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Monopole-Quadrupole Static Axisymmetric Solutions of Einstein Field Equations

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An infinite family of exact solutions of the Einstein vacuum equations for the static case with axial symmetry is presented in an explicit form. Each solution of this family contains two arbitrary parameters M and Q that represent the mass and quadrupole moment of the source. In addition, each solution can be interpreted physically as the pure relativistic quadrupole correction to the Schwarzschild solution at a given multipole order.

1. INTRODUCTION

In Newtonian gravitation, the field created by a celestial body of mass M with axial symmetry is determined at each point of the exterior space by the following potential:

$$\Phi = -G \sum_{n=0}^{\infty} \frac{D_n}{\hat{r}^{n+1}} P_n(\cos\hat{\theta}), \tag{1}$$

where G is the universal gravitation constant, $(\hat{r}, \hat{\theta})$ are the radial and polar coordinates of the point with respect to an origin situated on the symmetry axis, P_n are the Legendre polynomials and constants D_n are the multipolar moments of the source, defined as

$$D_n = \int_V \mu(\xi, \beta) \xi^n P_n(\cos\beta)\, d^3\vec{\xi}, \qquad D_0 \equiv M, \tag{2}$$

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where $\mu(\xi, \beta)$ is the density of the object and where the integral is extended to the volume of that object such that $\vec{\xi}$ represents the positional vector of the point of integration, and β is the corresponding polar angle. It is interesting to recall that, assuming that the source has equatorial symmetry (which models a celestial object reasonably well), and taking the origin of the coordinates on the symmetry plane, the multipolar moments of odd order prove to be null.

The description of the field through series (1) has the great advantage of disclosing an important physical characteristic of the source, that is its mass distribution. Additionally, since all the summands are solutions of the Laplace equation, it is possible to cut the whole series at any order n, thus obtaining a solution that admits the two following possible interpretations. Either it represents the field at a point sufficiently distant from the source to be able to dispense with the terms corresponding to moments higher than order n, or the source has a mass distribution sufficiently close to spherical symmetry for those moments to be considered negligible. This second assumption is the most natural one in astrophysics since celestial bodies do not differ excessively from spherical distribution. Thus, for example, if one considers a homogeneous revolution ellipsoid with half-axes (a, a, b), which forms a possible configuration of equilibrium [1], the multipolar moments are written as follows:[2]

$$\begin{cases} D_{2n} = \dfrac{(-2)^n \, 3 M a^{2n}}{(2n+1)(2n+3)} \epsilon^n (1 - \epsilon/2)^n , \\ D_{2n+1} = 0 \quad , \end{cases} \qquad \epsilon \equiv \frac{a-b}{a} . \tag{3}$$

This formula reveals the progressive decrease in the importance of the moments for small deviations from sphericity since the parameter ϵ measures this deviation and appears elevated to ever higher powers (note that this parameter will be positive or negative, depending on whether the ellipsoid is flattened or elongated, respectively).

Generalization of all the foregoing to general relativity, as is known, has serious problems even in the static case, which is the simplest one. The main reason for this stems from the fact that the general solution of the corresponding Einstein equations (vacuum, staticity and axial symmetry) are not known *in terms of the source*, since there is no Green function available for these equations; this however, does happen for the Poisson

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[2] It is obtained by simple calculation making use of integral 7.226/3 defined on p.822 of Ref. 2.

equation. Accordingly, for the time being it is not possible to develop a completely coherent program aimed at searching for solutions characterized by physical properties of the source linked to its mass distribution.

However, within the relativistic community there is consensus that a set of relativistic multipole moments can be associated to all static vacuum solutions; these were initially defined for this case by Geroch [3], later generalized to the stationary case by Hansen, Beig, Simon and others [4], and finally generalized to the general case by Thorne [5]. These moments characterize the mass distribution of the source, although they do not have integral expression of the type of (2). It should be noted, however, that attempts have been made [6] to connect the Geroch moments with the source using Tulczijew skeleton-sources [7].

In recent years, several methods have been developed to find exact solutions to the Einstein–Maxwell equations based on the work of Ernst [8]. These have led to an surfeit of works dealing with the topic. These solutions are interpreted *a posteriori* in physical terms, calculating their relativistic moments without these having played any role in the search process, such that some of the solutions are of doubtful value.

Our aim here is to invert this procedure by searching for solutions that will have *preset* multipole moments thus guaranteeing their physical interpretation from the very start. In this work we shall focus on the simple case of the static metrics with axial symmetry since in this case the general Weyl [9] solution is already known; this will allow us to set up the method we shall use. More explicitly, we shall search for and find the pure monopole-quadrupole solution (with equatorial symmetry), i.e. that all its multipole moments will be null with the exception of mass and the quadrupole moment, such that its classic equivalent will be the solution formed by the first two terms of series (1). As a result, it will be possible to consider it as a small deviation from the spherically symmetric solution; that is, from the Schwarzschild solution. With this, we generalize previous work [10] in which a solution containing two parameters representing the aforementioned moments was found.

In Section 2 we review the three known ways of representing the general Weyl solution, demonstrating directly and for the first time the equivalence between them as regards the component g_{00} of the metric, which is the one intervening in later calculation of the Geroch moments. This proof allows us to formulate explicitly the relationship among the three families of parameters involved, hitherto unsolved.

In Section 3 we offer a succinct description of the method employed to calculate the Geroch moments of the general Weyl solution, limiting ourselves to giving explicitly the first twelve since their extension becomes excessive for the following ones. The method is no more than that resulting from the definition of Thorne [5] of the relativistic moments for the stationary case. As has been demonstrated by Gursel [11], this is equivalent to that of Geroch. Owing to their extreme lengthiness the final explicit calculations were performed using the MATHEMATICA software program.

In Section 4, the results of the previous section are used to obtain the general structure of the pure monopole-quadrupole solution by means of an inductive process.

Finally, in Section 5, that solution is written explicitly in the form of an infinite series of powers of a certain dimensionless parameter q. However, this series features a major property in the sense that it is possible to cut off at any order n, giving rise to an *exact solution* whose multipole moments of order less or equal to $2(n+1)$ are all null with the exception of the mass and the quadrupole moment and such that the higher moments are of order q^{n+1}. This allows us to give a concrete meaning to the small deviations from the Schwarzschild solution.

In Appendix A we offer 4 Lemmas used in Sections 2 and 5. Finally, in Appendix B we list the packages used for performing the corresponding calculations with MATHEMATICA.

2. THE WEYL METRICS

The simplest way to describe the general solution to the Einstein vacuum equations for the static case with axial symmetry is to use the Weyl line element [9]

$$ds^2 = -e^{2\Psi} dt^2 + e^{-2\Psi}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right], \tag{4}$$

where the metric functions Ψ and γ, which depend only on the cylindrical coordinates $\{\rho, z\}$, are solutions of the differential set

$$\Delta\Psi \equiv \Psi_{\rho\rho} + \frac{1}{\rho}\Psi_\rho + \Psi_{zz} = 0, \tag{5}$$

$$\begin{cases} \gamma_\rho = \rho(\Psi_\rho^2 - \Psi_z^2) \\ \gamma_z = 2\rho\Psi_\rho\Psi_z \,. \end{cases} \tag{6}$$

These equations lead to the following asymptotically flat general solution:

$$\Psi = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\cos\theta), \tag{7}$$

$$\gamma = \sum_{n,k=0}^{\infty} \frac{(n+1)(k+1)}{n+k+2} \frac{a_n a_k}{r^{n+k+2}} (P_{n+1}P_{k+1} - P_n P_k), \tag{8}$$

where $r = (\rho^2 + z^2)^{1/2}$, $\cos\theta = z/r$ and $P_n(\cos\theta)$ are the Legendre polynomials. On the other hand, a_n are arbitrary real constants which will be called "Weyl's moments" since evidently they cannot be identified with the relativistic moments despite the formal equality between expression (7) and the classical potential (1). Indeed, as is known, the Schwarzschild solution, which defines a pure monopole, is defined in the Weyl structure as follows:

$$a_{2n} = -\frac{M^{2n+1}}{2n+1}, \qquad a_{2n+1} = 0, \tag{9}$$

which is very different from the "Weyl monopole" defined by the only non-null coefficient a_0 corresponding to the well-known Curzon solution [12].

