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Fluid Dynamics in Higher Order Gravity

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We examine the kinematic and dynamic properties of fluid spacetimes in higher order gravity. In particular we extend the general equations of Ehlers and Ellis governing relativistic fluid dynamics from general relativity to the higher order theory. We find exact results for the evolution of shear in Bianchi spacetimes with isotropic surfaces, thus generalising the general relativity results. Furthermore we show that the vanishing of vorticity, shear and acceleration does not imply FRW geometry in $R+\alpha R^2$ gravity without the further assumption of a barotropic equation of state, $p = p(\rho)$, $p'(\rho) \neq 0$. In particular, this result means that the Ehlers–Geren–Sachs theorem on cosmic background radiation also holds in the higher order theory.

1. INTRODUCTION

Higher order theories emerged from semi-classical quantum corrections to the Einstein theory [1]. Inclusion of terms proportional to R^2 and $R_{ab}R^{ab}$ in the gravitational Lagrangian produces a stabilisation of the divergences and permits renormalisation (at the expense of S-matrix unitarity). It turns out that these corrections which arise from the expectation value of the vacuum energy-momentum tensor take the geometrical form of additional curvature terms in the field equations. Moreover, the additional terms can produce inflationary expansion [1–3], without invoking an extraneous inflaton scalar field. (In fact the addition of an R^2 term to the

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Einstein Lagrangian leads to a theory conformally equivalent to Einstein's theory with scalar field source; Ref. 4.) Higher order theories also produce modifications to Newton's law in the weak field limit which have important implications for the problem of dark matter [5].

The fact that quantum vacuum correction terms take a purely geometrical form allows one to treat the higher order field equations as one would the Einstein vacuum equations—the difference being the far greater complexity of the former (in particular the fact that they lead in general to fourth-order differential equations). An extension to the non-vacuum case is then possible. Although the higher order theory arises from a study of the vacuum and is applicable to the very early universe, it is also possible and interesting to consider, in addition to the vacuum terms, a classical fluid energy-momentum tensor, which would represent the non-vacuum contribution to the gravitational field (treated classically) [6]. As the universe expands, the influence of the vacuum terms rapidly dies away and the (classical) non-vacuum energy-momentum terms emerge as dominant. In the early universe, the vacuum terms may be dominant.

We aim in this paper to pursue such an approach and, more specifically, to investigate some of the exact kinematic and dynamic properties of fluid spacetimes in higher order gravity. This represents an extension of aspects of the work of Ehlers [7] and Ellis [8] from Einstein's theory to the higher order theory. In particular, we generalise results on the kinematic characterisation of FRW universes and on the dissipation of shear anisotropy, showing how the higher order correction terms can significantly modify the standard results of Einstein's theory.

In Section 2, we extend the general equations governing relativistic fluid dynamics from General Relativity to higher order gravity. In particular, we present the space-time decomposition of the propagation and field equations (including the generalised Raychaudhuri equation), and the generalised Bianchi identities. These equations are applied in Section 3 to derive the main results of the paper.

Firstly, we generalise the integration of the shear (with and without viscosity) in Bianchi spacetimes with isotropic hypersurfaces, from General Relativity to higher order gravity. The results show that in general the dissipation of shear may differ markedly from the Einstein case, depending non-linearly on the evolution of the Ricci scalar. In the particular case of Bianchi I spacetimes we are able to give exact expressions for the shear in terms of the metric scale factor, and to reduce the problem to a single second-order ordinary differential equation. We show that shear dissipates more slowly (though negligibly so) due to the higher order corrections. Our exact results extend the vacuum numerical and qualitative results of

Berkin [9].

Secondly, we show that the kinematic characterisation (zero acceleration, vorticity and shear) of FRW geometry in General Relativity no longer applies in higher order gravity. In order to force spatial homogeneity and isotropy onto the metric, it is necessary to impose in addition the assumption that the fluid obeys a barotropic equation of state, $p = p(\rho)$ (with p non-constant). It is perhaps surprising that this is the only additional assumption needed, since the equations are far more complex (in particular, they do not automatically lead to conformal flatness). In the case of an ultra-relativistic fluid, we show that the Ehlers–Geren–Sachs theorem on cosmic background radiation generalises from Einstein gravity to higher order gravity thus averting a potentially serious constraint on the higher order theory, while also extending the depth and generality of the original theorem.

The additional non-linearity introduced by the higher order terms requires one to be more careful with conventions than in General Relativity. We follow the conventions of [9]: The metric signature is $(-+++)$ and the curvature tensors are defined by

$$R^a{}_{bcd} = -\Gamma^a{}_{bc,d} + \ldots, \qquad R_{ab} = R^c{}_{acb}\,.$$

In addition, we use units in which $c = 1 = 8\pi G$. Latin indices $a, b, \ldots = 0, 1, 2, 3$ refer to a general basis. In a coordinate system $\{x^0, x^\alpha\}$ ($\alpha = 1, 2, 3$), the spatial coordinates are denoted by Greek indices.

