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ON CONDITIONS FOR EXISTENCE OF UMBILICAL POINTS ON A CONVEX SURFACE†)

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Let F be a complete convex surface of class C^∞ in the three-dimensional Euclidean space $\mathbb{R}^3$. The convexity property of F implies that the surface is homeomorphic to either a circular cylinder, or a plane, or a sphere. In the last case, on F (as on any other surface homeomorphic to a sphere) there are at least two umbilical points. In the first case, umbilical points on F are either absent or fill some set of generators. We are left with considering the case of a complete convex surface homeomorphic to a plane. In this event, it is easy to exhibit an example of a surface on which there are no umbilical points. For instance, let γ be a convex planar curve whose curvature differs from zero at each point and let C be a cylinder whose director curve is γ and whose generators are perpendicular to the plane of the curve γ. As is easily seen, there are no umbilical points on C. Denote by $k_1(p)$ and $k_2(p)$ the principal curvatures of F at a point p, and assume the normal $n(p)$ to the surface F be directed so that $k_1(p)$ and $k_2(p)$ be nonnegative. For definiteness, assume that

$$0 \le k_1(p) \le k_2(p). \tag{1}$$

It is obvious that the following equality holds for the cylinder C:

$$\inf_{p\in C}(k_2(p) - k_1(p)) = 0. \tag{2}$$

In such a case, we can say that C "has" an umbilical point at "infinity." This example, as well as other examples of convex surfaces known to the author, leads to the conjecture: On a complete convex surface F homeomorphic to a plane, the following equality of type (2) holds: $\inf_{p\in F}(k_2(p)-k_1(p)) = 0$. It is possible to reformulated this as follows: every complete convex surface homeomorphic to a plane has an umbilical point which may lie at "infinity." However, the author did not succeed in proving this conjecture; hence, it is unknown whether the conjecture is true or not. The following two theorems are proven:

Theorem 1. *If the integral curvature of a complete convex surface F of class C^∞ and homeomorphic to a plane is strictly less than 2π, then*

$$\inf_{p\in F}(k_2(p) - k_1(p)) = 0.$$

Theorem 2. *If, for a complete convex surface F of class C^∞ and homeomorphic to a plane, the Gaussian curvature K of F and the moduli of the gradients of the functions $k_1(p)$ and $k_2(p)$ are bounded on F, then*

$$\inf_{p\in F}(k_2(p) - k_1(p)) = 0.$$

Theorems 1 and 2 are equivalent to the next Theorems 1* and 2* respectively.

Theorem 1*. *If the integral curvature of a complete convex surface F of class C^∞ is strictly less than 2π and $\inf_{p\in F}(k_2(p) - k_1(p)) = c_0 > 0$, then F is a cylinder homeomorphic to a circular cylinder.*

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Theorem 2*. *If, for a complete convex surface F of class C^∞, the relation $\inf_{p\in F}(k_2(p)-k_1(p))=c_0>0$ holds on F and the Gaussian curvature K of F and the moduli of the gradients of the functions $k_1(p)$ and $k_2(p)$ are bounded on F, then the surface F is a cylinder homeomorphic to a circular cylinder.*

REMARK. Under the conditions of Theorems 1* and 2* the director curve γ of the cylinder is a convex planar curve whose curvature is not less than c_0 at any point.

The proofs of Theorems 1 and 2 are based on the following lemmas:

Lemma 1. *If $\inf_{p\in F} k_2(p)=0$ then $\inf_{p\in F}(k_2(p)-k_1(p))=0$.*

The proof is obvious, since $k_1(p)\geq 0$.

Denote by $D(F)$ the convex domain of the Euclidean space $\mathbb{R}^3$ which is bounded by the surface F and denote by $D(\nu,F)$ the projection of $D(F)$ to some plane perpendicular to the vector ν.