Another interesting way of writing the general solution (7),(8) was obtained by Erez–Rosen [13] (see also Ref. 14) by integrating eqs. (5),(6) in prolate spheroidal coordinates x, y, defined by

$$\begin{gathered} x = \frac{r_+ + r_-}{2M}, \qquad y = \frac{r_+ - r_-}{2M} \\ r_\pm \equiv [\rho^2 + (z \pm M)^2]^{1/2} \equiv r(\lambda^2 \pm 2\lambda\cos\theta + 1)^{1/2} \\ x \geq 1, \qquad -1 \leq y \leq 1, \end{gathered} \tag{10}$$

where M is a constant that is identified with the mass (expressed in dimensions of length) of the body generating the field and where the variable $\lambda \equiv M/r$ has been introduced. In these coordinates, the expression for the metric function Ψ adopts the following simple form:

$$\Psi = \sum_{n=0}^{\infty} (-1)^{n+1} q_n Q_n(x) P_n(y), \tag{11}$$

where $Q_n(x)$ are the second type Legendre functions and where q_n are arbitrary constants which, as indicated in [10], are related to the Weyl moments a_n as follows:

$$\begin{gathered} a_n = \sum_{j=0}^{T} (-M)^{n+1} \frac{n!}{(n+k+1)!!(n-k)!!} q_k \\ \begin{cases} n \text{ even} & : \ k = 2j, \quad T = n/2 \\ n \text{ odd} & : \ k = 2j+1, \quad T = (n-1)/2. \end{cases} \end{gathered} \tag{12}$$

The expression of γ, obtained by Quevedo [14], proves to be highly complicated and as a result will not be explicitly given here.

It is possible to make a direct demonstration of the equivalence between series (7) and (11) by substituting expression (12) in (7), then rearranging the sums and finally taking into account Lemma 3 of Appendix A. Indeed, limiting ourselves to the case of equatorial symmetry, for which odd-order coefficients are null, according to (12) one has that

$$a_{2n} = -M^{2n+1} \sum_{k=0}^{n} \frac{(2n)!}{(2n+2k+1)!!(2n-2k)!!} q_{2k} \,. \tag{13}$$

However, through the usual technique of decomposition into simple fractions the following relationship can readily be proved:

$$\frac{(2n)!}{(2n+2k+1)!!(2n-2k)!!} = \sum_{j=0}^{k} \frac{L_{2k,2j}}{2n+2j+1} \,, \tag{14}$$

where the coefficients $L_{2k,2j}$ are defined as follows:

$$L_{2k,2j} \equiv (-1)^{k-j} 2^{j-k} \frac{(2k+2j-1)!!}{(k-j)!(2j)!} \tag{15}$$

and represent the coefficients of the Legendre polynomials; that is,

$$P_{2k}(\zeta) = \sum_{j=0}^{k} L_{2k,2j} \zeta^{2j} \,. \tag{16}$$

Substituting these results in (7), one obtains the following:

$$\Psi = -\sum_{n=0}^{\infty} \lambda^{2n+1} P_{2n}(\cos\theta) \sum_{k=0}^{n} q_{2k} \sum_{j=0}^{k} \frac{L_{2k,2j}}{2n+2j+1} \,. \tag{17}$$

Now taking into account that by virtue of Lemma 1 the upper limit of the second summatory can be made infinite, one can rearrange the sums in the form

$$\Psi = -\sum_{k=0}^{\infty} q_{2k} \left[\sum_{j=0}^{k} L_{2k,2j} \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2n+2j+1} P_{2n}(\cos\theta) \right] , \tag{18}$$

with which one can use Lemma 3 to add the series in powers of λ, thus obtaining the expression

$$\Psi = -\sum_{n=0}^{\infty} q_{2n} \sum_{k=0}^{n} L_{2n,2k} \sum_{j=0}^{k} C_{2k,2j} Q_{2j}(x) P_{2j}(y), \tag{19}$$

where coefficients $C_{2k,2j}$ are defined by the expression

$$C_{2k,2j} \equiv 2^j (4j+1) \frac{k!(2k-1)!!}{(k-j)!(2k+2j+1)!!}, \qquad C_{0,0} \equiv 1 \tag{20}$$

and represent the coefficients that appear on writing the powers of an arbitrary variable as a function of the Legendre polynomials of that variable, i.e.,

$$\zeta^k = \sum_{j=0}^{k} C_{k,j} \, P_j(\zeta). \tag{21}$$

Now rearranging the sums in (19) and grouping the Legendre polynomials and the functions of the second species of the same degree, one has

$$\Psi = -\sum_{n=0}^{\infty} Q_{2n}(x) P_{2n}(y) \sum_{k=n}^{\infty} q_{2k} \sum_{j=n}^{k} L_{2k,2j} C_{2j,2n}. \tag{22}$$

Again, by virtue of Lemma 1 one can consider the last summatory in (22) from $j = 0$ without altering the sum on doing so, taking into account the definition of the coefficients $C_{2k,2j}$. Now applying the results of Lemma 2, one ends with

$$\Psi = -\sum_{k=0}^{\infty} q_{2k} Q_{2k}(x) P_{2k}(y). \tag{23}$$

A similar proof demonstrates the equivalence between series (7) and (11) for the case of odd n.

A striking property of the Erez–Rosen–Quevedo representation is that the first two contributions of series (11) are directly connected with the solutions of Schwarzschild and Erez–Rosen [13]. We shall now analyze this property within the context of the equivalence between the Weyl and Erez–Rosen–Quevedo representations. From formula (12) it is easy to obtain the inverse relationship that expresses parameters q_n as a function of the Weyl moments a_n (in what follows we shall always assume equatorial symmetry):

$$q_{2n} = -(4n+1) \sum_{k=0}^{n} \frac{a_{2k}}{M^{2k+1}} L_{2n,2k}. \tag{24}$$

As is known, the Weyl moments of the Schwarzschild solution are given by (9), such that according to (24) the q_n parameters corresponding to that equation will be as follows:

$$q_{2n} = (4n+1)\sum_{k=0}^{n} \frac{L_{2n,2k}}{2k+1} \tag{25}$$

and since the Legendre polynomials fulfill the property proved in Lemma 1, one has that

$$\sum_{k=0}^{n} \frac{L_{2n,2k}}{2k+1} = \delta_{0n}, \tag{26}$$

and hence

$$q_0 = 1, \quad q_{2n} = 0 \quad \forall\, n \geq 1. \tag{27}$$

As is well known, the Schwarzschild solution is defined by the first summand of the Erez–Rosen–Quevedo representation

$$\Psi_{\mathrm{Sch}} = -q_0 Q_0(x) P_0(y) = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right). \tag{28}$$

In a similar way, one sees that the Erez–Rosen solution [13] is defined by

$$q_0 = 1, \quad q_2 \neq 0, \quad q_{2n} = 0 \quad \forall\, n \geq 2. \tag{29}$$

In order to obtain this result from the Weyl representation (7) the following expression [10], which gives the Weyls moments a_n of the Erez–Rosen solution, must be taken into account:

$$a_{2n}^{\mathrm{ER}} = -M^{2n+1}\left[\frac{q_0}{2n+1} + q_2 \frac{2n}{(2n+1)(2n+3)}\right], \qquad a a_{2n+1}^{\mathrm{ER}} = 0, \tag{30}$$

together with the following property of the Legendre polynomials, proved in Lemma 2:

$$(4n+1)\sum_{k=l}^{n} L_{2n,2k} \sum_{h=0}^{l} \frac{L_{2l,2h}}{2k+2h+1} = \delta_{nl}\,. \tag{31}$$

Finally, a third representation of the solution (7),(8) was obtained by Gutsunaev and Manko [15,16] generating, by a simple procedure, successive solutions from the Schwarzschild solution. In this representation one has, also using prolate spheroidal coordinates $\{x, y\}$, that

$$\Psi = \frac{1}{2}\ln\frac{x-1}{x+1} + \sum_{n=0}^{\infty} b_{n+1}\left[\frac{P_n^{+}}{(x+y)^{n+1}} - \frac{P_n^{-}}{(x-y)^{n+1}}\right] \tag{32}$$

where b_n are arbitrary real constants and $P_n^{\pm}$ are the following Legendre polynomials:

$$P_n^{\pm} \equiv P_n\left(\frac{xy \pm 1}{x \pm y}\right) . \tag{33}$$

The corresponding γ function proves to be considerably more compact [17] than the case of the Erez–Rosen–Quevedo representation.

