GORYACHEV—CHAPLYGIN TOP AND THE INVERSE SCATTERING METHOD

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It is shown that the inverse scattering method applies to both the classical and the quantum Goryachev—Chaplygin top. A new method, based on the R-matrix formalism, is proposed for deriving the equations determining the spectrum of the quantum integrals of motion. This method is of a rather general nature and may serve as an alternative to the so-called algebraic Bethe Ansatz.

INTRODUCTION

The Goryachev—Chaplygin (GC) top is the name given to the following Hamiltonian system. Consider the six-dimensional phase space with dynamical variables x_α, J_α $(\alpha=1,2,3)$ that generate the Lie algebra $e(3)$ with respect to the Poisson bracket

$$\begin{aligned} \{J_\alpha, J_\beta\} &= \varepsilon_{\alpha\beta\gamma} J_\gamma, \\ \{J_\alpha, x_\beta\} &= \varepsilon_{\alpha\beta\gamma} x_\gamma, \\ \{x_\alpha, x_\beta\} &= 0. \end{aligned} \tag{1}$$

For fixed values of the Casimir operators

$$\begin{aligned} \rho &= x_1^2 + x_2^2 + x_3^2 = 1, \\ \sigma &= x_1 J_1 + x_2 J_2 + x_3 J_3 = 0, \end{aligned} \tag{2}$$

we obtain a four-dimensional manifold on which the Poisson bracket (1) is nondegenerate.

The Hamiltonian of the GC top has the form

$$H = \tfrac{1}{2}(J_1^2 + J_2^2 + 4J_3^2) - b x_1 = \tfrac{1}{2}(J^2 + 3J_3^2) - b x_1, \tag{3}$$

where $J^2 = J_1^2 + J_2^2 + J_3^2$ and b is a parameter. For the physical interpretation of the quantities x_α, J_α, and of the Hamiltonian (3) see, e.g., [1].

As Chaplygin showed [2], for $\sigma=0$ the Hamiltonian (3) commutes relative to the Poisson bracket (1) with the integral of motion

$$G = 2J_3(J_1^2 + J_2^2) + 2b x_3 J_1 = 2J_3(J^2 - J_3^2) + 2b x_3 J_1, \tag{4}$$

and hence, the considered Hamiltonian system is completely integrable in the sense of Liouville. Chaplygin also established [2] that the equations of motions with Hamiltonian (3) can be integrated by quadratures by passing to the variables

$$\begin{aligned} u_1 &= J_3 + \sqrt{J^2}, \\ u_2 &= J_3 - \sqrt{J^2}. \end{aligned} \tag{5}$$

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 133, pp. 236-257, 1984.

0090-4104/85/3106-3417$09.50

In paper [3] it was shown that, along with the GC top, the so-called GC gyrostat [1] with the Hamiltonian

$$H_p = H + p\mathcal{J}_3 \tag{6}$$

is also completely integrable. Hamiltonian (6) commutes relative to the Poisson bracket (1) with the integral of motion

$$G_p = G + p(\mathcal{J}^2 - \mathcal{J}_3^2)\,. \tag{7}$$

The GC gyrostat is integrated in the same variables (5) as the GC top. It is clear that the GC gyrostat degenerates into the GC top for $p = 0$.

In the quantum case the Poisson brackets (1) are replaced by the corresponding commutation relations

$$\begin{aligned} [\mathcal{J}_\alpha, \mathcal{J}_\beta]_- &= -i\varepsilon_{\alpha\beta\gamma}\mathcal{J}_\gamma\,, \\ [\mathcal{J}_\alpha, x_\beta]_- &= -i\varepsilon_{\alpha\beta\gamma}x_\gamma\,, \\ [x_\alpha, x_\beta]_- &= 0 \end{aligned} \tag{8}$$

(hereafter we use the notation $[A,B]_\pm = AB \pm BA$). The Casimir operators (2) and the Hamiltonians (3) and (6) preserve their form. As in the classical case, we shall assume that $\rho = 1$ and $\sigma = 0$.

Henceforth, we will denote the quantum operators by the same symbols as their classical counterparts and use the symbol $\wedge$ to indicate those quantum quantities whose expression differs from the corresponding classical expression by the order of factors or the presence of quantum corrections.

Komarov showed [4] that in the quantum case Hamiltonian (3) commutes with the operator

$$\hat{G} = 2\mathcal{J}_3(\mathcal{J}^2 - \mathcal{J}_3^2 + \tfrac{1}{4}) + 6[x_3, \mathcal{J}_1]_+\,, \tag{9}$$

provided $\sigma = 0$. Moreover, he succeeded in reducing the problem of finding the spectrum of H to two independent one-dimensional spectral problems, which obviously corresponds to separation of variables in classical mechanics.

The results listed above were generalized to the quantum GC gyrostat in [5]. In particular, the quantum analog of the integral of motion (7) has the form

$$\hat{G}_p = \hat{G} + p(\mathcal{J}^2 - \mathcal{J}_3^2 + \tfrac{1}{4})\,. \tag{10}$$

This paper has two goals. The first is to show that the GC top (gyrostat), both classical and quantum, can be systematically investigated in the framework of the classical, respectively quantum inverse scattering method (ISM); the latter is perhaps the only universal method for studying complete integrable systems available previously. Actually, we refer to a modified ISM developed recently and known as the method of the R-matrix [6, 7]. Although the complete integrability of the GC top (gyrostat) was established earlier by different methods, such a result is nevertheless of methodological importance, as another proof of the universality of the ISM.

The second goal of this paper is to demonstrate, on the example of the quantum GC top (gyrostat), a new method for deriving the equation for the spectrum of H, which could serve as an alternative to the algebraic Bethe Ansatz [6] in those situations in which there is no local vacuum for the L -operator. The method proposed here generalizes the device used in [8] to investigate the quantum periodic Toda lattice. We emphasize that this generalization became available thanks to the formalism of the R -matrix.

The main text is divided into two sections: the first deals with the classical GC top (gyrostat), whereas the second is devoted to the quantum analogs.

I am deeply grateful to I. V. Komarov for fruitful discussion that stimulated the elaboration of this paper.

1. Classical Case

For the integration of the classical GC top (gyrostat) we appeal to the method of the classical r -matrix [7], which is essentially a version of the ISM. This method is applied to discrete completely integrable systems as follows. We attach to the given system a square $(N\times N)$ -matrix $L(u)$, called the L -operator, which depends on the dynamical variables and an auxiliary parameter u referred to as the spectral parameter. It is required that an $(N^2\times N^2)$ -matrix $r(u)$ exist, depending on the spectral parameter u, such that the following identity holds:

$$\{L(u) \overset{\otimes}{,} L(v)\} = [r(u-v), L(u)\otimes L(v)]_- . \tag{11}$$

It is readily verified [7] that (11) implies the involutiveness of the functions $t(u) = \operatorname{tr} L(u)$

$$\{t(u), t(v)\} = 0 . \tag{12}$$

If the Hamiltonian H of the given system and $t(u)$ are functionally dependent, then, by (12), $t(u)$ may be regarded as a generating function for commuting integrals of motion.