Lemma 2. *If, for some vector ν, the domain $D(\nu,F)$ includes a circle of radius R, then $\inf_{p\in F} k_2(p)\leq 1/R$.*

PROOF. Denote by K_R a circle of radius R which lies in $D(\nu,F)$. Introduce a Cartesian rectangular coordinate system x, y, z in $\mathbb{R}^3$ with origin O at the center of the circle K_R and the z-axis directed in parallel to the vector ν. Let C stand for the right circular cylinder whose director curve is the boundary circumference of the circle K_R. The cylinder C cuts out at least one surface F_1 from F which is projected onto the circle K_R in a one-to-one fashion. Let $z=f_1(x,y)$ be the equation of the surface F_1, where $x^2+y^2<R^2$, and let, for definiteness, the surface F_1 be convex downward. Now, take a sphere $S(R-\varepsilon)$ of radius $R-\varepsilon$ whose center is on the z-axis and which lies entirely higher than the surface F_1 (ε is an arbitrary number satisfying the inequality $0<\varepsilon<R$). We now lower the sphere $S(R-\varepsilon)$ until the first contact of $S(R-\varepsilon)$ and F_1. Let a point p belongs to $F_1\cap S(R-\varepsilon)$. The surface F_1, as is seen from our construction, lies entirely lower than $S(R-\varepsilon)$ and touches it at the point p which is an interior point of F_1. Therefore, all normal curvatures of F_1 at p do not exceed $1/R-\varepsilon$. Since ε can be taken arbitrarily close to zero, Lemma 2 is thus proven.

Lemma 3. *If $\inf_{p\in F}(k_2(p)-k_1(p))=c_0>0$ then, for each point in the domain $D(F)$, there is exactly one ray that passes through the point and lies entirely in the domain $D(F)$.*

PROOF. Since the domain $D(F)$ is convex and noncompact, for each point $p\in D(F)$ there is at least one ray that passes through the point and lies in $D(F)$. If we suppose that two rays r_1 and r_2 pass through some point $p_0\in D(F)$ and lie entirely in $D(F)$, then the projection of $D(F)$ to the plane of the rays r_1 and r_2 includes a circle of an arbitrarily large radius. In that event $\inf_{p\in F} k_2(p)=0$ by Lemma 2, contradicting the hypothesis of Lemma 3. The contradiction obtained proves Lemma 3.

Lemma 3 implies that all rays in $D(F)$ are parallel to one another. Let α stand for a plane perpendicular to all the rays and tangent to F at some point O. Construct a Cartesian rectangular coordinate system x, y, z, placing the origin at the tangency point O and directing the z-axis so that it be perpendicular to the plane α. In this coordinate system, the surface F can be given by an explicit equation

$$z=f(x,y),\quad (x,y)\in \operatorname{Int} D(\bar{k},F).$$

Lemma 4. *If $\inf_{p\in F} k_2(p)=c_0>0$ then the domain $D(\bar{k},F)$ is compact.*

PROOF. Suppose that the domain $D(\bar{k},F)$ is not compact. Since it is convex, we can draw a ray r_0 through the origin which lies entirely in $D(\bar{k},F)$. Let r_0 coincide with the positive half-axis of the x-axis. Consider the convex curve $z=f(x,0)$, $y=0$, $x>0$, lying entirely in the plane XOZ. Let R be an arbitrary positive number. In the plane XOZ, consider the domain $D_1(R)$ determined by the inequalities $0<x<2R$, $z>R_1=f(2R,0)$. Then we can place a circle K_R of radius R in the domain D_1 and assume that $K_R\subset D(\bar{\jmath},F)$. By Lemma 2, we then have $\inf_{p\in F} k_2(p)\leq 1/R$. In view of the arbitrariness of the number R, we obtain $\inf_{p\in F} k_2(p)=0$, which contradicts the hypothesis of Lemma 4. The contradiction proves Lemma 4.