2. FLUID KINEMATICS AND DYNAMICS IN HIGHER ORDER GRAVITY

We first give a brief summary of the relevant equations of relativistic fluid dynamics [7,8]. The fluid 4-velocity vector field u^a defines a projection tensor $h_{ab} = g_{ab} + u_a u_b$ which projects into the instantaneous rest space of a comoving observer. The Weyl tensor C_{abcd} can be split by u^a into an "electric" part E_{ab} and a "magnetic" part H_{ab}:

$$E_{ab} = C_{acbd}u^c u^d, \qquad H_{ab} = \tfrac{1}{2}\eta_{acde}u^c C^{de}{}_{bf}\, u^f \tag{1}$$

where $\eta^{abcd} = \eta^{[abcd]}$, $\eta^{0123} = |g|^{-1/2}$. E_{ab} and H_{ab} are trace-free, symmetric tensors in the rest space of u^a. The stress-energy-momentum tensor T_{ab} can be decomposed as

$$T_{ab} = \rho u_a u_b + p h_{ab} + q_a u_b + u_a q_b + \pi_{ab} \tag{2}$$

where ρ is the total energy density, p is the isotropic pressure, π_{ab} is the anisotropic pressure tensor ($\pi_{ab}u^b = \pi^a{}_a = \pi_{[ab]} = 0$), and q^a is the energy flux vector ($q_a u^a = 0$), all measured by a comoving observer. The 4-velocity gradient decomposes as

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - \dot{u}_a u_b \tag{3}$$

where ω_{ab} is the skew vorticity tensor ($\omega_{ab}u^b = 0$), σ_{ab} is the symmetric shear tensor ($\sigma_{ab}u^b = \sigma^a{}_a = 0$), θ is the volume expansion ($\theta = u^a{}_{;a}$), and $\dot{u}_a$ is the 4-acceleration ($\dot{u}_a = u_{a;b}u^b$). The tensors ω_{ab} and σ_{ab} define scalars $\omega^2 = \frac{1}{2}\omega^{ab}\omega_{ab}$ and $\sigma^2 = \frac{1}{2}\sigma^{ab}\sigma_{ab}$. The vorticity tensor defines the vorticity vector $\omega^a = \frac{1}{2}\eta^{abcd}u_b\omega_{cd}$, and the volume expansion defines a representative length ℓ along the flowlines by $\theta = 3\dot{\ell}/\ell$. The conservation equations $T^{ab}{}_{;b} = 0$ imply

$$\dot{\rho} + (\rho + p)\theta + \pi_{ab}\sigma^{ab} + q^a{}_{;a} + q^a\dot{u}_a = 0 \tag{4}$$

$$(\rho + p)\dot{u}_a + h_a{}^b(p_{;b} + \pi_b{}^c{}_{;c} + \dot{q}_b) + (\omega_{ab} + \sigma_{ab} + \tfrac{4}{3}\theta h_{ab})q^b = 0 \tag{5}$$

which are the equations of thermal energy conservation and momentum conservation respectively.

Quadratic corrections to the gravitational Lagrangian (motivated in part by quantum field theory) are of the form [1]

$$\mathcal{L} = R + \alpha R^2 + \beta R_{ab}R^{ab} \tag{6}$$

where α and β are constants which are usually taken as positive for quantum physical reasons [10]. (Inclusion of a term proportional to $R^{abcd}R_{abcd}$ is unnecessary due to the Gauss–Bonnet relation.) The field equations derived from (6) are then [11]

$$\begin{aligned} T_{ab} = {} & R_{ab} - \tfrac{1}{2}Rg_{ab} + 2\alpha[R(R_{ab} - \tfrac{1}{4}Rg_{ab}) - R_{;ab} + g_{ab}\Box R] \\ & + \beta[2(R_{acbd} - \tfrac{1}{4}g_{ab}R_{cd})R^{cd} - R_{;ab} + \tfrac{1}{2}g_{ab}\Box R + \Box R_{ab}] \end{aligned} \tag{7}$$

where $\Box = g^{ab}\nabla_a\nabla_b$ and T_{ab} represents a (classical) fluid. Equation (7) implies the conservation equations (4) and (5). It obviously reduces to Einstein's field equations when $\alpha = 0 = \beta$. However (7) also reduces to Einstein's field equations when g_{ab} is an Einstein metric, and therefore also in the special case when g_{ab} is of constant curvature. We can rewrite (7) as

$$\begin{aligned} R_{ab} = {} & (1 + 2\alpha R)^{-1}[T_{ab} - \beta\Box R_{ab} + (2\alpha + \beta)R_{;ab} - 2\beta R_{acbd}R^{cd} \\ & + \{-(2\alpha + \tfrac{1}{2}\beta)\Box R + \tfrac{1}{2}R(1 + \alpha R) + \tfrac{1}{2}\beta R_{cd}R^{cd}\}g_{ab}]. \end{aligned} \tag{8}$$

We define

$$S_{ab} = 2\alpha R_{;ab} + \beta(R_{;ab} - \Box R_{ab} - 2R_{acbd}R^{cd})$$

so $S_{ab} = 0$ when $\alpha = 0 = \beta$. Using eqs. (2) and (8) we can split the field equations as in the Einstein case [8], making clear the role of the higher order terms:

$$R = \rho - 3p + (6\alpha + 2\beta)\Box R \tag{9}$$

$$R_{ab}u^a u^b = (1 + 2\alpha R)^{-1}[\rho - \tfrac{1}{2}R(1 + \alpha R) + (2\alpha + \tfrac{1}{2}\beta)\Box R - \tfrac{1}{2}\beta R_{ab}R^{ab} + S_{ab}u^a u^b] \tag{10}$$

$$R_{ab}u^a h^b{}_c = (1 + 2\alpha R)^{-1}[-q_c + S_{ab}u^a h^b{}_c] \tag{11}$$

$$R_{ab}h^a{}_c h^b{}_d = (1 + 2\alpha R)^{-1}[\pi_{cd} + \{p + \tfrac{1}{2}R(1 + \alpha R) + (2\alpha + \tfrac{1}{2}\beta)\Box R + \tfrac{1}{2}\beta R_{ab}R^{ab}\}h_{cd} + S_{ab}h^a{}_c h^b{}_d]. \tag{12}$$

Note that by (9), $R = 0$ implies $\rho = 3p$. However, unlike the Einstein theory, the converse is not true: for a radiation fluid ($\rho = 3p$), the Ricci scalar satisfies a Klein–Gordon equation. In general (9) allows one to express $\Box R$ in terms of R, ρ and p.

