leq f'(\xi) \leq 1$. The fact that $g'(x) = f'(a)$ for $|x| > a$ implies $g'(\infty) \leq 1$.

Theorem 3.1 tells us that

$$\dot{x}(t) = -g(x(t-1)) \tag{3.1}$$

has exactly $n+1$ simple periodic solutions $x_l(t)$ with periods $\frac{4}{4l+1}, l = 0, 1, 2, \cdots, n$.

Suppose $\max|x_l(t)| = m$. We now prove $m < a$. Supposing to the contrary $m \geq a$, then $|g(x)| \leq f(a) + f'(a)(m-a) \leq m$. The equality does not hold for all $|x| < m$.

Let $x_l(t_0) = m$. Then

$$\dot{x}_l(t_0) = -g(x_l(t_0 - 1)) = 0.$$

Therefore $x_l(t_0 - 1) = 0$ and

$$x_l(t_0) = \int_{t_0-1}^{t_0} g\left(x_l\left(t - \frac{1}{4l+1}\right)\right) dt < m,$$

a contradiction.

Because $g(x) = f(x)$ for $|x| \le a$ and $|x_l(t)| < a, x_l(t)$ is also a periodic solution of Eq. (1.1). Obviously Eq. (1.1) has no other simple periodic solutions. The theorem is proved.

Example 3.1. Let $f(x) = (1 - |\sin x|)\sin x, x \in [-\pi/2, \pi/2]$. Consider

$$\dot{x}(t) = -\alpha f(x(t-1)), \tag{3.2}$$

where $\alpha > 0$ is a parameter. Clearly $f(x)$ is odd and continuous on $[-\pi/2, \pi/2]$.

Because $\lim_{x\to 0} f(x)/x = 1$ and $f(\pi/2) = 0, f(x) > 0$ for $x \in (0, \pi/2)$ Eq. (3.2) has periodic solutions when $\alpha > \pi/2$. Furthermore $f(x)$ reaches its maximum 1/4 at $x = \pi/6$ and $f'(x) \ge 0$ is monotonically decreasing on interval $(0, \pi/6)$. We can show as in the proof of Theorem 3.2 that all the periodic solutions of Eq. (3.2) have their maxima less than $\pi/6$ when $\alpha \le \frac{2}{3}\pi$. So when $\alpha \in \left(\frac{\pi}{2}, \frac{2}{3}\pi\right)$, Eq. (3.2) has one and only one simple periodic solution, which is 4-periodic.

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