Denote the boundary of the domain $D(\bar{k},F)$ by σ, and denote by C_σ the right cylinder that is constructed on σ as on the director set. The surface F lies entirely in the cylinder C_σ and may touch it only in some half-lines. Let us pass from the Cartesian coordinates (x,y,z) to cylindrical ones

(r, φ, z): $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$. In these coordinates, the surface F can be determined by the equations

$$r = r(z, \varphi), \quad z = z, \quad z \geq 0, \quad 0 \leq \varphi \leq 2\pi.$$

From convexity of F and compactness of $D(\bar{k}, F)$, we can easily infer the following properties of the function:

$$\lim_{z\to\infty} r(z,\varphi) = r_0(\varphi) < \infty, \quad \lim_{z\to\infty} r_z(z,\varphi) = 0, \tag{3}$$

$$r_z(z,\varphi) > 0, \quad r_{zz}(z,\varphi) \leq 0 \quad \text{for} \quad z > 0. \tag{4}$$

We can now prove Theorem 1 by using equalities (3). To prove Theorem 2, we need uniform upper estimates for the moduli of the functions r_φ, $r_{\varphi\varphi}$, r_{zz}, and $r_{z\varphi}$. These estimates are in general absent for the surface F itself. For this reason, we pass from the surface F to an equidistant surface Φ for which the mentioned estimates are now existent. Let a be some positive number. Define the surface Φ by lapping, from each point p of F, a segment of length a in the direction of the outward normal $(-n(p))$. Denote by $\varphi(p)$ the homeomorphism of F onto Φ that is generated by this construction.

Lemma 5. *If on a complete convex surface F of class C^∞ the next conditions hold:*
(i) *the Gaussian curvature K is bounded by some constant c_1;*
(ii) $|\operatorname{grad} k_1(p)|$ *and* $|\operatorname{grad} k_2(p)|$ *are bounded by a constant c_2;*
(iii) $\inf_{p\in F}(k_2(p) - k_1(p)) = b > 0$
then, for every a, the surface Φ is a complete convex surface of class C^∞ for which the same conditions (i)–(iii) *hold but possibly with other constants.*

PROOF. The principal radii of curvature $R_F(p)$ and $R_\Phi(\varphi(p))$ of the surfaces F and Φ are well known to be connected by the relation

$$R_\Phi(\varphi(p)) = R_F(p) + a. \tag{5}$$

It follows from (5) that Φ is convex and belongs to the class C^∞. Furthermore, it follows from (1) that the principal curvatures $\bar{k}_1(q) = \bar{k}_1(\varphi(p))$ and $\bar{k}_2(q) = \bar{k}_2(\varphi(p))$ of the surface F are expressed in terms of $k_1(p)$ and $k_2(p)$ by the formulas

$$\bar{k}_1(q) = \frac{k_1(p)}{1 + ak_1(p)}, \quad \bar{k}_2(q) = \frac{k_2(p)}{1 + ak_2(p)}. \tag{6}$$

From (6) we infer

$$K_\Phi(q) = \frac{k_1(p)k_2(p)}{(1 + ak_1(p))(1 + ak_2(p))} \leq k_1(p)k_2(p) \leq c_1. \tag{7}$$

Now, (7) implies the validity of item (i) of Lemma 6 for the surface Φ, and (6) implies the validity of item (ii) for Φ. Finally,

$$\inf_{q\in\Phi}(\bar{k}_2(q) - \bar{k}_1(q)) = \inf_{p\in F} \frac{k_2(p) - k_1(p)}{(1 + ak_1(p))(1 + ak_2(p))} \geq \frac{b}{(1 + c_3 a)(1 + \bar{c}_0 a)}, \tag{8}$$

where $c_3 = \sup_{p\in F} k_1(p)$ and $\bar{c}_0 = \inf_{p\in F} k_2(p) \geq 0$. We are left with observing that the boundedness of the function $k_1(p)$ ensues from the boundedness of the Gaussian curvature of the surface F. Lemma 5 is proven.