Following the Ehlers/Ellis formalism in General Relativity [7,8], we contract and project the Ricci identity for u^a to obtain a propagation equation for $v_{ab} = h_a{}^c h_b{}^d u_{c;d}$

$$h_a{}^c h_b{}^d \dot{v}_{cd} - \dot{u}_a \dot{u}_b - h_a{}^c h_b{}^d \dot{u}_{c;d} + v_{ad}v^d{}_b + R_{acbd}u^c u^d = 0. \tag{13}$$

The trace of (13) yields the *higher order generalisation of the Raychaudhuri equation* [the field equation (10)]:

$$\dot{\theta} + \tfrac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^a_{;a} + \tfrac{1}{2}(1 + 2\alpha R)^{-1}(\rho + 3p) = (1 + 2\alpha R)^{-1}[\tfrac{1}{2}\alpha R^2 + (\alpha + \tfrac{1}{2}\beta)\Box R + \tfrac{1}{2}\beta R_{ab}R^{ab} - S_{ab}u^a u^b]. \tag{14}$$

In general (i.e. $\alpha \neq 0 \neq \beta$) there is no obvious way of determining the effect each term has on the expansion. Even the simplification $\beta = 0$ does not yield any obvious generalisation of Raychaudhuri's theorem. Indeed the role of the higher order curvature terms in (14) could be to prevent the singularity predicted by the standard Raychaudhuri equation. Solutions with this behaviour are known [12].

The skew part of (13) is the *higher order vorticity propagation equation*, which is identical to the equation in the Einstein theory (since $R_{abcd}u^b u^d$ is symmetric):

$$h_a{}^c h_b{}^d(\dot{\omega}_{cd} - \dot{u}_{[c;d]}) - 2\sigma^d{}_{[a}\omega_{b]d} + \tfrac{2}{3}\theta\omega_{ab} = 0. \tag{15}$$

A further consequence of the generalised equation (14) is that, unlike in General Relativity, *static dust solutions are not ruled out by the Raychaudhuri equation in higher order gravity.* The static symmetry means that $\xi_a = \xi u_a$ is a hypersurface orthogonal Killing vector. Then from Killing's equation and the hypersurface orthogonality of u^a, $\omega = \theta = \sigma = 0$. From the conservation equation (5), vanishing pressure implies that the 4-acceleration is zero. Then, unlike in General Relativity Raychaudhuri's equation *does not* lead to vanishing energy density. Rather, eq. (14) gives

$$\rho = \alpha R^2 + (2\alpha+\beta)\Box R + \beta R_{ab}R^{ab} - 2S_{ab}u^a u^b$$

The symmetric and trace-free part of (13) is the *higher order shear propagation equation* which, using (1) and (14), is

$$\begin{aligned} &h_a{}^c h_b{}^d[\dot\sigma_{cd} - \dot u_{(c;d)}] - \dot u_a \dot u_b + \omega_a\omega_b + \sigma_{ac}\sigma^c{}_b + \tfrac{2}{3}\theta\sigma_{ab} \\ &+ \tfrac{1}{3}h_{ab}[\dot u^c_{;c} - \omega^2 - 2\sigma^2] + E_{ab} - \tfrac{1}{2}(1+2\alpha R)^{-1}\pi_{ab} \\ &+ \tfrac{1}{6}(1+2\alpha R)^{-1}(h_{ab}h^{cd} - 3h_a{}^c h_b{}^d)S_{cd} = 0. \end{aligned} \tag{16}$$

Directly from the Ricci identity for u^a we get three *higher order constraint equations*:

$$\begin{aligned} &h^{ab}(\tfrac{2}{3}\theta_{;b} - \sigma_{bc;d}h^{cd}) - \eta^{abcd}u_b(\omega_{c;d} + \omega_c\dot u_d) \\ &\quad = (1+2\alpha R)^{-1}[q^a - h^{ab}S_{bc}u^c] \end{aligned} \tag{17}$$

which is the field equation (11), and

$$\begin{aligned} &h^{ab}\omega_{a;b} = \dot u^a\omega_a \\ &H_{ab} = 2\dot u_{(a}\omega_{b)} - h_{(a}{}^c h_{b)d}[\omega_{ce;f} + \sigma_{ce;f}]\eta^{defg}u_g \end{aligned} \tag{18}$$

which have the same form as in the Einstein theory. The field equation (12) has yet to be given in terms of the fluid variables. We will do so shortly, in the case of irrotational flow ($\omega = 0$). First we give the equations governing the evolution of the Weyl tensor components E_{ab} and H_{ab}, i.e. the Bianchi identities, which are consistency equations for the field equations. From the definition of the Weyl tensor, following the Ehlers/Ellis formalism [7,8], we obtain the *higher order generalisation of the Bianchi identities.*