Now we have only to guess an L -operator and an r-matrix satisfying the above requirements for the GC top (gyrostat). To this end, it is convenient to enlarge the phase space by adding variables p and q with Poisson brackets

$$\begin{aligned} &\{p, q\} = 1 , \\ &\{p, J_\alpha\} = \{q, J_\alpha\} = 0 , \\ &\{p, x_\alpha\} = \{q, x_\alpha\} = 0 . \end{aligned} \tag{13}$$

Now set

$$L(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{14}$$

$$A(u) = (u+p+2J_3)K(u) + b(x_- u - x_3 J_-) , \tag{15a}$$

$$B(u) = be^{2iq}[(x_+ u - x_3 J_+)(u+p+2J_3) - bx_3^2] , \tag{15b}$$

$$C(u) = e^{-2iq} K(u) , \tag{15c}$$

and

$$D(u) = b(x_+ u - x_3 J_+) , \tag{15d}$$

where

$$K(u) = u^2 - 2J_3 u - (J^2 - J_3^2) = (u-u_1)(u-u_2) , \tag{16}$$

and

$$x_{\pm} = x_1 \pm i x_2 , \quad \mathcal{J}_{\pm} = \mathcal{J}_1 \pm i \mathcal{J}_2 . \tag{17}$$

One can verify by direct computation that the L -operator (14, 15) satisfies identity (11) with the r -matrix

$$r(u) = \frac{2i}{u} \mathcal{P}' , \qquad \mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{18}$$

A full list of Poisson brackets between entries of $L(u)$ is given in Appendix 1. We emphasize that equality (11) holds only for $\sigma = 0$.

As we have already remarked, identity (11) implies equality (12), which, in our case, holds with $t(u)$ the following cubic polynomial:

$$t(u) = A(u) + D(u) = u^3 + pu^2 - 2H_p u - G_p . \tag{19}$$

The coefficients H_p and G_p in (19) are the Hamiltonian (6) and respectively the integral motion (7) of the classical GC gyrostat, previously introduced. From (12) and (19) it follows that p, H_p , and G_p are in involution. Since p is an integral of motion, it can serve as a parameter in H_p and G_p . We have thus recovered the results of [2, 3] concerning the complete integrability of the GC top (gyrostat).

According to Chaplygin [2], in order to integrate the equations of motion of the GC top by quadratures, it is convenient to use the variables u_1, u_2 (5). Below we shall show that variables u_1, u_2 , together with the canonical conjugate variables v_1, v_2 admit a natural interpretation in the setting of the ISM. In our arguments we use the approach of papers [9, 10] translated in the language of the r -matrix formalism.

We first note that u_1 and u_2 are the roots of the quadratic polynomial $C(u)$ given by (15c, 16)*

$$C(u_n) = 0 , \quad n = 1, 2 . \tag{20}$$

Define the quantities $\lambda_n^{\pm}$ by the formulas

$$\lambda_n^- = A(u_n) , \quad \lambda_n^+ = D(u_n) , \quad n = 1, 2 . \tag{21}$$

Using formulas (15) and the chosen Casimir operators ρ and σ , it is readily verified that

$$\det L(u) = A(u) D(u) - B(u) C(u) = d(u) = \delta^2 u^2 . \tag{22}$$

Hence, in view of (20), (21)

$$\lambda_n^- \lambda_n^+ = A(u_n) D(u_n) = d(u_n) = \delta^2 u_n^2 . \tag{23}$$

Formulas (A1.1-16) of Appendix 1 and definitions (20), (21) permit us to compute the Poisson brackets between the quantities p, q, and $u_n, \lambda_n^{\pm}$:

$$\begin{gathered} \{p, u_n\} = \{p, \lambda_n^{\pm}\} = \{q, u_n\} = \{q, \lambda_n^{\pm}\} = 0 , \\ \{u_m, u_n\} = 0 , \quad \{\lambda_m^{\pm}, \lambda_n^{\pm}\} = 0 , \\ \{\lambda_m^{\pm}, u_n\} = \pm 2i \lambda_m^{\pm} \delta_{mn} , \\ \{\lambda_m^{+}, \lambda_n^{-}\} = 2i d'(u_m) \delta_{mn} = 4i \delta^2 u_m \delta_{mn} . \end{gathered} \tag{24}$$

*We owe this observation to I. V. Komarov.

For example, let us show how to compute the Poisson bracket $\{\lambda_m^+ u_n\}$. From (20), (21), and (A1.15) it follows that

$$\{\lambda_m^+ u_n\} = \{D(u_m), u_n\} = D'(u_m)\{u_m u_n\} + \{D(u), u_n\}\big|_{u=u_m} = -\frac{1}{c'(u_n)}\{D(u), C(v)\}\Big|_{\substack{u=u_m\\ v=u_n}} = -\frac{2i}{c'(u_n)}\frac{D(u)C(v)-C(u)D(v)}{u-v}\Big|_{\substack{u=u_m,\\ v=u_n}} = 2i\lambda_m^+\delta_{mn}.$$

The other equalities in (24) are verified in the same manner.

The variables p, q, u_n and $\lambda_n^{\pm}$ constitute a complete family of dynamical variables, meaning that every function on the phase space, in particular the entries $A(u)$, $B(u)$, $C(u)$, and $D(u)$ of the L-operator, is expressible in terms of these variables. In fact, the polynomials $C(u)$, $A(u)$, and $D(u)$ are uniquely specified by their values at the points u_n and the asymptotics

$$\begin{aligned} C(u) &= e^{-2iq}u^2 + O(u), \\ A(u) &= u^3 + pu^2 + O(u), \\ D(u) &= O(u) \end{aligned} \tag{25}$$

for $u \to \infty$. Using Lagrange's interpolation formula we find that

$$\begin{aligned} C(u) &= e^{-2iq}(u-u_1)(u-u_2), \\ A(u) &= (u+p+u_1+u_2)(u-u_1)(u-u_2) + \frac{u-u_2}{u_1-u_2}\lambda_1^- + \frac{u-u_1}{u_2-u_1}\lambda_2^-, \\ D(u) &= \frac{u-u_2}{u_1-u_2}\lambda_1^+ + \frac{u-u_1}{u_2-u_1}\lambda_2^+. \end{aligned} \tag{26}$$

The expression of $B(u)$ is obtained with the aid of formula (22):

$$B(u) = e^{2iq}(u+p+u_1+u_2)\left[\frac{u-u_2}{u_1-u_2}\lambda_1^+ + \frac{u-u_1}{u_2-u_1}\lambda_2^+\right] + \frac{e^{2iq}}{(u_1-u_2)}\left[2\sigma^2 u_1 u_2 - \lambda_1^+\lambda_2^- - \lambda_1^-\lambda_2^+\right]. \tag{27}$$