Now, assume that, in the above-introduced cylindrical coordinate system, the surface Φ is given by the equations $r = f(z, \varphi)$, $z = z$, $z \geq -a$, $0 \leq \varphi \leq 2\pi$. Let $\bar{z} > -a$ be some number. We say that some function $B(r, \varphi)$ *is bounded on* F, if there is a constant C such that $|B(z, \varphi)| \leq C$ for $z \geq \bar{z}$ and $0 \leq \varphi \leq 2\pi$.

Lemma 6. *If a surface F satisfies the conditions of Lemma 5, then the function $f(z,\varphi)$ possesses the following properties:*

(a) $f_z \geq 0$ *for* $z > -a$, (b) $\lim_{z\to\infty} f_z(z,\varphi) = 0$, (c) $f_{zz} < 0$ *for* $z > -a$, (d) *the functions* $|f_\varphi(z,\varphi)|$, $|f_{\varphi\varphi}(z,\varphi)|$, $|f_{zz}(z,\varphi)|$, $|f_{z\varphi}(z,\varphi)|$, *and* $|f_{zzz}(z,\varphi)|$ *are bounded on* Φ.

PROOF. Properties (a)–(c) of the function f follow from convexity of the surface Φ and compactness of the domain $D(\bar{k},\Phi)$, where $\bar{k}$ is the unit vector of the z-axis. We begin proving boundedness of $|f_\varphi(z,\varphi)|$. Let $\bar{\sigma}$ be the boundary of the domain $D(\bar{k},\Phi)$; assume it lying in the plane $z=-a$. Denote by $2r_0$ the minimal distance from the point $A(0,0,-a)$ to $\bar{\sigma}$ and by r_1, the maximal distance. Since the orthogonal projections of the curves γ_c: $r=f(c,\varphi)$, $z=c$, to the plane $z=-a$ converge to $\bar{\sigma}$ as $c\to\infty$, there is z_1 such that

$$f(z,\varphi) \geq r_0 \tag{9}$$

for $z \geq z_1$. Denote by $\alpha(\varphi)$ the acute angle between the tangent to γ_c at the point $r=f(c,\varphi)$ and the radius-vector from $A(c)(0,0,c)$ to this point. Let $d(\varphi)$ be the distance of $A(c)$ from the tangent straight line and let $b(\varphi)$ be the length of the radius-vector. Then

$$\sin\alpha(\varphi) = d/b, \quad \cos\alpha(\varphi) = \sqrt{b^2-d^2}/b. \tag{10}$$

On the other hand, $\cos\alpha(\varphi) = |f_\varphi|/\sqrt{b^2+f_\varphi^2}$, whence we infer

$$|f_\varphi| = b\operatorname{cotan}\alpha(\varphi) = \frac{\sqrt{b^2-d^2}}{d}b. \tag{11}$$

The convexity of the curve γ_c implies the inequalities

$$d(\varphi) \geq r_0, \quad b(\varphi) \leq r_1. \tag{12}$$

From (11) and (12) we obtain

$$|f_\varphi| \leq \frac{r_1}{r_0}\sqrt{r_1^2-r_0^2}. \tag{13}$$

Inequality (13) implies the first claim in item (d) of Lemma 6.

Now, we evaluate the curvature $k(\varphi)$ of the curve γ_c at an arbitrary point:

$$k(\varphi) = \frac{-f_{\varphi\varphi}f + 2f_\varphi^2 + f^2}{\left(f^2+f_\varphi^2\right)^{3/2}}.$$

From this equality we have

$$f_{\varphi\varphi} = -k(\varphi)\frac{\left(f^2+f_\varphi^2\right)^{3/2}}{f} + \frac{2f_\varphi^2}{f} + f.$$