The divergence equation for E_{ab}:

$$\begin{aligned} &h^{ab}E_{bc;d}h^{cd} - \eta^{abcd}u_b\sigma_{ce}H^e{}_d + 3H^{ab}\omega_b \\ &\quad = (1+2\alpha R)^{-1}[\tfrac{1}{3}h^{ab}\rho_{;b} - \tfrac{1}{2}h^{ab}\pi_b{}^c{}_{;c} + \tfrac{1}{2}\pi^{ab}\dot u_b + \tfrac{1}{2}(\sigma^{ab} - 3\omega^{ab})q_b - \tfrac{1}{3}\theta q^a] \\ &\quad + \tfrac{1}{2}(1+2\alpha R)^{-1}h^{ab}[h^{cd}\{\beta\Box R_{cd} - (2\alpha+\beta)R_{;cd} - 2\beta R_{cedf}R^{ef}\}_{;b} \\ &\qquad + \{(2\alpha + \tfrac{1}{2}\beta)\Box R + \tfrac{3}{2}\beta R_{cd}R^{cd}\}_{;b} \\ &\qquad - \dot S_{bc}u^c - 2\alpha\{(R_{cd}u^c u^d + \tfrac{1}{3}R_{bc}u^c\}]. \end{aligned} \tag{19}$$

The divergence equation for H_{ab}:

$$\begin{aligned}
&h^{ab}H_{bc;d}h^{cd} + \eta^{abcd}u_b\sigma_{ce}E^e_{\ d} - 3E^{ab}\omega_b \\
&\quad = (1+2\alpha R)^{-1}[(\rho+p)\omega^a + \tfrac{1}{2}\eta^{abcd}u_b q_{c;d} + \tfrac{1}{2}\eta^{abcd}u_b\pi_{ce}(\sigma^e{}_d + \omega^e{}_d)] \\
&\qquad - (1+2\alpha R)^{-1}[\tfrac{1}{2}\eta^{abcd}u_b S_{ce;d}u^e - \alpha\eta^{abcd}u_b R_{ce}u^e R_{;d}]. \qquad (20)
\end{aligned}$$

The propagation equation for E_{ab}:

$$\begin{aligned}
&h^{ac}h^{bd}\dot{E}_{cd} + h^{c(a}\eta^{b)def}H_{cd;e}u_f - 2H_c{}^{(a}\eta^{b)cde}u_d\dot{u}_e \\
&\quad + h^{ab}E^{cd}\sigma_{cd} + \theta E^{ab} - 3E^{c(a}\sigma^{b)}{}_c - E^{c(a}\omega^{b)}{}_c \\
&\quad = (1+2\alpha R)^{-1}[-\tfrac{1}{2}(\rho+p)\sigma^{ab} - q^{(a}\dot{u}^{b)} - \tfrac{1}{2}h^{c(a}h^{b)d}q_{c;d} - \tfrac{1}{2}h^{c(a}\tfrac{1}{2}\pi^{c(a}\omega^{b)}{}_c \\
&\qquad - \tfrac{1}{2}\pi^{c(a}\sigma^{b)}{}_c - \tfrac{1}{6}\theta\pi^{ab} + \tfrac{1}{6}(\pi_{cd}\sigma^{cd} + q^c{}_{;c} + \dot{u}^c q_c)] \\
&\qquad + \tfrac{1}{2}(1+2\alpha R)^{-1}h^{c(a}h^{b)d}[\{S_{de;c} - 2\alpha R_{;c}R_{de}\}u^e - \dot{S}_{cd} \\
&\qquad\quad + \{\tfrac{1}{6}\beta(\Box R)^{\cdot} - \tfrac{1}{2}\beta(R_{ef}R^{ef})^{\cdot}\}g_{cd} + 2\alpha\dot{R}(R_{cd} - \tfrac{1}{3}Rg_{cd}]. \qquad (21)
\end{aligned}$$

The propagation equation for H_{ab}:

$$\begin{aligned}
&h^{ac}h^{bd}\dot{H}_{cd} - h^{c(a}\eta^{b)def}E_{cd;e}u_f + 2E_c^{(a}\eta^{b)cde}u_d\dot{u}_e \\
&\quad + h^{ab}H^{cd}\sigma_{cd} + \theta H^{ab} - 3H^{c(a}\sigma^{b)}{}_c - H^{c(a}\omega^{b)}{}_c \\
&\quad = (1+2\alpha R)^{-1}[-\tfrac{1}{2}h^{c(a}\eta^{b)def}\pi_{cd;e}u_f \\
&\qquad - \tfrac{3}{2}q^{(a}\omega^{b)} + \tfrac{1}{2}\sigma_c{}^{(a}\eta^{b)cde}q_d u_e + \tfrac{1}{2}h^{ab}\omega^c q_c] \\
&\quad + \tfrac{1}{2}(1+2\alpha R)^{-1}h^{c(a}\eta^{b)def}u_f[S_{cd;e} - 2\alpha R_{cd}R_{;e}]. \qquad (22)
\end{aligned}$$

These equations show how the free gravitational field evolution is governed by the mass distribution. Note that all the Ricci tensor terms in (19)–(22) disappear in the Einstein theory. In higher order gravity, the Ricci tensor is directly linked to the Weyl tensor via the Bianchi identities.