Direct proof of the equivalence between the above series and the (7) series of Weyl is obtained using Lemma 4 and considering the following relationship between the constants b_n and the Weyl moments a_n:

$$\begin{aligned} a_{2n} &= -\frac{M^{2n+1}}{2n+1} - 2M^{2n+1}\sum_{k=1}^{n}\binom{2n}{2k-1}b_{2k} \\ a_{2n+1} &= -2M^{2n+2}\sum_{k=0}^{n}\binom{2n+1}{2k}b_{2k+1} \, . \end{aligned} \tag{34}$$

Indeed, limiting ourselves as before to the case of equatorial symmetry and substituting (34) in (7), one obtains

$$\Psi = \Psi_{\mathrm{Sch}} - 2\sum_{n=0}^{\infty}\lambda^{2n+1}P_{2n}(\cos\theta)\sum_{k=1}^{n}b_{2k}\sum_{j=1}^{2k-1}S_{kj}n^j \tag{35}$$

where coefficients S_{kj} are defined by the expression

$$\binom{2n}{2k-1} = \sum_{j=1}^{2k-1}S_{kj}n^j , \tag{36}$$

and are therefore only non-null for $k \leq n$. It is for this reason that the upper limit of the second summatory in expression (35) can be made infinite; that is, it is not necessary to constrain the index k since this is already done by the combinatory number. In this way, it is possible to rearrange expression (35) to give

$$\Psi = \Psi_{\mathrm{Sch}} - 2\sum_{k=1}^{\infty}b_{2k}\sum_{j=1}^{2k-1}S_{kj}\sum_{n=0}^{\infty}n^j\lambda^{2n+1}P_{2n}(\cos\theta), \tag{37}$$

with which, bearing in mind Lemma 4, one has

$$\begin{aligned} \Psi = \Psi_{\mathrm{Sch}} &- 2\sum_{k=1}^{\infty}b_{2k}\times \\ &\times \sum_{j=1}^{2k-1}S_{kj}\sum_{n=1}^{j}A_{jn}\left[\frac{P_n^+}{(x+y)^{n+1}} + (-1)^n\frac{P_n^-}{(x-y)^{n+1}}\right], \end{aligned} \tag{38}$$

where coefficients A_{jn} are defined by the expression

$$A_{jn} = \frac{n}{2^{j+1}} \sum_{h=1}^{n} (-1)^h h^{j-1} \binom{n-1}{h-1}. \tag{39}$$

Again reordering the sums in (38), one has

$$\Psi = \Psi_{\mathrm{Sch}} - 2 \sum_{k=1}^{\infty} b_{2k} \times$$
$$\times \sum_{n=1}^{2k-1} \left[\frac{P_n^+}{(x+y)^{n+1}} + (-1)^n \frac{P_n^-}{(x-y)^{n+1}} \right] \sum_{j=n}^{2k-1} S_{kj} A_{jn}. \tag{40}$$

Now then, the relationship

$$-2 \sum_{j=n}^{2k-1} S_{kj} A_{jn} = \delta_{n,2k-1} \tag{41}$$

is easy to prove, such that (40) finally leads to the following:

$$\Psi = \Psi_{\mathrm{Sch}} + \sum_{k=1}^{\infty} b_{2k} \left[\frac{P_{2k-1}^+}{(x+y)^{2k}} - \frac{P_{2k-1}^-}{(x-y)^{2k}} \right], \tag{42}$$

which is expression (32) of the Gutsunaev–Manko representation for the case of equatorial symmetry.

To end this section, we should like to underscore an important aspect of the works of Quevedo [14] and Manko [16] that could lead to confusion regarding the families of $\{q_k\}$ and $\{b_k\}$ parameters. In these works the Newtonian limit is calculated, by Ehler's procedure, of representations (11) and (32) respectively, concluding in both works that the corresponding Newtonian moments D_k are proportional in one case to the parameters q_k and, in the other, to the parameters b_k. In fact, if that Newtonian limit is applied to representation (7) of Weyl, one will also find that the moments D_k are proportional to parameters a_k. From this it could be deduced that the three families of parameters are proportional to one another which, as we have seen through formulas (12) and (34), is not true. The explanation to this apparent contradiction, as we shall see in Section 3, lies in the fact that each of the three families of parameters can be written in a non-trivial manner as a function of the Geroch moments such that in the limit they coincide with these with the exception of factors. As a result, there is no point in attempting to interpret any of the three parameter families as Newtonian moments.

3. MULTIPOLE MOMENTS OF THE WEYL METRICS

In this section we shall make explicit use of the procedure proposed by Thorne [5] for the calculation of the multipole moments of a stationary metric, consi-dering the particular case of the Weyl solution (7),(8). As will be seen, this affords the expressions of the Geroch moments M_n of that metric in terms of the parameters a_k. The procedure is reduced to determining a system of Thorne ACMC coordinates [5] at each order of the development of the metric in the inverse powers of the corresponding radial coordinate.

As is known, a system of Thorne ACMC coordinates is formed of a system $\{\hat{x}^\alpha\}$ of harmonic coordinates with good asymptotic behaviour, such that one should obtain::

$$\Box\, \hat{x}^{(\alpha)} \equiv \frac{1}{\sqrt{-g}}\, \partial_\lambda[\sqrt{-g}\, g^{\lambda\mu}\partial_\mu \hat{x}^{(\alpha)}] = 0, \tag{43}$$

where $\Box$ represents the D'Alembert operator associated with the metric, and g is its determinant. The simplest way of solving eq. (43) for the case under consideration is to introduce spherical coordinates $\{\hat{r}, \hat{\theta}, \hat{\varphi}\}$ associated with the $\{\hat{x}^\alpha\}$ coordinates; that is

$$\begin{cases} \hat{u} \equiv \hat{x} + i\hat{y} = \hat{r}e^{i\hat{\varphi}} \sin\hat{\theta} \equiv \hat{\rho}e^{i\hat{\varphi}} \\ \hat{z} = \hat{r}\cos\hat{\theta}. \end{cases} \tag{44}$$

However, taking into account (4), one readily sees that the time coordinate t is already harmonic and that the azimuth coordinate φ is a good harmonic spherical coordinate and hence eqs. (43) are reduced after a short calculation to the following:

$$\begin{aligned} &\partial_r(r^2 \sin\theta\partial_r\hat{\rho}) + \partial_\theta(\sin\theta\partial_\theta\hat{\rho}) = \frac{1}{\sin\theta}e^{2\gamma}\hat{\rho} \\ &\partial_r(r^2 \sin\theta\partial_r\hat{z}) + \partial_\theta(\sin\theta\partial_\theta\hat{z}) = 0, \end{aligned} \tag{45}$$

the unknown functions being the cylindrical coordinate $\hat{\rho}$ and the coordinate $\hat{z}$, respectively, both of them as functions of the variables (r,θ). From (45), it is easy to see that the solutions displaying good behaviour are the following:

$$\begin{aligned} &\hat{\rho}(r,\theta) = r\sin\theta \sum_{l=0}^{\infty} \frac{1}{r^l} H_l(\cos\theta) \\ &\hat{z}(r,\theta) = r\cos\theta \equiv z, \end{aligned} \tag{46}$$

where the functions $H_l(\omega)$ are the solutions of the following linear second order differential equation ($\omega \equiv \cos\theta$):

$$(1-\omega^2)\left\{l(l-3)H_l(\omega) - 4\omega H_l'(\omega) + (1-\omega^2)H_l''(\omega)\right\} = \\ = \begin{cases} 0, & l \leq 1 \\ \sum_{k+n=l-2} B_k(\omega)H_n(\omega), & l \geq 2 \end{cases} \tag{47}$$

and where the functions $B_k(\omega)$ are defined by the following set of expressions:

$$B_k(\omega) = \sum_{j=0}^{(k-\delta_k)/2} \frac{2^{j+1}}{(j+1)!} B_{k-2j}^{(j+1)}(\omega) \quad \begin{cases} \delta_k = 0 : & k \text{ even} \\ \delta_k = 1 : & k \text{ odd} \end{cases} \tag{48}$$

$$B_k^{(n)}(\omega) = \sum_{i_1+\ldots+i_n=k} B_{i_1}^{(1)} \ldots B_{i_n}^{(1)} \tag{49}$$

$$B_i^{(1)}(\omega) = \sum_{k+n=i} E_{kn}(\omega) \tag{50}$$

$$\gamma(r,\omega) = \sum_{k,n=0}^{\infty} \frac{(k+1)(n+1)}{k+n+2} \frac{a_k a_n}{r^{k+n+2}} \left[P_{k+1}(\omega)P_{n+1}(\omega) - P_k(\omega)P_n(\omega)\right] \\ \equiv \sum_{k,n=0}^{\infty} E_{kn}(\omega) \frac{1}{r^{k+n+2}} \,. \tag{51}$$