Estimate the function $k(\varphi)$ from above. According to Euler's formula, (5) implies that the normal curvature of the surface Φ at every point and in every direction does not exceed $1/a$. Therefore, from Meusnier's theorem we infer the inequality

$$k(\varphi) \leq 1/a\cos\beta, \tag{14}$$

where β is the angle between the principal normal to the curve γ_c and the normal to the surface Φ,

$$\cos\beta = \frac{f_\varphi^2+f^2}{\sqrt{f_\varphi^2+f^2}\sqrt{f_\varphi^2+f^2(1+f_z^2)}} = \frac{\sqrt{f_\varphi^2+f^2}}{\sqrt{f_\varphi^2+f^2(1+f_z^2)}}. \tag{15}$$

Since $\lim_{z\to\infty} f_z(z,\varphi) = 0$, it follows from (15) that there is z_2 such that $\cos\beta \geq 1/2$ for $z > z_2$. Inequalities (9) and (14) now imply the second claim in item (d) of Lemma 6.

Let $\bar{\gamma}_c$ be the curve determined by the equations $z = f(z,c)$, $z = z$, $\varphi = c$. The curvature $\bar{k}(z)$ of the curve is given by the formula

$$\bar{k}(z) = -f_{zz}/\left(1+f_z^2\right)^{3/2}. \tag{16}$$

By analogy with the above, from (16) we obtain

$$|f_{zz}| \leq \frac{1}{a\cos\bar{\beta}}\left(1+f_z^2\right)^{3/2}, \tag{17}$$

where $\bar{\beta}$ is the angle between the principal normal to the curve $\bar{\gamma}_c$ and the normal to the surface Φ,

$$\cos\bar{\beta} = \frac{f(1+f_z^2)}{\sqrt{1+f_z^2}\sqrt{f_\varphi^2+f^2(1+f_z^2)}}. \tag{18}$$

Relations (17), (18), (13), and (9) imply the third claim in item (d) of Lemma 6:

$$|f_{zz}| \leq \bar{c} \quad \text{for} \quad z > z_3. \tag{19}$$

We compute the coefficients E, F, G and L, M, N of the first and second quadratic forms of the surface Φ:

$$\begin{gathered} E = 1+f_z^2, \quad F = -f_z f_\varphi, \quad G = f^2+f_z^2, \quad L = \frac{-f_{zz}f}{A}, \\ M = \frac{-f_{z\varphi}f + f_z f_\varphi}{A}, \quad N = \frac{f^2+2f_\varphi^2 - f_{\varphi\varphi}f}{A}, \end{gathered} \tag{20}$$

where $A = \sqrt{f_\varphi^2+f^2(1+f_z^2)}$. Since the surface Φ is convex, we have $M^2 \leq LN$. Whence we infer the inequality

$$\left(f_{z\varphi}f - f_z f_\varphi\right)^2 \leq -f_{zz}f\left(f^2+2f_\varphi^2 - f_{\varphi\varphi}f\right). \tag{21}$$

From (21) and the inequalities proven earlier we infer the second claim of item (d) of Lemma 6:

$$|f_{z\varphi}| \leq \bar{c}_3 \quad \text{for} \quad z > z_4. \tag{22}$$

We turn to proving boundedness of the function $|f_{zzz}(z,\varphi)|$. The boundedness of the moduli of the gradients of the functions $k_1(p)$ and $k_2(p)$ implies the boundedness of the moduli of the gradients of the functions $\bar{k}_1(q)$ and $\bar{k}_2(q)$, and consequently the boundedness of the functions $|\bar{k}_1(q)+\bar{k}_2(q)|$ and $|\bar{k}_2(q)+\bar{k}_1(q)|$, which leads to the boundedness of the functions $|LG+EN-2MF|$ and $|LN-M^2|$. Involving the inequalities proven earlier and formulas (20), from the preceding conditions we obtain the equalities