We complete this system of equations in the case of zero rotation ($\omega = 0$), when there exist spacelike hypersurfaces $\{t = \text{constant}\}$ orthogonal to u^a. These surfaces have a 3-metric tensor $g^*{}_{ab} = h_{ab}$ which defines a 3-connection $\nabla_a{}^*$ and a 3-Riemann tensor $R^*{}_{abcd}$, where [8]

$$\begin{aligned}
R^*{}_{abcd} = {} & R^*{}_{ac}h_{bd} - R^*{}_{ad}h_{bc} + R^*{}_{bd}h_{ac} \\
& - R^*{}_{bc}h_{ad} - \tfrac{1}{2}R^*(h_{ac}h_{bd} - h_{ad}h_{bc}). \qquad (23)
\end{aligned}$$

The field equations (12) are then given by splitting the 3-Ricci tensor into its trace

$$\begin{aligned}
R^* = {} & 2\sigma^2 - \tfrac{2}{3}\theta^2 \\
& + (1+2\alpha R)^{-1}[2\rho + \alpha R^2 + \beta\Box R + 3\beta R_{ab}R^{ab} + 2h^{ab}S_{ab}] \quad (24)
\end{aligned}$$

and trace-free parts

$$R^*{}_{ab} - \tfrac{1}{3}R^* h_{ab} = h_a{}^c h_b{}^d[\dot{u}_{(c;d)} - \dot{\sigma}_{cd}] - \theta\sigma_{ab} + \dot{u}_a\dot{u}_b - \tfrac{1}{3}h_{ab}\dot{u}^c_{;c} + (1+2\alpha R)^{-1}\pi_{ab} + \tfrac{1}{3}(1+2\alpha R)^{-1}(3h_a{}^c h_b{}^d - h_{ab}h^{cd})S_{cd}. \tag{25}$$

These are the *higher order generalisations of the Gauss–Codazzi equations.*

3. SHEAR DISSIPATION AND THE KINEMATIC CHARACTERISATION OF FRW COSMOLOGIES IN HIGHER ORDER GRAVITY

The higher order field and evolution equations derived in Section 2 are extremely complex. The reduction of the Lagrangian, and consequently the field equations, in the case $\alpha \neq 0 = \beta$ leads to a somewhat more tractable set of equations. We turn now to apply the equations of Section 2 (with $\beta = 0$) to some questions relevant in cosmology. The results derived from these equations in the fluid model can differ markedly from those in the Einstein case ($\alpha = 0 = \beta$).

3.1. Shear Dissipation

Consider the Bianchi spacetimes whose homogeneous hypersurfaces have isotropic curvature $R^*{}_{ab} = \frac{1}{3}R^* h_{ab}$. These include the Bianchi models that can reach an FRW limit via dissipation of the shear anisotropy σ_{ab}. We also assume that the fluid has zero bulk viscosity and energy flux. In Einstein's theory, Ehlers [7] showed that the expansion itself dissipates shear, and Ellis [8] showed that shear viscosity accelerates the dissipation. We will generalise these results to the higher order theory $\alpha \neq 0 = \beta$.

Spatial homogeneity implies $\dot{u}^a = 0 = \omega$ and $h_a{}^b f_{;b} = 0$ for any geometrically or physically defined scalar f. From eq. (25)

$$\dot{\sigma}_{ab} + \theta\sigma_{ab} = (1+2\alpha R)^{-1}[\pi_{ab} + \tfrac{2}{3}\alpha(3h_a{}^c h_b{}^d - h_{ab}h^{cd})R_{;cd}]. \tag{26}$$

The shear propagation equation (16) determines the electric part of the Weyl tensor, E_{ab}. Using $R_{;ab} = \ddot{R}u_a u_b - \dot{R}(\sigma_{ab} + \frac{1}{3}\theta h_{ab})$ eq. (26) becomes

$$\ell^{-3}(\ell^3\sigma_{ab})^{\cdot} = (1+2\alpha R)^{-1}[\pi_{ab} - 2\alpha\dot{R}\sigma_{ab}]. \tag{27}$$

In the case of a *perfect fluid* (zero anisotropic pressure), (27) integrates along the flow lines to give

$$\sigma_{ab} = (1+2\alpha R)^{-1}\Sigma_{ab}\ell^{-3}, \qquad \dot{\Sigma}_{ab} = 0 \tag{28}$$

which reduces to Ehlers' result [7] when $\alpha = 0$ (or $R = 0$). It is surprising that we are able to integrate for the shear in the higher order theory. Now (28) implies

$$\sigma^2 = (1 + 2\alpha R)^{-2}\Sigma^2\ell^{-6} \tag{29}$$

where Σ is constant. In Einstein's theory, (29) implies that shear diverges at the big bang and then dissipates via expansion:

$$\sigma^2 \to \infty \quad \text{as} \quad \ell \to 0, \qquad \sigma^2 \to 0 \quad \text{as} \quad \ell \to \infty.$$

This behaviour is modified in the higher order theory. The nature of the modification depends non-linearly on the Ricci scalar R, which itself is linked to the shear via the generalised Raychaudhuri equation (14). Thus it is not in general possible to say how the higher order terms affect the evolution of shear. However we can give exact answers in the case of Bianchi I spacetime. Berkin [9] used numerical and phase plane techniques to investigate the consequences of shear anisotropy for inflation in the vacuum Bianchi I spacetimes. Berkin showed that the anisotropy enhances inflation, and that the anisotropy is dissipated by expansion. We can use (29) to obtain expressions for the shear which are exact and apply to the non-vacuum case.