The solution to eq. (47) has been found using the usual procedure of adding the general solution of the homogeneous equation to a particular solution of the complete equation. The homogeneous equation has as its general solution, at each order, a linear combination of a Gegenbauer polynomial and of a rational function in the ω variable with undesired behaviour, a function that will be ignored by considering the second integration constant equal to zero. As regards the complete equation, the procedure is to search for polynomial solutions of the type (we limit ourselves, as usual, to the case of equatorial symmetry)

$$H_{2n}(\omega) = \sum_{k=0}^{n-1} c_{2k}\omega^{2k}, \tag{52}$$

where the upper limit of the sum is determined by the degree of the polynomial in ω that appears in the non-homogeneous part of the differential equation at each order and that proves to be equal to $2n-2$. As is evident from formulas (48)–(51), calculation of the polynomial (52) becomes progressively longer as n increases and its final expression becomes hard to handle. Fortunately, the use of computer programs for symbolic calculations allows one to reach relatively high orders with ease, which, as we shall see, is extremely useful for the purposes of this work. However, to determine the Geroch moments of the Weyl solution up to a given order it does not suffice simply to calculate the harmonic coordinates up to that order; rather one must invert the corresponding formulas (46) up to that order to be able to express the $\{r,\theta\}$ coordinates as a function of the $\{\hat{r},\hat{\theta}\}$ coordinates,

$$\begin{aligned} r &= r(\hat{r},\hat{\theta}) \\ \omega &= \omega(\hat{r},\hat{\theta}). \end{aligned} \tag{53}$$

Thus, substituting (53) in (7) it is possible to obtain the expression g_{00} of the metric as a function of the harmonic coordinates, with which the Geroch M_{2n} coordinates will be fixed by the following Thorne structure [5]:

$$\begin{aligned} g_{00} = -1 + \frac{2G}{c^2}\Bigg[\frac{M_0}{\hat{r}} &+ \sum_{n=1}^{\infty} \frac{1}{\hat{r}^{2n+1}} M_{2n} P_{2n}(\cos\hat{\theta}) \\ &+ \sum_{n=1}^{\infty} \frac{1}{\hat{r}^{n+1}} R_{n-1}(\cos\hat{\theta})\Bigg], \end{aligned} \tag{54}$$

where R_{n-1} represents a polynomial of degree $n-1$.

All this process, even though elementary, becomes impracticable when n increases. We performed the calculation up to order 21 in the development in $1/\hat{r}$ using the MATHEMATICA V.2.1 software package, running on an IBM RS6000 machine. In this way it was possible to calculate the first 20 Geroch moments of the general Weyl solution, although explicit inclusion of these would probably demand a whole issue of the journal for this article alone. Nevertheless, we now offer the first 12 (let us remember that the odd numbers are null) for the reader to appreciate the type of complication involved, together with the truth in the words of Thorne [5] when he qualified these calculations as "horrendous".

$$M_0 = -a_0$$

$$M_2 = \frac{1}{3}a_0^3 - a_2$$

$$
\begin{aligned}
M_4 &= -\frac{19}{105} a_0^5 + \frac{8}{7} a_0^2 a_2 - a_4 \\
M_6 &= \frac{389}{3465} a_0^7 - \frac{23}{21} a_0^4 a_2 + \frac{60}{67} a_0 a_2^2 + \frac{17}{11} a_0^2 a_4 - a_6 \\
M_8 &= -\frac{257}{3465} a_0^9 + \frac{44312}{45045} a_0^6 a_2 - \frac{5204}{3003} a_0^3 a_2^2 + \frac{40}{143} a_2^3 - \frac{58}{33} a_0^4 a_4 \\
&+ \frac{226}{143} a_0 a_2 a_4 + 2 a_0^2 a_6 - a_8 \\
M_{10} &= \frac{443699}{8729721} a_0^{11} - \frac{17389}{20349} a_0^8 a_2 + \frac{226580}{88179} a_0^5 a_2^2 - \frac{459700}{323323} a_0^2 a_2^3 \\
&+ \frac{12902}{7293} a_0^6 a_4 - \frac{193130}{46189} a_0^3 a_2 a_4 + \frac{39150}{46189} a_2^2 a_4 + \frac{30870}{46189} a_0 a_4^2 \\
&- \frac{2624}{969} a_0^4 a_6 + \frac{566}{323} a_0 a_2 a_6 + \frac{47}{19} a_0^2 a_8 - a_{10} \\
M_{12} &= -\frac{1253390771}{35137127025} a_0^{13} + \frac{1703959024}{2342475135} a_0^{10} a_2 - \frac{15070540}{4732273} a_0^7 a_2^2 \\
&+ \frac{182660360}{52055003} a_0^4 a_2^3 - \frac{4259400}{7436429} a_0 a_2^4 - \frac{186209501}{111546435} a_0^8 a_4 \\
&+ \frac{4756282}{676039} a_0^5 a_2 a_4 - \frac{34924620}{7436429} a_0^2 a_2^2 a_4 - \frac{2192292}{1062347} a_0^3 a_4^2 \\
&+ \frac{79650}{96577} a_2 a_4^2 + \frac{2417306}{780045} a_0^6 a_6 - \frac{14888}{2737} a_0^3 a_2 a_6 + \frac{6600}{7429} a_2^2 a_6 \\
&+ \frac{9774}{7429} a_0 a_4 a_6 - \frac{11995}{3059} a_0^4 a_8 + \frac{6060}{3059} a_0 a_2 a_8 \\
&+ \frac{68}{23} a_0^2 a_{10} - a_{12}
\end{aligned}
\tag{55}
$$

4. STRUCTURE OF A PURE MONOPOLE-QUADRUPOLE SOLUTION

As was stated in the introduction, the main aim of the present work is to search for the pure monopole-quadrupole solution of the Einstein vacuum equations for the static case with axial symmetry. This means that from among all the Weyl solutions defined by formulas (7),(8) we must discover the one that has all the Geroch multipole moments null with the exception of mass and the quadrupole moment.

To achieve this aim, one must first obtain the Weyl a_n constants as a function of the moments M_n, which can readily be done from formulas (55) owing to their triangular structure; that is, because each moment M_n only contains the Weyl a_k constants with $k \leq n$, and is also linear in the constant a_n. Following this it is necesary to nullify all moments of order greater than 2 to obtain, thus, the constants a_n as functions of the two unique parameters $M \equiv M_0$ and $Q \equiv M_2$. The result obtained in this process, which once again was done with MATHEMATICA, is as follows: (coherent with the previous section, we only write the expressions corresponding to the first 12 Weyl constants, although the set is now much lighter):

$$
\begin{aligned}
a_0 &= -M_0 \\
a_2 &= -\frac{1}{3} M_0^3 - M_2 \\
a_4 &= -\frac{1}{5} M_0^5 - \frac{8}{7} M_0^2 M_2 \\
a_6 &= -\frac{1}{7} M_0^7 - \frac{25}{21} M_0^4 M_2 - \frac{60}{77} M_0 M_2^2 \\
a_8 &= -\frac{1}{9} M_0^9 - \frac{40}{33} M_0^6 M_2 - \frac{820}{429} M_0^3 M_2^2 - \frac{40}{143} M_2^3 \\
a_{10} &= -\frac{1}{11} M_0^{11} - \frac{175}{143} M_0^8 M_2 - \frac{460}{143} M_0^5 M_2^2 - \frac{300}{187} M_0^2 M_2^3 \\
a_{12} &= -\frac{1}{13} M_0^{13} - \frac{16}{13} M_0^{10} M_2 - \frac{60}{13} M_0^7 M_2^2 - \frac{1000}{221} M_0^4 M_2^3 \\
&\quad - \frac{19800}{29393} M_0 M_2^4 .
\end{aligned} \tag{56}
$$

From formulas (56) it is concluded trivially that the generic structure of the constants a_n as a function of the mass M and the quadrupole moment Q is as follows: (this structure was already mentioned in Ref. 10):

$$
a_{2n} = -M^{2n+1} \sum_{\alpha=0}^{\alpha_n} q^\alpha F(\alpha, n), \qquad a_{2n+1} = 0 \tag{57}
$$

where the dimensionless parameter $q \equiv Q/M^3$ has been introduced and where the upper limit α_n depends on each value of n as follows:

$$
\alpha_n \equiv \tfrac{1}{3}(2n + h_n), \tag{58}
$$

where h_n is a discrete function of n defined in terms of the classes of remains modulo 3 in the following way:

$$
h_n = \begin{cases} 0 & : n \in [0] \\ 1 & : n \in [1] \\ -1 & : n \in [2]. \end{cases} \tag{59}
$$

The main difficulty lies in determining the numerical function $F(\alpha, n)$, for which there is no canonical procedure. In reference [10] we were able to deduce from (56) the expressions of this function for the first four values of α and n arbitrary, a circumstance which allowed us to find an Erez–Rosen type solution [13] or a Gutsunaev–Manko [15] type solution, although with more interesting properties than these. Currently, thanks to the help of a powerful computer, we have obtained formulas (56) up to order 20, which has allowed us to deduce a generic expression for the $F(\alpha, n)$ function, which was also proved to be correct for some higher order. Accordingly, in our opinion one is dealing with the *exact* expression that sets the pure monopole-quadrupole expresion, which is as follows:

$$F(\alpha, n) = \frac{n!(2n-1)!!}{(2n+2\alpha+1)!!} \sum_{k=k_\alpha}^{\alpha} \frac{3^{\alpha-k} 5^k (4\alpha - 3k + 1)}{(n-\alpha-k)!(\alpha-k)!(2k-\alpha+1)!} \tag{60}$$

where the lower limit k_α of the sum is defined as

$$k_\alpha = \begin{cases} (\alpha-1)/2, & \alpha \text{ odd} \\ \alpha/2, & \alpha \text{ even} . \end{cases} \tag{61}$$