A comparison of (26) and (15) shows that

$$u_1 + u_2 = 2J_3, \qquad u_1 u_2 = J_3^2 - J^2; \tag{28}$$

$$\frac{\lambda_1^{\pm} - \lambda_2^{\pm}}{u_1 - u_2} = \sigma x_{\pm}, \quad \frac{u_2\lambda_1^{\pm} - u_1\lambda_2^{\pm}}{u_1 - u_2} = \sigma x_3 J_{\pm}; \tag{29}$$

$$2H = u_1^2 + u_1u_2 + u_2^2 - (u_1-u_2)^{-1}(\lambda_1^+ + \lambda_1^- - \lambda_2^+ - \lambda_2^-), \tag{30a}$$

and

$$G = -u_1u_2(u_1+u_2) - (u_1-u_2)^{-1}\left[u_1(\lambda_2^+ + \lambda_2^-) - u_2(\lambda_1^+ + \lambda_1^-)\right]. \tag{30b}$$

From the relations $\bar{x}_+ = x_-$, $\bar{J}_+ = J_-$, and (29) we obtain the reality conditions for $\lambda_n^{\pm}$

$$\bar{\lambda}_n^{\pm} = \lambda_n^{\mp}. \tag{31}$$

Equalities (23), (24), and (31) allow us to represent $\lambda_n^{\pm}$ in the form

$$\lambda_n^{\pm} = \sigma u_n e^{\pm 2iv_n}, \tag{32}$$

where v_n are the momenta canonically conjugate with the coordinates u_n:

$$\{v_1, v_2\} = 0, \quad \{v_m, u_n\} = \delta_{mn}.$$

Since the technique of integration by quadratures of the equations of motion of the GC top is described in detail in references [1, 2] (see also papers [9, 10], where the transition from the canonical variables u_n, v_n to action-angle variables is described in the case of the periodic Toda lattice), we end here the investigation of the classical GC top and turn to the quantum case.

2. Quantum Case

In the quantum case, as in the classical one, we must introduce the auxiliary operators p and q which obey the canonical commutation relation $[p,q]_-=-i$ and commute with the dynamical variables x_α and J_α (8).

The quantum L -operator is a matrix $\hat{L}(u)$ of the form

$$\hat{L}(u)=\begin{pmatrix}\hat{A}(u) & \hat{B}(u)\\ \hat{C}(u) & \hat{D}(u)\end{pmatrix}, \tag{33}$$

with entries given by the formulas

$$\hat{A}(u)=(u+p+2J_3)\hat{K}(u)+b(x_-u-\tfrac{1}{2}[x_3,J_-]_+), \tag{34a}$$

$$\hat{B}(u)=be^{2iq}\left[(x_+u-\tfrac{1}{2}[x_3,J_+]_+)(u+p+2J_3)-bx_3^2\right], \tag{34b}$$

$$\hat{C}(u)=e^{-2iq}\hat{K}(u), \tag{34c}$$

and

$$\hat{D}(u)=b(x_+u-\tfrac{1}{2}[x_3,J_+]_+), \tag{34d}$$

in which

$$\hat{K}(u)=u^2-2J_3u-(J^2-J_3^2+\tfrac{1}{4}). \tag{35}$$

By straightforward calculation one verifies that $\hat{L}(u)$ satisfies the equality [6, 7]

$$R(u-v)(\hat{L}(u)\otimes\hat{L}(v))=(I\otimes\hat{L}(v))(\hat{L}(u)\otimes I)R(u-v), \tag{36}$$

where

$$R(u)=u(1+ir(u))=u-2\mathcal{P}. \tag{37}$$

The full list of commutation relations for the operators $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$, derived from (36), is given in Appendix 2. Equality (36) plays the same role in quantum ISM as equality (11) plays in the classical case, ensuring the commutativity

$$[\hat{t}(u), \hat{t}(v)]_-=0 \tag{38}$$

of the values of the generating function of the integral of motion

$$\hat{t}(u)=\mathrm{tr}\,\hat{L}(u)=\hat{A}(u)+\hat{D}(u)=u^3+pu^2-2(H_p+\tfrac{1}{8})u-\hat{G}_p, \tag{39}$$

where H_p and $\hat{G}_p$ denote respectively the Hamiltonian (6) and the integral of motion (10) of the quantum GC gyrostat.

We are now prepared to approach the most interesting problem in the quantum case: the determination of the joint spectrum of the commuting self-adjoint operators H_p and $\hat{G}_p$. The customary approach to this problem in the setting of the quantum ISM is to use the so-called algebraic Bethe Ansatz [6], which in its simplest version can be described as follows. Suppose that there is a vector Ω (pseudovacuum) which is annihilated by the operator $\hat{C}(u)$ for any value of u :

$$\hat{C}(u)\,\Omega = 0 . \tag{40}$$

Then we seek the eigenvectors of the generating function of integrals of motion, $\hat{t}(u)$, in the form $\hat{B}(u_1)\ldots\hat{B}(u_N)\,\Omega$. The task of determining the spectrum and constructing the eigenvectors of $\hat{t}(u)$ is thus reduced to solving a certain system of equations for the parameters $u_1,\ldots,u_N$.

Unfortunately, the above method does not apply to the GC top (gyrostat), because Eq. (40) does not admit a solution Ω which does not depend on u ; indeed, the operator e^{-2iq} does not have zero as an eigenvalue. For this reason, in order to solve the problem formulated above we apply a new device, the idea of which is to extend the arguments of the preceding section to the quantum case. Proceeding by analogy with the classical case, let us try to decompose the polynomial $\hat{C}(u)$ into factors linear in u :

$$\hat{C}(u) = e^{-2iq}(u-\hat{u}_1)(u-\hat{u}_2) . \tag{41}$$

by (A2.11) operators $e^{-2iq}, \hat{u}_1$, and $\hat{u}_2$ must commute. From (34c) and (35) we obtain the following system of equations for $\hat{u}_1$ and $\hat{u}_2$:

$$\begin{cases} \hat{u}_1 + \hat{u}_2 = 2J_3 \\ \hat{u}_1\hat{u}_2 = J_3^2 - J^2 - \frac{1}{4} \end{cases} \tag{42}$$

Since the "momentum" p is an integral of motion, we may always confine our analysis to eigenstates corresponding to a fixed eigenvalue of p . For this reason, hereafter we shall not distinguish between the operator p and its eigenvalues.

Consider in the state space of the quantum GC top (gyrostat) for a fixed eigenvalue of p the basis given by the joint eigenvectors $|j,m\rangle$ of the operators J^2 and J_3 :

$$\begin{cases} J^2|j,m\rangle = j(j+1)|j,m\rangle, \\ J_3|j,m\rangle = m|j,m\rangle , \end{cases} \tag{43}$$

where j and m assume the values

$$\begin{cases} j = 0,1,2,\ldots \\ m = -j,-j+1,\ldots,0,\ldots,j . \end{cases} \tag{44}$$

We normalize the vectors $|j,m\rangle$ by the condition

$$\langle j,m|j',m'\rangle = \delta_{jj'}\,\delta_{mm'} . \tag{45}$$

From (42) and (43) we obtain the following equations for the eigenvalues $u_n(j,m)$ of the operators $\hat{u}_n$ on the vector $|j,m\rangle$:

$$\begin{cases} u_1(j,m) + u_2(j,m) = 2m, \\ u_1(j,m)\,u_2(j,m) = (m+j+\frac{1}{2})(m-j-\frac{1}{2}) . \end{cases} \tag{46}$$