It is straightforward to show that the Ricci scalar for the Bianchi I metric is

$$R = 2\dot{\theta} + \tfrac{4}{3}\theta^2 + 2\sigma^2. \tag{30}$$

Then (29) implies

$$\sigma^6 + A\sigma^4 + \tfrac{1}{4}A^2\sigma^2 - \Sigma^2/(16\alpha^2\ell^6) = 0 \tag{31}$$

where $A = 2\dot{\theta} + \frac{4}{3}\theta^2 + \frac{1}{2}\alpha$. The cubic (31) can be solved to give $\sigma^2 = f(\ell, \dot{\ell}, \ddot{\ell})$. This solution can then be substituted into the Raychaudhuri equation (14). For the linear equation of state $p = (\gamma - 1)\rho$, we get $\rho = \rho_0\ell^{-3\gamma}$ from (4). Then, (30) and Raychaudhuri's equation (14) reduce after lengthy calculations to a single autonomous third-order equation in $\ell(t)$. Via the transformation $y = (\dot{\ell}/\ell)^2$, $x = \log \ell$, this can be reduced to the second-order equation

$$\begin{aligned} 3yy'' - \tfrac{3}{4}y'^2 + 12yy' + 6y^2 + \left(3y - \rho_0 e^{-3\gamma x}\right)/6\alpha \\ + 2f'y - f^2 - f(2y' + 6y + 1/6\alpha) = 0 \end{aligned} \tag{32}$$

where the anisotropy function f is given by the roots of the cubic (31):

$$\begin{aligned} f &= -\tfrac{1}{3}A + B_+ + B_- \\ &= -\tfrac{1}{3}A - \tfrac{1}{2}(B_+ + B_-) \pm \tfrac{1}{2}(B_+ - B_-)\sqrt{-3} \end{aligned}$$

with

$$\begin{aligned} A &= 3y' + 12y + 1/2\alpha \\ B_\pm &= [A^3/216 + \Sigma^2 e^{-6x}/(32\alpha^2) \\ &\qquad \pm (\Sigma^4 e^{-12x}/(1024\alpha^4) + \Sigma^2 A^3 e^{-6x}/(3456\alpha^2))^{1/2}]^{1/3}. \end{aligned}$$

Thus we have derived a ***non-vacuum and anisotropic generalisation of the higher order dynamical equation for vacuum*** FRW ***universes*** [3]. In principle, we can solve for $\ell(t)$ from (32) and then find the exact evolution of σ^2 from (31). In practice of course, the complexity of the equations prevents closed form solutions for ℓ. This is true both in the FRW case $f = 0$ of (32) and even if we further assume a vacuum, $\rho_0 = 0$. However, it is possible to solve (31) at late times in an expanding universe. When $\theta, \dot\theta \to 0$ we obtain

$$\sigma^2 = \Sigma^2 \ell^{-6} + 72\alpha\, \Sigma^4 \ell^{-12} + O(\ell^{-18}) \tag{33}$$

which implies that the ***expansion dissipates shear more slowly than in the Einstein theory***. However, the rate of retardation of the shear dissipation is effectively negligible. The effect of the higher order terms on the shear at early times ($\ell \to 0$) cannot readily be deduced from (31), since the relative behaviour of θ and $\dot\theta$, which is needed to determine the behaviour of A, can only be found from (32), and this appears not to be feasible.

At late times we can use (32) to find the approximate behaviour of $\ell(t)$, and then (33) gives $\sigma(t)$. The matter and shear terms in (32) can both be neglected as $\ell \to \infty$. Then (32) reduces to an Abel equation of the second kind via the transformation $Y = y'/6y$, $X = \frac{1}{2}\log 9y$:

$$2YY' + 3Y^2 + 2Y + ae^{-2X} = 0, \qquad a = \text{constant}.$$

Unfortunately we are unable to find exact solutions even of this late-time equation.

In the case of a ***viscous fluid***, π_{ab} is governed by thermodynamic laws. According to the second-order (and causal) theory of Israel and Stewart [13],

$$\pi_{ab} + \tau\dot\pi_{ab} = -\eta\sigma_{ab} \tag{34}$$

where $\eta \geq 0$ is the shear viscosity and $\tau \geq 0$ is a relaxation time. In the first-order (and non-causal) Eckart theory [13], $\tau = 0$. In this case, (34) and (27) give

$$\sigma^2 = (1 + 2\alpha R)^{-2} \Sigma^2 \ell^{-6} \exp[-2 \int \eta (1 + 2\alpha R)^{-1} dt] \tag{35}$$

where we have used $\eta = \eta(t)$, which follows from spatial homogeneity. In a realistic model, η is a complicated function deduced from thermodynamics or kinetic theory [13]. In Einstein gravity, (35) immediately shows that shear viscosity accelerates the dissipation of shear, regardless of the form of η —provided that η is positive. However in higher order gravity the Ricci scalar terms mean that this is no longer true. In fact since R in general depends on σ^2, (35) is not a solution for σ^2, but an implicit condition on σ^2. In order to get an idea of some of the possibilities, we make an ansatz that allows us to perform the integral in (35):

$$\eta = k(1 + 2\alpha R) \tag{36}$$

where k is a positive constant. Since $\eta \geq 0$, this requires $1 + 2\alpha R \geq 0$. Then (35) gives

$$\sigma^2 = (1 + 2\alpha R)^{-2}\Sigma^2 \ell^{-6} e^{-2kt}, \qquad \dot{\Sigma} = 0 \tag{37}$$

which reduces to Ellis' result [8] when $\alpha = 0$ (or $R = 0$). Note that (37) approaches (29) as $t \rightarrow 0$. Thus the presence of anisotropic pressure obeying Eckart thermodynamics and with shear viscosity given by (36) becomes irrelevant at early times. Using (37) in the Bianchi I case, we could proceed as before to get a cubic equation for σ^2 [simply replace ℓ^6 by $\ell^6 e^{2kt}$ in (31)]. Of course the viscosity ansatz (36) cannot be regarded as realistic. (In particular, it makes the viscosity directly dependent on the shear via the Ricci scalar.) In Bianchi I spacetimes, (30) shows that $R \rightarrow 0$ with the expansion of the universe, so that $\eta \rightarrow k$. Thus the ansatz (36) is less unrealistic at late times, when it approaches Ellis' model [8].