In order to show the structure of expression (60) for different values of arbitrary α and n we wrote the formulas corresponding to first five values of α:

$$\begin{aligned} F(0,n) &= \frac{1}{2n+1} \\ F(1,n) &= 5\frac{n(n+2)}{(2n+1)(2n+3)} \\ F(2,n) &= \frac{5}{2}\frac{n(n-1)(n-2)(5n+21)}{(2n+1)(2n+3)(2n+5)} \\ F(3,n) &= \frac{25}{6}\frac{n(n-1)(n-2)(n-3)(5n^2+18n-98)}{(2n+1)(2n+3)(2n+5)(2n+7)} \\ F(4,n) &= \frac{25}{24}\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(25n^2+155n-642)}{(2n+1)(2n+3)(2n+5)(2n+7)(2n+9)} . \end{aligned} \tag{62}$$

The interest in these formulas lies firstly in the fact that they show us for which value of n the function $F(\alpha, n)$ begins to be different from zero when α increases. Secondly, they suggest a considerable simplification of that function, consisting in decomposing into simple fractions the part that depends on n, after having made the division of the corresponding polynomials in n when the degree of the numerator is greater than that of the denominator. This operation is not very difficult to carry out and finally leads to the following interesting expression for the $F(\alpha, n)$ function, an expression that, as we shall see in the next section, is of great use for the aims of this work:

$$F(\alpha,n) = \sum_{j=0}^{\alpha-1} g_j(\alpha)\, n^j + \sum_{j=0}^{\alpha} \frac{h_j(\alpha)}{2n+2j+1}, \tag{63}$$

where coefficients $g_j(\alpha)$ y $h_j(\alpha)$ are defined by the following expressions:

$$\begin{aligned} g_j(\alpha) &= \sum_{k=k_j(\alpha)}^{\alpha} \frac{3^{(\alpha-k)}5^k(4\alpha-3k+1)}{(\alpha-k)!(2k-\alpha+1)!} J_{kj}(\alpha) \\ h_j(\alpha) &= \left(\frac{3}{2}\right)^{\alpha} \frac{L_{2\alpha,2j}}{(2\alpha+2j-1)!!} \times \\ &\times \sum_{k=k(\alpha)}^{\alpha} \left(-\frac{5}{6}\right)^k \frac{(4\alpha-3k+1)(2\alpha+2j+2k-1)!!}{(\alpha-k)!(2k-\alpha+1)!} \end{aligned} \tag{64}$$

with the functions $J_{kj}(\alpha)$ defined recurrently, as follows:

$$\begin{aligned} J_{kj}(\alpha) &= \sum_{i=j}^{k-1} J_{i,j-1}(\alpha)\,(-1)^{i+k+1} \frac{(\alpha+k-1)!}{(\alpha+i)!} \\ J_{k0}(\alpha) &= (-1)^{k+1} \frac{(2k-3)!!}{2^{\alpha+k}} \binom{2\alpha+2k-1}{2k-2} \end{aligned} \tag{65}$$

where the lower limits of the sums (64) are the following:

$$\begin{aligned} k(\alpha) &= \begin{cases} (\alpha-1)/2, & \alpha \text{ odd} \\ \alpha/2, & \alpha \text{ even.} \end{cases} \\ k_j(\alpha) &= \begin{cases} k(\alpha), & j \le k(\alpha)-1 \\ j+1, & j \ge k(\alpha). \end{cases} \end{aligned} \tag{66}$$

5. EXACT SOLUTIONS

Once the expression of the "Weyl moments" is known as a function of the mass M and the quadrupole moment Q, it is possible to analyze the pure monopole-quadrupole solution we are searching for. To do this it suffices to substitute result (57) in formula (7), thus obtaining the corresponding Ψ function that is (let us remember the hypothesis of axial symmetry)

$$\Psi = -\sum_{n=0}^{\infty} M^{2n+1} \sum_{\alpha=0}^{\alpha_n} q^{\alpha} F(\alpha, n) \frac{P_{2n}(\cos\theta)}{r^{2n+1}} . \tag{67}$$

A first observation of the above series gives us an interesting decomposition of it. Indeed, if one takes into account the form of the $F(\alpha, n)$ function, it is possible to separate the part that depends only on mass M and the part that depends only on the quadrupole moment Q, thus giving rise to the following three summands:

$$\Psi = \Psi_{\mathrm{M}} + \Psi_{\mathrm{Q}} + \Psi_{\mathrm{MQ}} , \tag{68}$$

where the following definitions have been used:

$$\Psi_{\mathrm{M}} = -\sum_{n=0}^{\infty} \frac{M^{2n+1}}{2n+1} \frac{P_{2n}(\omega)}{r^{2n+1}} \tag{69}$$

$$\Psi_{\mathrm{Q}} = -\sum_{k=0}^{\infty} Q^{2k+1} F(2k+1, 3k+1) \frac{P_{6k+2}(\omega)}{r^{6k+3}} \tag{70}$$

$$\Psi_{\mathrm{MQ}} = -\sum_{\nu=0}^{2} \sum_{j=0}^{\infty} \sum_{\alpha=1}^{2j+\nu} Q^{\alpha} M^{6j-3\alpha+2\nu+3} F(\alpha, 3j+\nu+1) \frac{P_{6j+2\nu+2}(\omega)}{r^{6j+2\nu+3}} . \tag{71}$$

The Ψ_{M} function is well known since it provides the Schwarzschild solution, which can be termed the pure monopole solution. On the other hand the Ψ_{Q} function is a solution, written in series form, that represents the pure quadrupole. Finally, Ψ_{MQ} should be considered as the monopole-quadrupole interaction, which points to the non-linear nature of the theory.

The expression for the $F(\alpha, n)$ function, α being odd and $n = 3k + 1$, proves to be

$$\begin{aligned} F(2k+1, 3k+1) &= \frac{15^{k+1}}{2^{2k+1}} \sum_{j=0}^{2k+1} \frac{L_{4k+2,2j}}{6k+2j+3} \\ &= \frac{15^{k+1}}{2^{2k+1}} \frac{(6k+2)!}{(10k+5)!!(2k)!!} , \end{aligned} \tag{72}$$

that is, in the same way that choice (9) of the "Weyl moments" affords the Schwarzschild solution, which described spherical symmetry, the following choice of these parameters,

$$a_{6k+2} = \frac{15^{k+1}}{2^{2k+1}} \frac{(6k+2)!}{(10k+5)!!(2k)!!}, \tag{73}$$

leads to a solution that represents the pure quadrupole, whose interest lies in the possibility of being able to write static vacuum solutions as the sum of multipole contributions, as happens in Newtonian gravitation.

A much more interesting view of solution (67) is obtained by inverting the order of the sums and grouping the terms in powers of the dimensionless parameter q, that is,

$$\Psi = -\sum_{\alpha=0}^{\infty} q^{\alpha} \sum_{n=0}^{\infty} F(\alpha, n)\lambda^{2n+1} P_{2n}(\cos\theta). \tag{74}$$

The Ψ function thus obtained shows that the solution sought can be understood as an infinite sum of the following contributions:

$$\Psi_{\mathrm{M-Q}} = \Psi_{q^0} + q\Psi_{q^1} + q^2\Psi_{q^2} + \ldots = \sum_{\alpha=0}^{\infty} q^{\alpha}\Psi_{q^{\alpha}}, \tag{75}$$

where

$$\Psi_{q^{\alpha}} = -\sum_{n=0}^{\infty} F(\alpha, n)\lambda^{2n+1} P_{2n}(\cos\theta) \tag{76}$$

and where order zero is no more than the Schwarzschild solution,

$$\Psi_{q^0} = -\sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2n+1} P_{2n}(\cos\theta), \tag{77}$$

such that each power in q adds a quadrupole correction to the solution with spherical symmetry. Now then, it should be born in mind that as a result of the linearity of the Laplace equation, these corrections give rise to successive exact solutions. That is, the series in powers of q is susceptible to being cut off at any order, giving rise to an exact solution that represents a certain quadrupole correction of the Schwarzschild solution.