To be specific, let us fix one of the two solutions of system (46), say

$$\begin{cases} u_1(j,m) = m+j+\frac{1}{2} \\ u_2(j,m) = m-j-\frac{1}{2} \end{cases} . \tag{47}$$

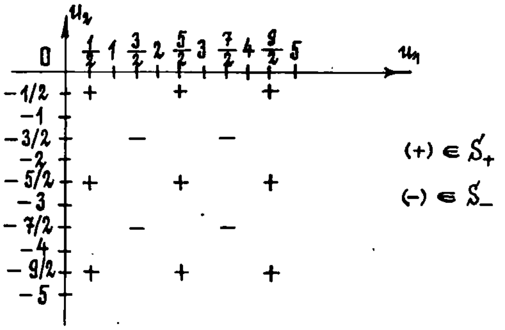

Fig. 1

As (44) and (47) show, the joint spectrum S of the commuting self-adjoint operators $\hat{u}_1$ and $\hat{u}_2$ thus defined is simple and is representable as the union $S = S_+ \cup S_-$ of the square lattices $S_\pm$ (see Fig. 1):

$$S_\pm = \{(u_1,u_2)\in\mathbb{R}^2 : (u_1,u_2) = (1\mp\tfrac{1}{2}+2n_1,\ -1\pm\tfrac{1}{2}-2n_2),\quad n_1,n_2 = 0,1,2,\dots\}\ , \tag{48}$$

which lie in the lower right quadrant (u_1,u_2) and correspond to the even (S_+) and odd (S_-) states relative to the involution p :

$$p|j,m\rangle = (-1)^{j+m}|j,m\rangle\ ,$$
$$pJ_3p = J_3\ ,\qquad px_3p = -x_3\ ,$$
$$pJ_\pm p = -J_\pm\ ,\qquad px_\pm p = x_\pm\ .$$

Thus, we can realize the state space of the quantum GC top (gyrostat) as the space $\mathcal{L}_2(S)$ of square-integrable functions on the spectrum S :

$$\mathcal{L}_2(S) = \{f(u_1,u_2) : (u_1,u_2)\in S,\ \sum_{(u_1,u_2)\in S}|f(u_1,u_2)|^2 < \infty\}\ .$$

The next step is to represent the generating function $\hat{t}(u)$ of integrals of motion as an operator in $\mathcal{L}_2(S)$. The most direct way of achieving this is to use the formulas given in [11], which describes explicitly an irreducible representation of the Lie algebra $e(3)$ with the prescribed values (2) of the Casimir operators:

$$\begin{aligned}
&J_3|j,m\rangle = m|j,m\rangle\ ,\\
&J_\pm|j,m\rangle = \pm i\sqrt{(j\pm m)(j\mp m+1)}\ |j,m\mp1\rangle\ ,\\
&x_3|j,m\rangle = i\sqrt{\frac{(j-m)(j+m)}{(2j-1)(2j+1)}}\,|j-1,m\rangle - i\sqrt{\frac{(j-m+1)(j+m+1)}{(2j+1)(2j+3)}}\,|j+1,m\rangle\ ,\\
&x_\pm|j,m\rangle = \sqrt{\frac{(j\pm m-1)(j\pm m)}{(2j-1)(2j+1)}}\,|j-1,m\mp1\rangle + \sqrt{\frac{(j\mp m+1)(j\mp m+2)}{(2j+1)(2j+3)}}\,|j+1,m\mp1\rangle\ .
\end{aligned} \tag{49}$$

Substituting expressions (49) and (34) in (39) we obtain the action of the operator $\hat{t}(u)$ on the wave function $f(u_1,u_2) = \langle j,m|f\rangle \in \mathcal{L}_2(S)$:

$$\begin{aligned}
(\hat{t}(u)f)(u_1,u_2) = {}&(u+p+u_1+u_2)(u-u_1)(u-u_2)f(u_1,u_2) + \frac{(u-u_2)\,\hat{d}^{1/2}(u_1+1)}{\sqrt{(u_1-u_2)(u_1-u_2+2)}}f(u_1+2,u_2) +\\
&\frac{(u-u_2)\,\hat{d}^{1/2}(u_1-1)}{\sqrt{(u_1-u_2)(u_1-u_2-2)}}f(u_1-2,u_2) + \frac{(u-u_1)\,\hat{d}^{1/2}(u_2+1)}{\sqrt{(u_1-u_2)(u_1-u_2-2)}}f(u_1,u_2+2) + \frac{(u-u_1)\,\hat{d}^{1/2}(u_2-1)}{\sqrt{(u_1-u_2)(u_1-u_2+2)}}f(u_1,u_2-2)\ ,
\end{aligned} \tag{50}$$

where $\hat{d}(u)$ designates the quantum determinant (A2.19) of the L -operator (33).

It is convenient to make the substitution

$$f(u_1,u_2) = i^{\frac{2u_2+1}{2}}\sqrt{u_1-u_2}\ \varphi(u_1,u_2)\ .$$

and thus simplify considerably formulas (50):

$$\begin{aligned}(\hat{t}(u)\varphi)(u_1,u_2) &= (u+p+u_1+u_2)(u-u_1)(u-u_2)\varphi(u_1,u_2)+\\ &+\frac{u-u_2}{u_1-u_2}\left[\hat{d}^{1/2}(u_1+1)\varphi(u_1+2,u_2)+\hat{d}^{1/2}(u_1-1)\varphi(u_1-2,u_2)\right]+\\ &+\frac{u-u_1}{u_2-u_1}\left[\hat{d}^{1/2}(u_2+1)\varphi(u_1,u_2+2)+\hat{d}^{1/2}(u_2-1)\varphi(u_1,u_2-2)\right].\end{aligned} \tag{51}$$

For the wave function $\varphi(u_1,u_2)$ the inner product has the form

$$<\chi|\varphi> = \sum_{(u_1,u_2)\in S}(u_1-u_2)\,\overline{\chi}(u_1,u_2)\varphi(u_1,u_2). \tag{52}$$

Note that $(u_1,u_2)\in S$ whenever $u_1-u_2>0$. We also remark that although the support of function $\varphi(u_1,u_2)$ lies in the square $\{u_1>0,\ u_2<0\}$, by shifting the arguments u_n by ± 2 in formula (51) we do not take $\varphi(u_1,u_2)$ out of the indicated state space because the factor $\hat{d}^{1/2}(u_n\pm 1)=b\sqrt{(u_n\pm\frac{1}{2})(u_n\pm\frac{3}{2})}$ vanishes for $u_n=\mp\frac{1}{2},\ \mp\frac{3}{2}$. Hereafter it will be convenient to assume that $\varphi(u_1,u_2)=0$ for $(u_1,u_2)\notin S$.