$$\begin{aligned} L_zG + N_zE - 2M_zF &= \bar{h}_1(z,\varphi), \quad L_\varphi G + N_\varphi E - 2M_\varphi F = \bar{h}_2(z,\varphi), \\ L_zN + N_zL - 2M_zM &= \bar{h}_3(z,\varphi), \quad L_\varphi N + N_\varphi L - 2M_\varphi M = \bar{h}_4(z,\varphi), \end{aligned} \tag{23}$$

where the functions $\bar{h}_1$, $\bar{h}_2$, $\bar{h}_3$, and $\bar{h}_4$ are bounded on the surface Φ. Further, using the Peterson-Codazzi equations, we express the functions M_z and M_φ through the functions L_φ and N_z respectively, and insert their expressions into (23). We obtain

$$\begin{aligned} L_zG - L_\varphi 2F + N_zE &= h_1(z,\varphi), \quad L_\varphi G - N_z 2F + N_\varphi E = h_2(z,\varphi), \\ L_zN - L_\varphi 2M + N_zL &= h_3(z,\varphi), \quad L_\varphi N - N_z 2M + N_\varphi L = h_4(z,\varphi), \end{aligned} \tag{24}$$

where the functions h_1, h_2, h_3, and h_4 are bounded on the surface Φ. Consider equalities (24) as a system of equations in the functions L_z, L_φ, N_z, and N_φ. The determinant Δ of the system equals

$$\Delta = L^2G^2 + N^2E^2 - 4MFLG - 4MFNE - 2EGLN + 4M^2EG + 4F^2LN$$
$$= (LG + NE - 2FM)^2 - 4(EG - F^2)(LN - M^2) = \left(\frac{\bar{k}_2 - \bar{k}_1}{2}\right)^2 (EG - F^2)^2. \tag{25}$$

From (25) and (8) we infer that Δ is bounded above on the surface Φ:

$$\Delta \geq \bar{b} > 0. \tag{26}$$

But then (24), (26), and (20) imply that

$$L_z = \left(\frac{-f_{zz}f}{\sqrt{f_\varphi^2 + f^2(1 + f_z^2)}}\right)_z = h_5(z, \varphi), \tag{27}$$

where $h_5(x, \varphi)$ is bounded on the surface Φ. Rewrite equality (27) as follows:

$$-f_{zzz}(z, \varphi)f(z, \varphi) = h_6(z, \varphi), \tag{28}$$

where $h_6(z, \varphi)$ is bounded on the surface Φ. The last claim of Lemma 6 follows from (9) and (28). Lemma 6 is proven.

PROOF OF THEOREM 1. Let us prove that under the conditions of Theorem 1 $\inf_{p\in F} k_2(p) = 0$, and thereby in view of Lemma 1 $\inf_{p\in F}(k_2(p) - k_1(p)) = 0$. Assume the contrary. Let $\inf_{p\in F}(k_2(p) = c_0 > 0$. Then Lemma 4 implies that, in the cylindrical coordinate system introduced in the lemma, the surface F is determined by the equations $r = r(z, \varphi)$, $z = z$, $z > 0$, $0 \leq \varphi \leq 2\pi$. The function $r(z, \varphi)$ meets the second of equalities (3).

In a standard way we find the components of the normal vector $n(p)$ of the surface F in the Cartesian coordinate system relating to the cylindrical coordinate system:

$$h(p) = \left(\frac{-(r_\varphi \sin\varphi + 2\cos\varphi}{\sqrt{r_\varphi^2 + r^2(1 + r_z^2)}}, \frac{r_\varphi \cos\varphi - r\sin\varphi}{\sqrt{r_\varphi^2 + r^2(1 + r_z^2)}}, \frac{r_z r}{\sqrt{r_\varphi^2 + r^2(1 + r_z^2)}}\right). \tag{29}$$

It follows from (29) that the spherical image of the surface F coincides with the upper half-sphere. Therefore, by the Gauss theorem on the area of the spherical image, the integral curvature of the surface F equals 2π, contradicting the hypothesis of Theorem 1. The contradiction obtained completes the proof of Theorem 1.