The interest in the result obtained would only be formal if we limited ourselves to presenting solution (74) in the form of a double series; that is, if it were not possible to add series (76) that determine the summands of the main series (75). Fortunately, owing to the structure of $F(\alpha, n)$ defined by (63), this series (76) can be added, thus obtaining finite expressions for each of the contributions Ψ_{q^α}. Indeed, substituting (63) in (76) and rearranging the sums, one has

$$\Psi_{q^\alpha} = -\sum_{j=0}^{\alpha-1} g_j(\alpha) \sum_{n=0}^{\infty} n^j \lambda^{2n+1} P_{2n}(\cos\theta) - \sum_{j=0}^{\alpha} h_j(\alpha) \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2n+2j+1} P_{2n}(\cos\theta). \tag{78}$$

Now making use of Lemmas 3 and 4, one arrives at the following finite expression:

$$\Psi_{q^\alpha} = -\sum_{j=0}^{\alpha-1} g_j(\alpha) \sum_{n=1-\delta_{j0}}^{j} A_{jn} \left[\frac{P_n^+}{(x+y)^{n+1}} + (-1)^n \frac{P_n^-}{(x-y)^{n+1}} \right] - \sum_{j=0}^{\alpha} h_j(\alpha) \sum_{n=0}^{j} C_{2j,2n} Q_{2n}(x) P_{2n}(y). \tag{79}$$

Rearranging these finite sums one finally obtains the following interesting expression:

$$\Psi_{q^\alpha} = -\sum_{k=0}^{\alpha-1} b_k(\alpha) \left[\frac{P_k^+}{(x+y)^{k+1}} + (-1)^k \frac{P_k^-}{(x-y)^{k+1}} \right] - \sum_{k=0}^{\alpha} q_k(\alpha) Q_{2k}(x) P_{2k}(y) \tag{80}$$

where the coefficients

$$b_k(\alpha) = \sum_{j=k}^{(\alpha-1)(1-\delta_{k0})} g_j(\alpha) A_{jk},, \qquad q_k(\alpha) = \sum_{j=k}^{\alpha} h_j(\alpha) C_{2j,2k}, \tag{81}$$

have been defined. Thus, with expression (80) one has that contributions (76) to the pure monopole-quadrupole solution (75) can be written as the sum of finite solutions of the type of Gutsunaev–Manko [16] and Erez–Rosen [13].

The most elementary way of physically interpreting the exact solutions derived from the above quadrupole corrections consists of analyzing the corresponding multipole moments. To this effect, using expressions (55) it is possible, cutting off solution (75) at a certain order, to obtain an exact solution with the following properties: the monopole and quadrupole moments are different from zero; all the following moments are null up to order 3α (α is even) and $3\alpha+1$ (α is odd), whereas all higher moments are of order $q^{\alpha+1}$. Accordingly, one is dealing with the pure quadrupole correction to the Schwarzschild solution up to the preset multipole order. In our opinion, these are the first realistic attempts to describe small deviations from spherical symmetry since the static solutions obtained up to now, such as that of Erez–Rosen [13] and that of Gutsunaev–Manko [15], with the same arbitrary parameters M and Q, have a multipole structure in which moments above the quadrupole are of the same order in this.

Below we give simplest explicit example of the solution, first obtained in [10], which describes correctly the quadrupole deformation of a massive source and is defined by the functions Ψ and γ of the form

$$
\begin{aligned}
\Psi_{M-Q^1} = {} & \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + \frac{5}{8}q(3y^2-1)\times \\
& \times\left[\left(\frac{3x^2-1}{4} - \frac{1}{3y^2-1}\right)\ln\left(\frac{x-1}{x+1}\right)\right. \\
& \left. - \frac{2x}{(x^2-y^2)(3y^2-1)} + \frac{3x}{2}\right],
\end{aligned}
\tag{82a}
$$

$$
\begin{aligned}
\gamma = {} & \frac{1}{2}\left(1+\frac{225}{24}q^2\right)\ln\left(\frac{x^2-1}{x^2-y^2}\right) \\
& - \frac{15}{8}qx(1-y^2)\left[1 - \frac{15}{32}q\left(x^2+7y^2-9x^2y^2\right.\right. \\
& \qquad \left.\left. +1-\frac{8}{3}\frac{x^2+1}{x^2-y^2}\right)\right]\ln\left(\frac{x-1}{x+1}\right) \\
& + \frac{225}{1024}q^2(x^2-1)(1-y^2)(x^2+y^2-9x^2y^2-1)\ln^2\left(\frac{x-1}{x+1}\right) \\
& - \frac{15}{4}q(1-y^2)\left[1-\frac{15}{64}q(x^2+4y^2-9x^2y^2+4)\right] \\
& - \frac{75}{16}q^2x^2\frac{1-y^2}{x^2-y^2} - \frac{5}{4}q(x^2+y^2)\frac{1-y^2}{(x^2-y^2)^2}
\end{aligned}
$$

$$-\frac{25}{64}q^2(2x^6-x^4+3x^4y^2-6x^2y^2+4x^2y^4-y^4-y^6)\frac{1-y^2}{(x^2-y^2)^4}. \quad (82b)$$

From the expressions of its first twelve relativistic multipole moments which were found to be ($M_1 = M_3 = M_5 = M_7 = M_9 = M_{11} = 0$)

$$M_0 = M, \qquad M_2 = M^3 q, \qquad M_4 = 0, \qquad M_6 = -\frac{60}{77}, M^7 q^2$$

$$M_8 = -\frac{1060}{3003}M^9 q^2 - \frac{40}{143}M^9 q^3, \; M_{10} = -\frac{19880}{138567}M^{11}q^2 + \frac{146500}{323323}M^{11}q^3$$

$$M_{12} = -\frac{23600}{437437}M^{13}q^2 + \frac{517600}{1062347}M^{13}q^3 + \frac{4259400}{7436429}M^{13}q^4 \quad (83)$$

it can be clearly seen that the parameter Q representing the mass-quadrupole moment is contained in the moments M_{2n}, $n \geq 2$, only in the second and higher orders, so that the solution (82) with a small Q describes the exterior field of a static source possessing only the mass monopole and quadrupole moments, in contradistinction to the solutions [13,15] which cannot be interpreted as describing small quadrupole deformation of a spferically symmetric source in the perturbation theory.

In conclusion we would like to write down the function Ψ for the second solution of the family discussed in this paper which can be shown to have the form

$$\begin{aligned}\Psi_{M-Q^2} &= \Psi_{M-Q^1} \\ &+ q^2\Bigg\{\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)\left[\frac{165}{32} + \frac{225}{16}P_2(x)P_2(y) - \frac{765}{32}P_4(x)P_4(y)\right] \\ &\quad - \frac{675}{32}xP_2(y) + \frac{765}{32}\frac{55x-105x^3}{24}P_4(y) \\ &\quad + \frac{165}{32}\frac{x}{x^2-y^2} + \frac{25}{32}\frac{-3x^3y^2-xy^4+x^3+3x^2y^2}{(x^2-y^2)^3}\Bigg\} \quad (84)\end{aligned}$$

with the multipole moments defined by the following expressions ($M_{2k+1} = 0$, $k = 0, 1, \ldots$)

$$M_0 = M, \quad M_2 = M^3 q, \quad M_4 = 0, \quad M_6 = 0$$

$$M_8 = -\frac{40}{143}M^9 q^3, \quad M_{10} = -\frac{42140}{46189}M^{11}q^3$$

$$M_{12} = -\frac{38800}{55913}M^{13}q^3 - \frac{888600}{7436429}M^{13}q^4. \quad (85)$$

From (85) one can see that the parameter Q enters into M_{2n}, $n \geq 2$, already in the third and higher orders, the solution (84) providing a more correct version of the monopole-quadrupole solution than the previous one.

6. CONCLUSION

We have succeeded in finding an infinite family of static vacuum solutions of the Einstein equations which have a clear physical interpretation as describing the exterior field of a massive source possessing an arbitrary quadrupole moment. The principal difference of our family of solutions from the already known Erez–Rosen and Gutsunaev–Manko metrics lies in the possibility of using our solution in the perturbation theory, which suggests that they could be more appropriate for the description of the exterior gravitational fields of the deformed astrophysical objects than the aforementioned metrics.

We also have been able to establish the interrelation between different representations of the general Weyl solution obtained up to now, thus providing a procedure for the interpretation of the results obtained with the aid of each representation from the point of view of the original Weyl solution.

As a final remark we would like to say that it seems likely to extend our approach to the stationary case what, as we believe, could bring an additional insight into the nature and physical interpretation of the stationary metrics.

APPENDIX A

We shall now prove the Lemmas used in Sections 2 and 5.

Lemma 1. For all pairs of non–negative whole numbers n and k such that $n < k$, the following equivalence is fulfilled:

$$\sum_{j=0}^{k} \frac{L_{2k,2j}}{2n+2j+1} = 0, \qquad (A.1)$$

where $L_{2k,2j}$ are the coefficients of the Legendre polynomials defined by (15) and (16).