We derive the equations for the determination of the spectrum of $\hat{t}(u)$ by the following simple argument (cf. [8]). Let $\varphi(u_1,u_2)$ be an eigenfunction of $\hat{t}(u)$ corresponding to the eigenvalue $\tau(u)=u^3+pu^2-2(h_p+\frac{1}{8})u-g_p$, where h_p and g_p are eigenvalues of the operators H_p (6) and $\hat{G}_p$ (10). Then in formula (51) the left-hand side becomes $\tau(u)\varphi(u_1,u_2)$. Letting in (51) $u=u_1$ and then $u=u_2$, we get for $\varphi(u_1,u_2)$ the system of equations

$$\begin{cases}\tau(u_1)\varphi(u_1,u_2)=\hat{d}^{1/2}(u_1+1)\varphi(u_1+2,u_2)+\hat{d}^{1/2}(u_1-1)\varphi(u_1-2,u_2)\\ \tau(u_2)\varphi(u_1,u_2)=\hat{d}^{1/2}(u_2+1)\varphi(u_1,u_2+2)+\hat{d}^{1/2}(u_2-1)\varphi(u_1,u_2-2),\end{cases} \tag{53}$$

which is equivalent to the original system (51): indeed, the cubic polynomial $\tau(u)$ is uniquely specified by its values at the points u_1, u_2 and the asymptotics $\tau(u)=u^3+pu^2+O(u)$, as $u\to\infty$ via Lagrange interpolation.

Now consider the following spectral problem:

$$\tau(u)\chi(u)=\hat{d}^{1/2}(u+1)\chi(u+2)+\hat{d}^{1/2}(u-1)\chi(u-2) \tag{54}$$

for function χ of the variable u, which runs through the lattice $u=\pm\frac{1}{2},\ \pm\frac{3}{2}\ \pm\frac{5}{2}\ldots$, and with the boundary conditions for $\chi(u)$

$$\begin{cases}\chi(u)=0, & u<0\\ \sum_{\{u\}}u\,|\chi(u)|^2<\infty\end{cases}. \tag{55}$$

The three-term recursion relation (54) with boundary conditions (55) is studied in detail in [4] (for $p=0$) and [5] (in the general case). It is shown there that for $b\neq 0$ the eigenvalues h_p and g_p (p fixed), for which problem (54), (55) has a solution, form a discrete set, and that each point of the spectrum (h_p,g_p) has multiplicity one.

It is readily verified that the one-dimensional spectral problem (54), (55) is equivalent to the two-dimensional problem (51). In fact, to every solution $\chi(u)$ of (54), (55) there corresponds an eigenfunction $\varphi(u_1,u_2)$ of operator $\hat{t}(u)$ (51) of the form

$$\varphi|u_1,u_2| = \chi(u_1)\chi(-u_2) \tag{56}$$

and conversely, every eigenfunction of problem (51), (53) is necessarily of the form (56) because the spectrum of problem (54), (55) is simple.*

Thus we achieved separation of variables for the quantum GC top (gyrostat), that is, we have reduced the problem of determining the joint spectrum of the integrals of motion to a set of independent one-dimensional spectral problems, which may be subsequently investigated, say, by numerical methods [4, 5].

The tactic used above to derive formula (51) has, however, the drawback of resting too heavily on the explicit form (34) of the operators $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$. We conclude this section by describing a different tactic which uses only the fundamental relation (36) and hence may work for other models that can be treated by the inverse scattering method.

The idea used below in deriving formula (51) is to construct a quantum analog of the variables u_n (20) and $\lambda_n^{\pm}$ (21): as we have seen, these variables are quite useful in investigating the classical GC top. The quantum operators $\hat{u}_n$ were already defined in (41) and (47). We now introduce the operators $\hat{\lambda}_n^{\pm}$ by the formulas

$$\hat{\lambda}_n^- = \hat{A}(u \Leftarrow \hat{u}_n), \; \hat{\lambda}_n^+ = \hat{D}(u \Leftarrow u_n) ; \quad n = 1, 2; \tag{57}$$

where $\Leftarrow$ indicates the operators $\hat{u}_n$ are substituted in the operator polynomials $\hat{A}(u)$ and $\hat{D}(u)$ from the *left*: for example, the polynomial $\hat{A}(u) = \sum_{k=0}^{3} u^k \hat{A}_k$ becomes the operator $\hat{A}(u \Leftarrow \hat{u}_n) = \sum_{k=0}^{3} \hat{u}_n^k \hat{A}_k$. This convention is important since, generally, speaking, the operators $\hat{A}(u)$ and $\hat{D}(u)$ do not commute with $\hat{u}_n$.

Using formulas (A2.1-16) and definitions (41) and (57) we can write the commutation relations between the operators $p, q, \hat{u}_n$, and $\hat{\lambda}_n^{\pm}$:

$$[p, \hat{u}_n]_- = [p, \hat{\lambda}_n^{\pm}]_- = [q, \hat{u}_n]_- = [q, \hat{\lambda}_n^{\pm}]_- = 0 , \tag{58a}$$

$$[\hat{u}_m, \hat{u}_n]_- = 0, \; [\hat{\lambda}_m^{\pm}, \hat{\lambda}_n^{\pm}]_- = 0, \tag{58b}$$

$$\hat{\lambda}_m^{\pm} \hat{u}_n = (\hat{u}_n \pm 2\delta_{mn}) \hat{\lambda}_m^{\pm} . \tag{58c}$$

Moreover, relations (A2.20-27) yield the following equalities:

$$\hat{\lambda}_n^- \hat{\lambda}_n^+ = \hat{d}(\hat{u}_n - 1) = b^2 (\hat{u}_n - \tfrac{1}{2})(\hat{u}_n - \tfrac{3}{2}) , \tag{59a}$$

$$\hat{\lambda}_n^+ \hat{\lambda}_n^- = \hat{d}(\hat{u}_n + 1) = b^2 (\hat{u}_n + \tfrac{1}{2})(\hat{u}_n + \tfrac{3}{2}) . \tag{59b}$$

Let us show, for example, how to obtain the commutation relation (58c) between $\hat{\lambda}_m^-$ and $\hat{u}_n$. First, rewrite equality (A2.3) in the form

$$(u-v)\hat{A}(u)\hat{C}(v) = (u-v-2)\hat{C}(v)\hat{A}(u) + 2\hat{C}(u)\hat{A}(v) . \tag{60}$$

Now set here $u = \hat{u}_m$. Using definition (57) and the fact that, in view of (41), $[\hat{C}(v), \hat{u}_m]_- = 0$, and

$$\hat{C}(u \Leftarrow \hat{u}_m) = 0 \tag{61}$$

*Note that the condition that the spectrum be simple is not essential. If the eigenvalue (h_p, g_p) of problem (54), (55) has multiplicity μ, then it obviously has multiplicity μ^2 as an eigenvalue of problem (51), (53).

we find that

$$(\hat{u}_m - v)\,\hat{\lambda}_m^-\,\hat{C}(v) = (\hat{u}_m - v - 2)\,\hat{C}(v)\hat{\lambda}_m^- \ . \tag{62}$$

Suppose, to take a specific case, that $m=1$. Then, upon inserting in (62) the expression (41) for $\hat{C}(v)$ and simplifying at left by $e^{-2iq}(v-\hat{u}_1)$ (which is always possible when v does not belong to the spectrum of $\hat{u}_1$), we get

$$\hat{\lambda}_1^-\,(v-\hat{u}_1)(v-\hat{u}_2) = (v-\hat{u}_1+2)(v-\hat{u}_2)\,\hat{\lambda}_1^- \ . \tag{63}$$

This equality immediately yields

$$\hat{\lambda}_1^-\, s(\hat{u}_1,\hat{u}_2) = s(\hat{u}_1-2;\hat{u}_2)\,\hat{\lambda}_1^- \tag{64}$$

for any *symmetric* function $s(u_1,u_2)=s(u_2,u_1)$ because, as it is known, every such function u_1,u_2 is uniquely expressible in terms of the symmetric polynomials u_1+u_2 and u_1u_2. But since the joint spectrum $\mathcal{S}$ of the operators $\hat{u}_1$ and $\hat{u}_2$ lies, as we observed earlier, in the right-lower quadrant of the plane $\mathbb{R}^2$ equality (64) is actually valid for *any* function $s(u_1,u_2)$, in particular, for $s(u_1,u_2)=u_1$ and $s(u_1,u_2)=u_2$. The proof is complete.