PROOF OF THEOREM 2. Assume, under conditions of Theorem 2, that $\inf_{p\in F}(k_2(p) - k_1(p)) = c_0 > 0$. Then the condition

$$\inf_{q\in\Phi}(\bar{k}_2(q) - \bar{k}_1(q)) = \bar{c}_0 > 0 \tag{30}$$

too holds on Φ. It follows from (30) that there are no umbilical points on Φ. Hence, on Φ there are two continuous unit vector fields $e_1(q)$ and $e_2(q)$, the fields of principal directions of the surface Φ. Let $r = f(z, \varphi)$, $z = z$, be the equations of the surface Φ in the cylindrical coordinate system introduced in Lemma 5. Were there at least one value $z = z_0$ for which the field $e_1(q)$ has a nonzero angle with the curve $\bar{\gamma}_{z_0}$: $r = f(z_0, \varphi)$, $z = z_0$, we would obtain, in the domain $z \leq z_0$ on the surface Φ homeomorphic to a circle, existence of a continuous vector field of unit vectors which is not tangent to the boundary of the circle and vanish nowhere in the circle, which contradicts the Brauer fixed-point theorem. Consequently, for every $z = c$, $z > -a$, on the curve $\bar{\gamma}_c$ there is a point $q(c)$ at which the vector $e_2(q(c))$ is orthogonal to the curve $\bar{\gamma}_c$. Furthermore, it follows from Lemma 1 and inequality (30) that

$$\inf_{q\in\Phi} \bar{k}_2(q) = \bar{k}_0(q) > 0. \tag{31}$$

From (31), Euler's formulas, (20), and claim (b) of Lemma 6 we derive that the normal curvature $k(c)$ of the surface Φ at the point $q(c)$ in the direction of the vector $(1,0)$ is not less than $\bar{k}_0/2$ for a sufficiently large c:

$$k(c) = L/E \geq \bar{k}_0/2. \tag{32}$$

From (32), (20), and (18) we infer that

$$-f_{zz} \geq \frac{\bar{k}_0}{2}\sqrt{EG-F^2} \geq \frac{\bar{k}_0}{2}\sqrt{r_0}. \tag{33}$$

We now evaluate the size of the interval in which $-f_{zz} \geq \bar{k}_0\sqrt{r_0}/4$. Let the point $q(c)$ have coordinates (c, φ_0). Then

$$-f_{zz}(z,\varphi_0) = -f_{zz}(c,\varphi_0) - \frac{1}{2}f_{zzz}(\xi,\varphi_0)(z-z_0) \geq \frac{\bar{k}_0}{4}\sqrt{r_0}. \tag{34}$$

Here ξ lies between c and z. Solving inequality (34), we find

$$|z-c| \geq \bar{k}_0\sqrt{r_0}/4\bar{c}, \tag{35}$$

where $\bar{c}$ is a constant bounding the modulus of f_{zzz} (see Lemma 6). From (35) and (34) we obtain the inequality

$$\int_c^\infty (-f_{zz})\,dz \geq \int_c^{c+\bar{k}_0\sqrt{r_0}/4\bar{c}} (-f_{zz})\,dz \geq \int_c^{c+\bar{k}_0\sqrt{r_0}/4\bar{c}} \frac{k_0\sqrt{r_0}}{4}\,dz = \frac{\bar{k}_0^2 r_0}{16\bar{c}}. \tag{36}$$

On the other hand,

$$\int_c^\infty (-f_{zz})\,dz = -f_z|_c^\infty = f_z(c,\varphi_0) < \varepsilon, \tag{37}$$

where ε is an arbitrarily small number for c large. Choose c large enough to have $\varepsilon < \frac{1}{32\bar{c}}\bar{k}_0^2 r_0$. Then inequality (37) contradicts inequality (36), which completes the proof of Theorem 2.

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