Proof. Let us consider expression (21) of the power of an arbitrary variable as a function of the Legendre polynomials in this variable:

$$\zeta^{2n} = \sum_{k=0}^{n} C_{2n,2k} P_{2k}(\zeta), \qquad (A.2)$$

from which the following expression may be deduced for the coefficients $C_{2n,2k}$, taking into account the orthogonality of the polynomials:

$$C_{2n,2k} = \frac{4k+1}{2} \int_{-1}^{1} P_{2k}(\zeta)\zeta^{2n} d\zeta. \qquad (A.3)$$

If we now employ expression (16) of the Legendre polynomials in powers of their argument, we shall have

$$C_{2n,2k} = \frac{4k+1}{2} \int_{-1}^{1} \sum_{j=0}^{k} L_{2k,2j} \zeta^{2j+2j} d\zeta, \tag{A.4}$$

with which we obtain trivially

$$C_{2n,2k} = (4k+1) \sum_{j=0}^{k} \frac{L_{2k,2j}}{2n+2j+1}, \tag{A.5}$$

from which one deduces what was stated in the Lemma if one bears in mind that

$$C_{2n,2k} = 0 \quad \forall\, k > n. \qquad \blacksquare \tag{A.6}$$

Lemma 2. The following orthogonality relationship is fulfilled:

$$\sum_{j=0}^{k} L_{2k,2j} C_{2j,2n} = \delta_{kn} \,. \tag{A.7}$$

Proof. The above relation is a direct result of the orthogonality of the Legendre polynomials. Indeed, using expression ($A.3$) for the coefficients $C_{2j,2n}$, one has that

$$\sum_{j=0}^{k} L_{2k,2j} C_{2j,2n} = \frac{4n+1}{2} \sum_{j=0}^{k} L_{2k,2j} \int_{-1}^{1} P_{2n}(\zeta) \zeta^{2j} d\zeta \tag{A.8}$$

and, commuting the integral and the sum, one obtains

$$\sum_{j=0}^{k} L_{2k,2j} C_{2j,2n} = \frac{4n+1}{2} \int_{-1}^{1} P_{2n}(\zeta) P_{2k}(\zeta) d\zeta, \tag{A.9}$$

a relation that is equivalent to the statement of the Lemma if one takes into account the orthogonality of the Legendre polynomials. $\blacksquare$

Lemma 3. For all non-negative whole numbers j the following equalities are fulfilled:

$$\sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{2n+2j+1} P_{2n}(\cos\theta) = \sum_{n=0}^{j} C_{2j,2n} Q_{2n}(x) P_{2n}(y) \tag{A.10a}$$

$$\sum_{n=0}^{\infty} \frac{\lambda^{2n+2}}{2n+2j+3} P_{2n+1}(\cos\theta) = \sum_{n=0}^{j} C_{2j+1,2n+1} Q_{2n+1}(x) P_{2n+1}(y), \tag{A.10b}$$

where the notations of Section 2 and the equivalent ones for odd subindices have been used.

Proof. Detailed proof of this Lemma is long and laborious, such that we shall only indicate the main steps involved. In this sense, and as a previous step, let us consider the generating function of the Legendre polynomials:

$$\frac{1}{(\lambda^2 - 2\lambda\cos\theta + 1)^{1/2}} = \sum_{n=0}^{\infty} \lambda^n P_n(\cos\theta) \qquad (A.11)$$

and also, by carrying out the transformation $\lambda \to -\lambda$,

$$\frac{1}{(\lambda^2 + 2\lambda\cos\theta + 1)^{1/2}} = \sum_{n=0}^{\infty} (-1)^n \lambda^n P_n(\cos\theta), \qquad (A.12)$$

with which by adding and subtracting both expressions, one obtains

$$2\sum_{n=0}^{\infty} \lambda^{2n} P_{2n}(\cos\theta) = \frac{1}{\Delta_-(\lambda)} + \frac{1}{\Delta_+(\lambda)} \qquad (A.13a)$$

$$2\sum_{n=0}^{\infty} \lambda^{2n+1} P_{2n+1}(\cos\theta) = \frac{1}{\Delta_-(\lambda)} - \frac{1}{\Delta_+(\lambda)}, \qquad (A.13b)$$

where $\Delta_\pm(\lambda) = r_\pm / r$, having used definitions (10). Let us now define the following integrals:

$$I_j(\lambda) \equiv \int_0^\lambda \frac{\mu^j}{\Delta_-(\mu)}\, d\mu + (-1)^j \int_0^\lambda \frac{\mu^j}{\Delta_+(\mu)}\, d\mu. \qquad (A.14)$$

It therefore becomes evident, multiplying both terms of expressions $(A.13)$ by λ^{2j} and by λ^{2j+1}, respectively, and integrating the series term by term, that expressions $(A.10)$ of the Lemma are equal to the following:

$$I_{2j}(\lambda) = 2\lambda^{2j} \sum_{n=0}^{j} C_{2j,2n} Q_{2n}(x) P_{2n}(y) \qquad (A.15a)$$

$$I_{2j+1}(\lambda) = 2\lambda^{2j+1} \sum_{n=0}^{j} C_{2j+1,2n+1} Q_{2n+1}(x) P_{2n+1}(y). \qquad (A.15b)$$

Proof of these latter equalities is done simply, although it is time-consuming, by using the procedure of complete induction; that is, checking first that both are fulfilled for $j = 0, 1$ and then demonstrating that if they hold for $j = l$ then they also hold for $j = l + 1$. For this last step it is necessary to use the following recurrence relations among the I_j integrals (see formula 2.263/1, p. 82, Ref. 2):

$$I_j = \frac{2}{j}\lambda^j(\epsilon_{j-1}x - \epsilon_j y) + \frac{2j-1}{j}\cos\theta\, I_{j-1} - \frac{j-1}{j} I_{j-2}, \qquad (A.16)$$

where

$$\epsilon = \begin{cases} 0 : & j \text{ even} \\ 1 : & j \text{ odd} \end{cases} \qquad (A.17)$$

and where, in agreement with (10),

$$\Delta_\pm = \lambda(x \pm y), \qquad \begin{cases} \lambda^{-2} = x^2 + y^2 - 1 \\ \lambda^{-1}\cos\theta = xy \end{cases} \qquad (A.18)$$

has been taken into account. It is also necessary to use the usual recurrence relations for the Legendre polynomials and the Legendre functions of the second type; that is,

$$\begin{aligned} (2n+1)xQ_n(x) &= (n+1)Q_{n+1}(x) + nQ_{n-1}(x) \\ (2n+1)yP_n(y) &= (n+1)P_{n+1}(y) + nP_{n-1}(y) \end{aligned} \quad \blacksquare \qquad (A.19)$$

Lemma 4. For every non-negative whole number j the following equality is fulfilled:

$$\sum_{n=0}^{\infty} n^j \lambda^{2n+1} P_{2n}(\cos\theta) = \sum_{n=1-\delta_{j0}}^{j} A_{jn}\left[\frac{P_n^+}{(x+y)^{n+1}} + (-1)^n \frac{P_n^-}{(x-y)^{n+1}}\right] \qquad (A.20)$$

where, as in the previous Lemma, the notations of Sections 2 and 6 have been used.

Proof. Taking the derivative of the two terms of $(A.13a)$ with respect to the variable λ and then multiplying by $\lambda^2/4$, one obtains the following equivalence:

$$\sum_{n=0}^{\infty} n\lambda^{2n+1} P_{2n}(\cos\theta) = \frac{\lambda^2}{4}\frac{\partial}{\partial\lambda}(\Delta_+^{-1} + \Delta_-^{-1}). \qquad (A.21)$$

From this expression, one proceeds recurrently, dividing first by λ, then taking derivative with respect to λ and finally multiplying by $\lambda^2/2$. One thus obtains the following result:

$$\sum_{n=0}^{\infty} n^j \lambda^{2n+1} P_{2n}(\cos\theta) = \sum_{k=1-\delta_{jo}}^{j} \frac{1}{2^{j+1}} N_{jk} \lambda^{k+1} \frac{\partial^k}{\partial \lambda^k} (\Delta_+^{-1} + \Delta_-^{-1}), \quad (A.22)$$

where the numbers N_{jk} fulfill the following properties (similar to those of combinatorial numbers):

$$\begin{aligned} N_{j1} &= N_{jj} = 1, \qquad N_{j,j-1} = \binom{j}{j-2} \\ N_{jk} &= kN_{j-1,k} + N_{j-1,k-1} \,. \end{aligned} \quad (A.23)$$