The Eckart theory is unsatisfactory because of its non-causal and unstable features [14]. It is not really applicable to a non-quasistationary situation such as the expansion of a Bianchi universe. In order to get a realistic model of viscous dissipation of shear, one needs the Israel–Stewart theory. In general, (27) and the Israel–Stewart law (34) can be combined to yield a single second-order linear differential equation for π_{ab} (or σ_{ab}):

$$\begin{aligned}\ddot{\pi}_{ab} &+ [\tau^{-1} + \{\log \tau\eta^{-1}\ell^3(1 + 2\alpha R)\}^{\cdot}]\dot{\pi}_{ab} \\ &+ [\tau^{-1}\{\log \eta^{-1}\ell^3(1 + 2\alpha R)\}^{\cdot} + \tau\{\eta(1 + 2\alpha R)\}^{-1}]\pi_{ab} = 0.\end{aligned} \tag{38}$$

The thermodynamic quantities η and τ may be taken as functions of ℓ. If these are known, the solution of (38) depends in general on knowledge of $\ell(t)$. In principle this follows from Raychaudhuri's equation (14). However, in addition to all the complexities described in the perfect fluid case ($\eta =$

$0 = \tau$), there are the further complications that σ_{ab} is determined as a function of π_{ab} by (34), and that the conservation equation (4) involves $\pi_{ab}\sigma^{ab}$. Thus one is forced in general to consider the coupled system of Raychaudhuri's equation (14) and (38) in the variables ℓ and π_{ab}. Even in the Einstein theory, no exact solutions are known for $\tau \neq 0$.

In summary:
In $R + \alpha R^2$ gravity the shear in Bianchi perfect fluid spacetimes with isotropic surfaces is given exactly by (29). *The shear depends non-linearly on the Ricci scalar and it is therefore not known in general how the shear dissipation compares with the General Relativity case. However, in Bianchi I spacetimes the shear dissipation is shown to be slower than in Einstein gravity* [*eq.* (33)].*Furthermore, the field equations for Bianchi I are reduced to a single second-order equation* (32).

3.2. Kinematic Characterisation of FRW Cosmologies

The kinematic conditions

$$\dot{u}_a = \omega_{ab} = \sigma_{ab} = 0 \tag{39}$$

are usually taken to imply an FRW geometry , i.e. to imply that there exist coordinates $x^a = (t, x^\mu)$ such that

$$ds^2 = -dt^2 + A(t)^2 \Lambda_{\mu\nu}(x^\tau) dx^\mu dx^\nu \tag{40}$$

where $\Lambda_{\mu\nu}(x^\tau)$ has constant curvature. However, (39) implies (40) only if (a) the fluid is perfect, and (b) Einstein's field equations hold [15]. An important question, which appears not to have been answered until now, is whether this carries through to the higher order theories. We will show that (39) is almost, but not quite, enough to imply (40). In the higher order theory (with $\alpha \neq 0 = \beta$), we require in addition to (39) that the fluid obey a barotropic equation of state

$$p = p(\rho), \qquad p'(\rho) \neq 0 \tag{41}$$

which means that the surfaces $\{p = \text{constant}\}$ and $\{\rho = \text{constant}\}$ coincide. In a sense it is surprising that given the complexity of the field equations, (40) is the *only* additional assumption required, and that essentially the kinematic characterisation (39) of the FRW geometry (40) holds also in the fourth-order gravity.

In Einstein's theory, the kinematic conditions (39) lead directly via (16) and (18) to the vanishing of the Weyl tensor. However, in the higher order case, only the magnetic part of $C^a{}_{bcd}$ is immediately zero: $H_{ab} = 0$.

To show that $E_{ab} = 0$ requires us first to show that the Ricci scalar is spatially homogeneous. The kinematic conditions (39) imply that there exist coordinates such that [16]

$$ds^2 = -dt^2 + S(t, x^\tau)^2 \lambda_{\mu\nu}(x^\tau) dx^\mu dx^\nu, \qquad u^a = \delta^a{}_0 \tag{42}$$

where $\theta = S_{,0}/S$. In the Einstein theory Anderson shows [15] that the field equations $R_{0\mu} = 0$ (which follow from zero heat flow) imply that $(\log S)_{,0\mu} = 0$, which reduces the metric (42) to the form (40). Then the $(\mu\nu)$ field equations show that $\Lambda_{\mu\nu}(x^\tau)$ has constant curvature.

In the higher order theory the (0μ) field equations (11) imply that

$$(\log S)_{,0\mu} = \tfrac{1}{3}\theta_{,\mu} = -\alpha(1 + 2\alpha R)^{-1} R_{;0\mu}. \tag{43}$$

In general, this is not zero. If we further assume an equation of state (41), the conservation equations (4) and (5) imply that

$$h_a{}^b p_{;b} = h_a{}^b \rho_{;b} = h_a{}^b \theta_{;b} = 0 \tag{44}$$

using (39). Note that for dust, or more generally $p =$ constant, we do not get ρ and θ homogeneous. It would be interesting to find an explicit non-FRW dust solution without acceleration, vorticity or shear. In the coordinates of (42), (44) gives $\theta = \theta(t)$ and (43) implies $S = A(t)B(x^\tau)$, which reduces the metric (42) to the form (40). It is now considerably more difficult than in the Einstein case to show that $\Lambda_{\mu\nu}$ has constant curvature. We proceed as follows.