Similarly, the substitution $u \triangleq u_n$ in equality (A2.12) yields the commutation relation (58c) between $\hat{\lambda}_m^+$ and $\hat{u}_n$.

The proof of equality (59a) is more tedious. We first remark that, as in the classical case, definitions (57) and asymptotics (25) yield the equalities

$$\begin{gathered}\hat{A}(u) = (u+p+\hat{u}_1+\hat{u}_2)(u-\hat{u}_1)(u-\hat{u}_2) + \frac{u-\hat{u}_2}{\hat{u}_1-\hat{u}_2}\,\hat{\lambda}_1^- + \frac{u-\hat{u}_1}{\hat{u}_2-\hat{u}_1}\,\hat{\lambda}_2^- \ ,\\ \hat{\mathfrak{D}}(u) = \frac{u-\hat{u}_2}{\hat{u}_1-\hat{u}_2}\,\hat{\lambda}_1^+ + \frac{u-\hat{u}_1}{\hat{u}_2-\hat{u}_1}\,\hat{\lambda}_2^+ \ ,\end{gathered} \tag{65}$$

in which, unlike formulas (26), the order of the operator factors is important.

Now consider the equality

$$\hat{A}(u)\,\hat{\mathfrak{D}}(u-2) - \hat{C}(u)\,\hat{B}(u-2) = \hat{d}(u-1)\,.$$

The second left-hand term vanishes on substituting $u \triangleq \hat{u}_n$ in view of (61). To evaluate the first term, replace in it $\hat{A}(u)$ and $\hat{\mathfrak{D}}(u)$ by their expressions in terms of $\hat{u}_n$ and $\hat{\lambda}_n^{\pm}$, which yields:

$$\begin{gathered}\hat{A}(u)\,\hat{\mathfrak{D}}(u-2) = (u+p+\hat{u}_1+\hat{u}_2)(u-\hat{u}_1)(u-\hat{u}_2)\hat{\mathfrak{D}}(u-2) + \\ + \frac{u-\hat{u}_2}{\hat{u}_1-\hat{u}_2}\hat{\lambda}_1^-\frac{u-\hat{u}_2-2}{\hat{u}_1-\hat{u}_2}\hat{\lambda}_1^+ + \frac{u-\hat{u}_2}{\hat{u}_1-\hat{u}_2}\hat{\lambda}_1^-\frac{u-\hat{u}_1-2}{\hat{u}_2-\hat{u}_1}\hat{\lambda}_2^+ + \frac{u-\hat{u}_1}{\hat{u}_2-\hat{u}_1}\hat{\lambda}_2^-\frac{u-\hat{u}_2-2}{\hat{u}_1-\hat{u}_2}\hat{\lambda}_1^+ + \frac{u-\hat{u}_1}{\hat{u}_2-\hat{u}_1}\hat{\lambda}_2^-\frac{u-\hat{u}_1-2}{\hat{u}_2-\hat{u}_1}\hat{\lambda}_2^+ \ .\end{gathered}$$

Next, use relations (58c) to move the operators $\hat{\lambda}_n^-$ to the right:

$$\begin{gathered}\hat{A}(u)\hat{\mathfrak{D}}(u-2) = (u+p+\hat{u}_1+\hat{u}_2)(u-\hat{u}_1)(u-\hat{u}_2)\hat{\mathfrak{D}}(u-2) + \\ + \frac{(u-\hat{u}_2)(u-\hat{u}_2-2)}{(\hat{u}_1-\hat{u}_2)(\hat{u}_1-\hat{u}_2-2)}\hat{\lambda}_1^-\hat{\lambda}_1^+ + \frac{(u-\hat{u}_2)(u-\hat{u}_1)}{(\hat{u}_1-\hat{u}_2)(\hat{u}_2-\hat{u}_1+2)}\hat{\lambda}_1^-\hat{\lambda}_2^+ + \frac{(u-\hat{u}_1)(u-\hat{u}_2)}{(\hat{u}_2-\hat{u}_1)(\hat{u}_1-\hat{u}_2+2)}\hat{\lambda}_2^-\hat{\lambda}_1^+ + \frac{(u-\hat{u}_1)(u-\hat{u}_1-2)}{(\hat{u}_2-\hat{u}_1)(\hat{u}_2-\hat{u}_1-2)}\hat{\lambda}_2^-\hat{\lambda}_2^+ \ .\end{gathered}$$

Substituting here $u \triangleq \hat{u}_n$, we get $\hat{\lambda}_n^-\hat{\lambda}_n^+$, as claimed. Proceeding similarly with equality (A2.27) one gets (59b).

Next we need the representations of relations (58b, c) and (59) in the space of functions $\varphi(u_1,u_2)$ on $\mathcal{S}$ (we leave open for the moment the problem of specifying the metric in this space). As we already know, $\hat{u}_n$ are realized as the multiplication operators

$$(\hat{u}_n\varphi)(u_1,u_2) = u_n\varphi(u_1,u_2)\,. \tag{66}$$

It is readily verified that the operators $\hat{\lambda}_n^{\pm}$ defined by the rules

$$\begin{aligned}(\hat{\lambda}_1^{\pm}\varphi)(u_1,u_2) &= \hat{d}^{1/2}(u_1\pm 1)\,\varphi(u_1\pm 2,u_2),\\ (\hat{\lambda}_2^{\pm}\varphi)(u_1,u_2) &= \hat{d}^{1/2}(u_2\pm 1)\,\varphi(u_1,u_2\pm 2),\end{aligned} \tag{67}$$

satisfy, together with the operators $\hat{u}_n$ (66), all relations (58) and (59). Moreover, one can show that representation (67) for $\hat{\lambda}_n^{\pm}$ is unique to within similarity transformations $\hat{\lambda}_n^{\pm}\to W^{-1}\hat{\lambda}_n^{\pm}W$ with operators W of the form

$$(W\varphi)(u_1,u_2) = w(u_1,u_2)\,\varphi(u_1,u_2)\,.$$

We omit the easy proof of this fact, which repeats the proof of the uniqueness of the irreducible representation of the Heisenberg group.

To obtain representation (51) for the operator $\hat{t}(u)=\hat{A}(u)+\hat{D}(u)$ it remains to substitute expression (67) in formula (65).