Now taking into account the well known relation [16]

$$\frac{\partial^k}{\partial \lambda^k} \Delta_\pm^{-1} = \frac{(\mp 1)^k \, k!}{\lambda^{k+1} \, (x \pm y)^{k+1}} P_k \left(\frac{xy \pm 1}{x \pm y} \right) \quad (A.24)$$

one has, by substituting in $(A.22)$,

$$\begin{aligned} &\sum_{n=0}^{\infty} n^j \lambda^{2n+1} P_{2n}(\cos\theta) \\ &\qquad = \sum_{k=1}^{j} \frac{(-1)^k \, k!}{2^{j+1}} N_{jk} \left[\frac{P_k^+}{(x+y)^{k+1}} + (-1)^k \frac{P_k^-}{(x-y)^{k+1}} \right] . \end{aligned} \quad (A.25)$$

Finally, the coefficients N_{jk} are determined as follows. By the rules of recurrence, $(A.23)$ proves to be

$$\begin{aligned} N_{j2} &= \sum_{r=0}^{j-2} 2^r \\ N_{j3} &= \sum_{r=0}^{j-3} 3^r \sum_{s=0}^{j-3-r} 2^s \\ N_{j4} &= \sum_{r=0}^{j-4} 4^r \sum_{s=0}^{j-4-r} 3^s \sum_{h=0}^{j-4-r-s} 2^h \end{aligned} \quad (A.26)$$

and adding up these geometric successions, one readily obtains

$$N_{jk} = \frac{1}{(k-1)!}\sum_{r=1}^{k}(-1)^{r+k} r^{j-1}\binom{k-1}{r-1}, \qquad j \geq k. \qquad (A.27)$$

APPENDIX B

We now detail the programs used in MATHEMATICA to perform each of the tasks listed.

1. Search for particular solutions H_l, at each order, of the differential equation (47).

The non-homogeneous part of this equation involves, at each order, solutions H_l of lower order. This information is read in **HACHES[w]**, the archive where each solution must later be stored.

The command **Timing[harmonic[n]]** affords, together with the calculation time employed, the coefficients of the particular solution of order n of type (52).

The functions B[a,j_] defined below must be included in successive orders of **a** depending on the order n of the solution one is searching for. In particular, the solution H_{12} only requires the first six B[a,j_] functions

```
A[m_,n_] := ((m + 1) * (n + 1)/(m + n + 2)) * a[m] * a[n]*
        (LegendreP[m + 1, w] * LegendreP[n + 1, w]-
        LegendreP[m, w] * LegendreP[n, w]);
B1[s_] := Sum[A[mm, s - mm], {mm, 0, s}];
B[1,j_] := B1[j];
B[2,j_] := Sum[B1[v] * B1[j - v], {v, 0, j}];
B[3,j_] := Sum[B1[v] * B1[i] * B1[j - v - i], {v, 0, j}, {i, 0, j - v}];
B[4,j_] := Sum[B1[v] * B1[i] * B1[k] * B1[j - v - i - k],
        {v, 0, j}, {i, 0, j - v}, {k, 0, j - v - i}];
B[5,j_] := Sum[B1[v] * B1[i] * B1[k] * B1[l] * B1[j - v - i - k - l],
        {v, 0, j}, {i, 0, j - v}, {k, 0, j - v - i}, {l, 0, j - v - i - k}];
B[6,j_] := Sum[B1[v] * B1[i] * B1[k] * B1[l] * B1[t]*
        B1[j - v - i - k - l - t], {v, 0, j}, {i, 0, j - v}, {k, 0, j - v - i},
        {l, 0, j - v - i - k}, {t, 0, j - v - i - k - l}];
top[ll_] := Switch[Mod[ll, 2], 0, ll/2, 1, (ll - 1)/2];
BG[l_] := Sum[((2^(k + 1))/((k + 1)!))
```

```
        * B[k + 1, l - 2 * k], {k, 0, top[l]}];
Needs["HACHES[w]`"];
Inhomog[e_] := Collect[Sum[H[s] * BG[e - 2 - s], {s, 0, e - 2}], w]
grado[j_] := Exponent[Inhomog[j], w]
Solu[a_] := Sum[coef[a, 2i] * w^(2i), {i, 0, (grado[a] - 2)/2}]
Ecua[b_] := Collect[(1 - w^2) * ((1 - w^2)D[Solu[b], w, 2]-
        4 * w * D[Solu[b], w] + b(b - 3) * Solu[b]), w]
vari[t_] := Table[coef[t, i], {i, 0, t - 2}]
armonic[p_] := Solve[Ecua[p] == Inhomog[p], vari[p], w]
Timing[armonic[12]  ]
```

2. Definition of $\hat{u} = 1/\hat{r}$ as a series of terms in powers of $u = 1/r$.

```
Coefinizu := Block[{l, k},
suma = Sum[u^(2l)H[2l], {l, 1, orden/2}];
rt2 = 1 + 2(1 - w^2)suma + (1 - w^2)(suma^2 + O[u]^(orden + 1));
invrt = rt2^(-1/2) + O[u]^(orden + 1);
Do[ciut[2k] = Coefficient[invrt, u, 2k], {k, 0, orden/2}];   ]
```

3. Inversion of the previous series invrt, *to obtain $1/r$ as a series of powers of $1/\hat{r}$.*

```
Coefinvunavar := Block[{i, l, k, j},
fu = Sum[b[2 * l] * u^(2 * l), {l, 0, orden/2}];
desa = Expand[fu * (Sum[c[2 * k] * (u * fu)^(2 * k)
      + O[u]^(orden + 1), {k, 0, orden/2}])];
c[0] = (1/b[0]);
Do[c[2i] = Expand[-Coefficient[desa, u, 2i]/
        b[0]^(2i + 1) + c[2i]], {i, 1, orden/2}];
Do[cf1v[2j] = c[2j], {j, 0, orden/2}];   ]
```

4. Definition of $\hat{\omega} = \cos\hat{\theta}$ as a series of powers in $\omega = \cos\theta$, where each power $1/r$ has been substituted by the previous series cf1v[ ].

```
Coefinizw := Block[{i, j, k, l},
Do[cf1vu[2j] = cf1v[2j]   /.{b[0] -> 1}, {j, 0, orden/2}];
```

```
omg = Sum[b[2l] * (ut * Sum[ut^(2k) * cf1vu[2k],
      {k, 0, orden/2}])^(2l), {l, 0, orden/2}] + O[ut]^(orden + 1);
Needs["HACHES[w]`"];
Do[b[2i] = ciut[2i], {i, 0, orden/2}];
omgf = Collect[Normal[omg], w];
Do[ciwt[2k] = Coefficient[omgf, w, 2k], {k, 0, orden/2}];   ]
```

5. Inversion of the series $\hat{\omega}$, to obtain ω as a series of powers in $1/\hat{r}$ with dependent coefficients in $\hat{\omega}$. Definitive obtention of $1/\hat{r}$ as a function of ω and $1/r$.

```
Sustfinal := Block[{i, j, k, l},
Clear[b];
Do[b[2k] = ciwt[2k], {k, 0, orden/2}];
wfinal = wt * Sum[cf1v[2l]wt^(2l), {l, 0, orden/2}]
      + O[ut]^(orden + 1);
Clear[b];
Do[b[2i] = ciut[2i], {i, 0, orden/2}];
ucf = ut * Sum[cf1v[2j]ut^(2j), {j, 0, orden/2}] + O[ut]^(orden + 1);
ucfpw = Collect[ucf, w];
ufinal = Sum[(Coefficient[ucfpw, w, 2l] * wfinal^(2l))
      + O[ut]^(orden + 1), {l, 0, orden/2}];   ]
```

6. Calculation of the time component of the metric, $g_{00} = -exp(2U)$, where U is the solution (7) of Weyl, substituting the expressions ω and $1/r$ by their corresponding series in powers of $1/\hat{r}$ with dependence on $\hat{\omega}$.

```
ComputeExpo2U := Block[i, j, k, l,
Uf = Sum[a[2i]u^(2i + 1)LegendreP[2i, w], {i, 0, (orden - 1)/2}];
expo2U = -Sum[(1/(j!))(2 * Uf)^j + O[u]^(orden + 1), {j, 0, orden}];
expcf = Collect[Sum[(Coefficient[expo2U, u, l]ufinal^(l))
      + O[ut]^(orden + 1), {l, 0, orden}], w];
exp2Uf = Sum[(Coefficient[expcf, w, k]wfinal^(k))
      + O[ut]^(orden + 1), {k, 0, orden}];]
```

7. Joint execution of each task at the order considered and reading of the Multipole Moments in the expression obtained for g_{00}.

```
Timing[
        orden = 13;
        Coefinizu;
        Coefinvunavar;
        Coefinizw;
        Sustfinal;
        ComputeExpo2U;]

Do[MM[2k] = Expand[Coefficient[Coefficient[exp2Uf, ut, 2k + 1],
   wt, 2k]/(2 * Coefficient[LegendreP[2k, x], x, 2k])],
   {k, 0, orden/2}];
Do[Print["M[", 2z, "] = ", MM[2z]], {z, 0, orden/2}]
```

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