From (43) we have $R_{;0\mu} = 0$ which, using (40), integrates to

$$R = A(t) f(x^\tau) + B(t) \tag{45}$$

where f and B are arbitrary. Using the metric (40) and the form of the Ricci scalar (45), the (00) field equation (10) reduces after lengthy calculations to

$$Q(t) = P(t) f(x^\tau) + L(t) f(x^\tau)^2 \tag{46}$$

where

$$\begin{aligned}
Q(t) &= \alpha[-6\ddot{A}B/A + 6\dot{A}\dot{B}/A + \tfrac{1}{2}B^2] - 3\ddot{A}/A + \tfrac{1}{2}B - \rho \\
P(t) &= \alpha[8\ddot{A} - AB] - \tfrac{5}{6}A \\
L(t) &= -\tfrac{1}{2}\alpha A^2.
\end{aligned}$$

Equation (46) is crucial. It implies that either $f_{,\mu} = 0$ or $P = 0 = L$. However, $L = 0$ is clearly equivalent to $f_{,\mu} = 0$ by (45). Then the only possibility allowed by (46) is $f_{,\mu} = 0$. Hence

$$h_a{}^b R_{;b} = 0. \tag{47}$$

Now the homogeneity of the Ricci scalar in turn leads to the constant curvature property of $\Lambda_{\mu\nu}$, which may be seen as follows. Equation (47) implies [using (39)]

$$R_{;ab} = \ddot{R} u_a u_b - (\dot{R}\dot{A}/A) h_{ab} \tag{48}$$

which, by (16) and (25) implies

$$E_{ab} = 0 = R^*{}_{ab} - \tfrac{1}{3} R^* h_{ab}.$$

Thus the surfaces $\{t = \text{constant}\}$ are intrinsically isotropic. We need finally to show that they are isotropically embedded. The $(\mu\nu)$ field equation (12), using (9),(10),(48) and the form of the metric (40), can be written as

$$R_{\mu\nu} = {}^3R_{\mu\nu}(x^\tau) + (A\ddot{A} + 2\dot{A}^2)\Lambda_{\mu\nu} = (\tfrac{1}{3}A^2 R - A\ddot{A})\Lambda_{\mu\nu} \tag{49}$$

where ${}^3R_{\mu\nu}(x^\tau)$ is the three-dimensional Ricci tensor formed from $\Lambda_{\mu\nu}(x^\tau)$ ($R^*{}_{\mu\nu}$ is the three-dimensional Ricci tensor formed from $h_{\mu\nu}(t, x^\tau)$). It is obvious from (49) that it was first necessary to prove (47) before we could show isotropic embedding. Equation (49) implies that ${}^3R_{\mu\nu}(x^\tau) = 2\varepsilon\Lambda_{\mu\nu}(x^\tau)$, since $R = R(t)$ and $A = A(t)$, where

$$\tfrac{1}{3}A^2 R - 2A\ddot{A} - 2\dot{A}^2 = 2\varepsilon = \text{constant}.$$

Thus ε is the constant curvature of $\Lambda_{\mu\nu}$ (${}^3R = 6\varepsilon$), and $\varepsilon = 0, \pm 1$. So the surfaces $\{t = \text{constant}\}$ are isotropic and isotropically embedded. Then the metric is FRW. The higher order Bianchi identities (19) to (22) are all identically satisfied, except for the E_{ab} propagation equation (21), which reduces to the $(\mu\nu)$ field equation (12).

In the Einstein theory, Ellis shows [8] that an equivalent set of conditions to the kinematic conditions (39) (for a perfect fluid) which will result in the FRW geometry is

$$h_a{}^b \theta_{;b} = \dot{u}^a = C^a{}_{bcd} = 0, \qquad \theta \neq 0.$$

This equivalence relies on the Bianchi identities and the assumption that $\rho + p \neq 0$ at any time. Because of the complex form of the higher order

Bianchi identities, this equivalence does *not* carry through to the higher order theory. The Bianchi identities instead set conditions on the geometry and the kinematic quantities ω^a and σ^{ab}.

Note that the homogeneity conditions (44), which follow from the equation of state and the conservation equations, lead to an alternative assumption to (41): instead of an equation of state, we could assume homogeneous ρ, i.e. $h_a{}^b \rho_{;b} = 0$. Alternatively, (41) may also be replaced by the assumption that u^a is a Ricci eigenvector. (Of course for the Einstein equations this is a trivial consequence of the vanishing of heat flow.)

In summary:
In $R + \alpha R^2$ gravity, a perfect fluid spacetime with vanishing vorticity, shear and acceleration is FRW *only if the fluid obeys in addition a barotropic equation of state, or alternatively if the energy density is spatially homogeneous.*

Flowing from this general result is an important corollary, namely that the Ehlers–Geren–Sachs theorem [17] on cosmic background radiation holds also for higher order gravity. This theorem has long been considered as one of the key results motivating the standard big-bang model of the background radiation. The fact that it extends to higher order gravity could not have been predicted without detailed calculation, and is a significant point in favour of the higher order theory. The crucial point is that radiation obeys the barotropic equation of state $p = \rho/3$ in general [16]. This means that our kinematic results are applicable, and a careful study of the proof in [17] leads to the following generalised Ehlers–Geren–Sachs theorem:

If a congruence of freely-falling observers measures an isotropic distribution of self-gravitating photons in equilibrium, then in $R + \alpha R^2$ gravity the spacetime is FRW.

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