The last aspect that should be considered is the form of the metric in the space of functions φ. Comparing expressions (65) and (41) with (34), we see that

$$\begin{gathered}\hat{u}_1+\hat{u}_2 = 2J_3\,, \quad \hat{u}_1\hat{u}_2 = J_3^2 - J^2 - \tfrac{1}{4}\,,\\ (\hat{u}_1-\hat{u}_2)^{-1}(\hat{\lambda}_1^{\pm}-\hat{\lambda}_2^{\pm}) = b\,x_{\pm}\,,\\ (\hat{u}_1-\hat{u}_2)^{-1}(\hat{u}_2\hat{\lambda}_1^{\pm}-\hat{u}_1\hat{\lambda}_2^{\pm}) = \tfrac{b}{2}[x_3,J_{\pm}]_+\,.\end{gathered} \tag{68}$$

For reader's convenience we give also the expressions for the integrals of motion

$$\begin{aligned}2H &= \hat{u}_1^2+\hat{u}_1\hat{u}_2+\hat{u}_2^2-(\hat{u}_1-\hat{u}_2)^{-1}(\hat{\lambda}_1^{+}+\hat{\lambda}_1^{-}-\hat{\lambda}_2^{+}-\hat{\lambda}_2^{-})-\tfrac{1}{4},\\ \hat{G} &= -\hat{u}_1\hat{u}_2(\hat{u}_1+\hat{u}_2)+(\hat{u}_1-\hat{u}_2)^{-1}\left[\hat{u}_2(\hat{\lambda}_1^{+}+\hat{\lambda}_1^{-})-\hat{u}_1(\hat{\lambda}_2^{+}+\hat{\lambda}_2^{-})\right]\,.\end{aligned}$$

(The order of the operator factors is again essential!)

From the equalities $x_+^*=x_-$, $J_+^*=J_-$, $x_3^*=x_3$, $J_3^*=J_3$ and commutations relations (58c) we obtain the conjugation relation for the operators $\hat{u}_n$ and $\hat{\lambda}_n^{\pm}$:

$$\hat{u}_n^* = u_n\,, \tag{69a}$$

$$\hat{\lambda}_1^{\pm *} = \frac{\hat{u}_1-\hat{u}_2\mp 2}{\hat{u}_1-\hat{u}_2}\,\hat{\lambda}_1^{\mp}\,, \quad \hat{\lambda}_2^{\pm *} = \frac{\hat{u}_1-\hat{u}_2\pm 2}{\hat{u}_1-\hat{u}_2}\,\hat{\lambda}_2^{\mp}\,. \tag{69b}$$

We look for the inner product for the functions $\varphi(u_1,u_2)$ in the form

$$\langle \chi|\varphi\rangle = \sum_{(u_1,u_2)\in S} \rho(u_1,u_2)\,\bar{\chi}(u_1,u_2)\,\varphi(u_1,u_2)\,, \tag{70}$$

such that the operators $\hat{u}_n$ (66) and $\hat{\lambda}_n^{\pm}$ (67) satisfy relations (69). Relation (69a) is automatically satisfied, whereas (69b) gives for function $\rho(u_1,u_2)$ the equations

$$\rho(u_1+2,u_2) = \frac{u_1-u_2+2}{u_1-u_2}\,\rho(u_1,u_2)\,,$$

$$\rho(u_1,u_2+2) = \frac{u_1-u_2-2}{u_1-u_2}\,\rho(u_1,u_2)\,,$$

whose unique (modulo equivalence) solution is $\rho(u_1,u_2)=u_1-u_2$, which leads to the metric (52).

CONCLUSIONS

We list a number of open problems related to the results discussed above.

For the moment the relationship between the GC top and the other completely integrable models possessing the same $sl(2)$-invariant R-matrix (37) is still not clear. Very likely, the GC top is a degenerate case of some model of a lattice ferromagnet on chain with three sites.

It would be also interesting to generalize the GC top for Lie algebras of dynamical variables other than $e(3)$ (8) or for the N-dimensional rigid body.

We wish to emphasize that the method proposed here for reducing the determination of the joint spectrum of the integrals of motion of the quantum GC top to a one-dimensional spectral of the form (54) is based almost exclusively on relations (36). For this reason its domain of applicability extends far beyond the specific model of the GC top considered here. At the present time the author is working on the application of this method to completely integrable models such as the Toda lattice and the sh-Gordon equation [6, 7], the L-operators of which do not possess a local vacuum.

APPENDIX 1

Below we give a full list of the matrix entries of relation (11), which in view of formula (18) for the r-matrix can be rewritten in the form

$$\{L_{\alpha_1\beta_1}(u), L_{\alpha_2\beta_2}(v)\} = \frac{2i}{u-v}\left[L_{\alpha_2\beta_1}(u)L_{\alpha_1\beta_2}(v) - L_{\alpha_1\beta_2}(u)L_{\alpha_2\beta_1}(v)\right],$$

where, according to (14),

$$L_{11}(u)=A(u),\ L_{12}(u)=B(u),\ L_{21}(u)=C(u),\ L_{22}(u)=D(u).$$

For each relation we indicate, at left, the values of the indices $(\alpha_1\alpha_2\beta_1\beta_2)$.

$$(1111):\{A(u),A(v)\}=0 \tag{A1.1}$$

$$(1112):\{A(u),B(v)\}=\frac{2i}{u-v}[A(u)B(v)-B(u)A(v)] \tag{A1.2}$$

$$(1211):\{A(u),C(v)\}=\frac{2i}{u-v}[-A(u)C(v)+C(u)A(v)] \tag{A1.3}$$

$$(1212):\{A(u),D(v)\}=\frac{2i}{u-v}[C(u)B(v)-B(u)C(v)] \tag{A1.4}$$

$$(1121):\{B(u),A(v)\}=\frac{2i}{u-v}[B(u)A(v)-A(u)B(v)] \tag{A1.5}$$

$$(1122):\{B(u),B(v)\}=0 \tag{A1.6}$$

$$(1221):\{B(u),C(v)\}=\frac{2i}{u-v}[D(u)A(v)-A(u)D(v)] \tag{A1.7}$$

$$(1222):\{B(u),D(v)\}=\frac{2i}{u-v}[-B(u)D(v)+D(u)B(v)] \tag{A1.8}$$

$$(2111):\{C(u),A(v)\}=\frac{2i}{u-v}[-C(u)A(v)+A(u)C(v)] \tag{A1.9}$$

$$(2112):\{C(u),B(v)\}=\frac{2i}{u-v}[A(u)D(v)-D(u)A(v)] \tag{A1.10}$$

$$(2211):\{C(u),C(v)\}=0 \tag{A1.11}$$

$$(2212):\{C(u),D(v)\}=\frac{2i}{u-v}[C(u)D(v)-D(u)C(v)] \tag{A1.12}$$

$$(2121):\{D(u),A(v)\}=\frac{2i}{u-v}[B(u)C(v)-C(u)B(v)] \tag{A1.13}$$

$$(2122):\{D(u),B(v)\}=\frac{2i}{u-v}[-D(u)B(v)+B(u)D(v)] \tag{A1.14}$$

$$(2221): \{\mathfrak{D}(u), C(v)\} = \frac{2i}{u-v}\left[\mathfrak{D}(u)C(v) - C(u)\mathfrak{D}(v)\right] \tag{A1.15}$$

$$(2222): \{\mathfrak{D}(u), \mathfrak{D}(v)\} = 0 \tag{A1.16}$$

APPENDIX 2

Using formula (37) we write relation (36) in the form

$$(u-v)\cdot \hat{L}_{\alpha_1\beta_1}(u)\hat{L}_{\alpha_2\beta_2}(v) - 2\cdot\hat{L}_{\alpha_2\beta_1}(u)\hat{L}_{\alpha_1\beta_2}(v) = (u-v)\cdot\hat{L}_{\alpha_2\beta_2}(v)\hat{L}_{\alpha_1\beta_1}(u) - 2\cdot\hat{L}_{\alpha_2\beta_1}(v)\hat{L}_{\alpha_1\beta_2}(u) \, .$$

