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Solutions of LRS Bianchi I Space-time Filled with a Perfect Fluid

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LRS Bianchi I space-time filled with a perfect fluid is considered and it is shown that the field equations are solvable for any arbitrary cosmic scale function. Solutions for a particular form of cosmic sclae functions are presented and all solutions, except for some cases, are shown to represent an empty universe for large time.

1. INTRODUCTION

The metric for LRS Bianchi I space-time is of the form [1]

$$ds^2 = -dt^2 + A^2dx^2 + B^2(dy^2 + dz^2) \tag{1}$$

where A and B are functions of cosmic time t.

In the case of an energy momentum tensor of a perfect fluid type, i.e.

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \qquad u_\mu u^\mu = -1 \tag{2}$$

where u^μ is the four-vector velocity, p the pressure and ρ the mass-energy density, the Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = KT_{\mu\nu} \tag{3}$$

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are as follows:

$$-Kp = \frac{2B''}{B} + \frac{B'^2}{B^2} \tag{4}$$

$$-Kp = \frac{B''}{B} + \frac{A''}{A} + \frac{A'B'}{AB} \tag{5}$$

$$K\rho = \frac{2A'B'}{AB} + \frac{B'^2}{B^2} \tag{6}$$

where K is Einstein's gravitational constant and the prime indicates derivative with respect to time t.

In recent papers Hajj-Boutros and and Sfeila [2] and Shri Ram [3] obtained some solutions of the above field equations by using their solution-generating techniques. In the present note, we show that field equations (4)–(6) are solvable for any arbitrary cosmic scale function $A(t)$ or $B(t)$. For a particular form of each $A(t)$ and $B(t)$, some solutions are presented spearately and the solutions of Refs. 2 and 3 are shown to be special cases of our solutions. Kinematic properties of all solutions are also studied.

2. SOLUTIONS AND THEIR PROPERTIES

From (4) and (5), the condition of isotropy of pressure is

$$\frac{B''}{B} + \frac{B'^2}{B^2} - \frac{A''}{A} - \frac{A'B'}{AB} = 0$$

which can be integrated to give

$$B^2A' - ABB' = \ell \tag{7}$$

where ℓ is an integrating constant.

Considering (7) as a linear differential equation for $A(t)$, where $B(t)$ is an arbitrary function, we obtain

$$A = c_1B + \ell B \int \frac{dt}{B^3(t)} \tag{8}$$

where c_1 is an integrating constant.

Similarly from (7) we obtain

$$B^2 = c_2A^2 - 2\ell A^2 \int \frac{dt}{A^3(t)} \tag{9}$$

where c_2 is an integrating constant and $A(t)$ is an arbitrary function.

Therefore, for any given $B(t)$ from (8) one can obtain $A(t)$ and then from (4) and (6), p and ρ can be calculated, i.e. for any given function $B(t)$, field equations are solvable. Similarly by using (9), field equations (4)–(6) can be solved for any given $A(t)$.

Choosing $B = t^{1/2(1-n)}$ (where n is a real number satisfying $n \neq \frac{1}{3}$) from (8) we obtain

$$A = c_1 t^{1/2(1-n)} + \frac{2\ell}{3n-1} t^n .$$

For this solution, the metric is

$$ds^2 = -dt^2 + \left(c_1 t^{1/2(1-n)} + \frac{2\ell}{3n-1} t^n \right)^2 dx^2 + t^{1-n}(dy^2 + dz^2) \qquad (10)$$

where $n \neq \frac{1}{3}$.

For the model (10), from (4) and (6) we obtain

$$Kp = \frac{1 + 2n - 3n^2}{4t^2} \qquad (11)$$

$$K\rho = \frac{2\ell(1 + 2n - 3n^2)t^{1/2(3n-1)} + 3c_1(n-1)^2)3n-1)}{4t^2[2\ell t^{1/2(3n-1)} + c_1(3n-1)]} . \qquad (12)$$

For this model, all the fluids are acceleration and rotation free.

For solution (10), spatial volume V^3 $(= AB^2)$, scalar expansion θ $(= 3V'/V)$ and shear scalar σ $(= \frac{1}{2}\sigma_{ab}\sigma^{ab})$ are given by

$$V^3 = \frac{t[c_1(3n-1) + 2t^{1/2(3n-1)}]}{(3n-1)t^{1/2(3n-1)}} \qquad (13)$$

$$\theta = \frac{4\ell t^{1/2(3n-1)} + 3c_1(1-n)(3n-1)}{2t[2\ell t^{1/2(3n-1)} + c_1(3n-1)]} \qquad (14)$$

$$\sigma = \left(\frac{3}{2}\right)^{1/2} \frac{2\ell(n - 1/3)t^{1/2(3n-1)}}{t[2\ell t^{1/2(3n-1)} + c_1(3n-1)]} . \qquad (15)$$

For $n = 1$ and for $n = -\frac{1}{3}$ from (11) we obtain $p = 0$ which represents a dust universe.

For $n \neq 1, -\frac{1}{3}, \frac{1}{3}$ from (11)–(15) it is seen that at the singularity state $t = 0$, $V^3 \to 0$ and p,ρ,θ and σ are infinitely large. At $t \to \infty$, $V^3 \to \infty$ and p,ρ,θ and σ vanish. Therefore, for $n \neq 1, -\frac{1}{3}, \frac{1}{3}$. the solution (10) represents an anisotropic universe exploding from $t = 0$, which expands for $0 < t < \infty$ and after an infinitely large time t, would give essentially an isotropic empty universe.

Choosing $A = k_1 t^{1/2(1-n)} + k_2 t^n$ in (9) we obtain

$$ds^2 = -dt^2 + (k_1 t^{1/2(1-n)} + k_2 t^n)^2 dx^2 + (\ell_1 t^{1-n} + \ell_2 t^{1/2(1+n)} + \ell_3 t^{2n})(dy^2 + dz^2) \quad (16)$$

where $\ell_1 = c_2 k_1^2 + (2\ell/3n - 1)$, $\ell_2 = 2c_2 k_1 k_2$, $\ell_3 = k_2^2$, $n \neq \frac{1}{3}$.

Expressions for p,ρ,θ and σ for the model (16) are not given here, but it is seen that properties of (16) are the same as that of the solution (10).

Choosing $B = t^{1/3}$ in (8) we obtain

$$ds^2 = -dt^2 + t^{2/3}(c_1 + \ell \ln t)^2 dx^2 + t^{2/3}(dy^2 + dz^2). \quad (17)$$

For $n = 0$ and $n = -\frac{1}{3}$ from (10) one can obtain the solutions of Hajj-Boutros and Sfeila [2]. For $n = 0$ and $n = -\frac{1}{3}$ from (16) we obtain the solutions of Shri Ram [3]. For $\ell = 0$, $c_1 = 1$, the solution (17) represents the Einstein–de Sitter universe.

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