A complete list of the matrix entries of relation (36) is given below, where, as in Appendix 1, we indicate at left the values of the indices $(\alpha_1\alpha_2\beta_1\beta_2)$.

$$(1111): (u-v-2)\hat{A}(u)\hat{A}(v) = (u-v-2)\hat{A}(v)\hat{A}(u) \tag{A2.1}$$

$$(1112): (u-v-2)\hat{A}(u)\hat{B}(v) = (u-v)\hat{B}(v)\hat{A}(u) - 2\hat{A}(v)\hat{B}(u) \tag{A2.2}$$

$$(1211): (u-v)\hat{A}(u)\hat{C}(v) - 2\hat{C}(u)\hat{A}(v) = (u-v-2)\hat{C}(v)\hat{A}(u) \tag{A2.3}$$

$$(1212): (u-v)\hat{A}(u)\hat{\mathfrak{D}}(v) - 2\hat{C}(u)\hat{B}(v) = (u-v)\hat{\mathfrak{D}}(v)\hat{A}(u) - 2\hat{C}(v)\hat{B}(u) \tag{A2.4}$$

$$(1121): (u-v-2)\hat{B}(u)\hat{A}(v) = (u-v)\hat{A}(v)\hat{B}(u) - 2\hat{B}(v)\hat{A}(u) \tag{A2.5}$$

$$(1122): \hat{B}(u)\hat{B}(v) = \hat{B}(v)\hat{B}(u) \tag{A2.6}$$

$$(1221): (u-v)\hat{B}(u)\hat{C}(v) - 2\hat{\mathfrak{D}}(u)\hat{A}(v) = (u-v)\hat{C}(v)\hat{B}(u) - 2\hat{\mathfrak{D}}(v)\hat{A}(u) \tag{A2.7}$$

$$(1222): (u-v)\hat{B}(u)\hat{\mathfrak{D}}(v) - 2\hat{\mathfrak{D}}(u)\hat{B}(v) = (u-v-2)\hat{\mathfrak{D}}(v)\hat{B}(u) \tag{A2.8}$$

$$(2111): (u-v)\hat{C}(u)\hat{A}(v) - 2\hat{A}(u)\hat{C}(v) = (u-v-2)\hat{A}(v)\hat{C}(u) \tag{A2.9}$$

$$(2112): (u-v)\hat{C}(u)\hat{B}(v) - 2\hat{A}(u)\hat{\mathfrak{D}}(v) = (u-v)\hat{B}(v)\hat{C}(u) - 2\hat{A}(v)\hat{\mathfrak{D}}(u) \tag{A2.10}$$

$$(2211): \hat{C}(u)\hat{C}(v) = \hat{C}(v)\hat{C}(u) \tag{A2.11}$$

$$(2212): (u-v-2)\hat{C}(u)\hat{\mathfrak{D}}(v) = (u-v)\hat{\mathfrak{D}}(v)\hat{C}(u) - 2\hat{C}(v)\hat{\mathfrak{D}}(u) \tag{A2.12}$$

$$(2121): (u-v)\hat{\mathfrak{D}}(u)\hat{A}(v) - 2\hat{B}(u)\hat{C}(v) = (u-v)\hat{A}(v)\hat{\mathfrak{D}}(u) - 2\hat{B}(v)\hat{C}(u) \tag{A2.13}$$

$$(2122): (u-v)\hat{\mathfrak{D}}(u)\hat{B}(v) - 2\hat{B}(u)\hat{\mathfrak{D}}(v) = (u-v-2)\hat{B}(v)\hat{\mathfrak{D}}(u) \tag{A2.14}$$

$$(2221): (u-v-2)\hat{\mathfrak{D}}(u)\hat{C}(v) = (u-v)\hat{C}(v)\hat{\mathfrak{D}}(u) - 2\hat{\mathfrak{D}}(v)\hat{C}(u) \tag{A2.15}$$

$$(2222): (u-v-2)\hat{\mathfrak{D}}(u)\hat{\mathfrak{D}}(v) = (u-v-2)\hat{\mathfrak{D}}(v)\hat{\mathfrak{D}}(u) \tag{A2.16}$$

In the main text of the paper we have used a number of formulas related to the notion of quantum determinant [7]. With the aid of the explicit form (34) of the quantum L -operator we may verify the relation

$$\sigma_2\hat{L}(u)\sigma_2\hat{L}(u-2) = \hat{d}(u-1), \tag{A2.17}$$

as well as the equivalent relation

$$\hat{L}(u)\sigma_2\hat{L}(u+2)\sigma_2 = \hat{d}(u+1), \tag{A2.18}$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{d}(u) = b^2\left(u^2 - \frac{1}{4}\right) \, . \tag{A2.19}$$

The quantity $\hat{d}(u)$ is called the quantum determinant [7] of the L-operator.

Finally, we write down the matrix entries of the relation (A2.17):

$$\hat{D}(u)\hat{A}(u-\hbar)-\hat{B}(u)\hat{C}(u-\hbar)=\hat{d}(u-1), \tag{A2.20}$$

$$\hat{D}(u)\hat{B}(u-\hbar)-\hat{B}(u)\hat{D}(u-\hbar)=0, \tag{A2.21}$$

$$\hat{A}(u)\hat{C}(u-\hbar)-\hat{C}(u)\hat{A}(u-\hbar)=0, \tag{A2.22}$$

$$\hat{A}(u)\hat{D}(u-\hbar)-\hat{C}(u)\hat{B}(u-\hbar)=\hat{d}(u-1) \tag{A2.23}$$

and of relation (A2.18)

$$\hat{A}(u)\hat{D}(u+\hbar)-\hat{B}(u)\hat{C}(u+\hbar)=\hat{d}(u+1), \tag{A2.24}$$

$$\hat{B}(u)\hat{A}(u+\hbar)-\hat{A}(u)\hat{B}(u+\hbar)=0, \tag{A2.25}$$

$$\hat{C}(u)\hat{D}(u+\hbar)-\hat{D}(u)\hat{C}(u+\hbar)=0, \tag{A2.26}$$

$$\hat{D}(u)\hat{A}(u+\hbar)-\hat{C}(u)\hat{B}(u+\hbar)=\hat{d}(u+1). \tag{A2.27}$$

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