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Newtonian Limit of Conformal Gravity and the Lack of Necessity of the Second Order Poisson Equation

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We study the interior structure of a locally conformal invariant fourth order theory of gravity in the presence of a static, spherically symmetric gravitational source. We find, quite remarkably, that the associated dynamics is determined exactly and without any approximation at all by a simple fourth order Poisson equation which thus describes both the strong and weak field limits of the theory in this static case. We present the solutions to this fourth order equation and find that we are able to recover all of the standard Newton–Euler gravitational phenomenology in the weak gravity limit, to thus establish the observational viability of the weak field limit of the fourth order theory. Additionally, we make a critical analysis of the second order Poisson equation, and find that the currently available experimental evidence for its validity is not as clearcut and definitive as is commonly believed, with there not apparently being any conclusive observational support for it at all either on the very largest distance scales far outside of fundamental sources, or on the very smallest ones within their interiors. Our study enables us to deduce that even though the familiar second order Poisson gravitational equation may be sufficient to yield Newton's Law of Gravity it is not in fact necessary.

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1. INTRODUCTION

One of the most attractive features of the standard second order Einstein theory of gravity is that it provides a covariant description of not only the exterior Newtonian gravitational potential but also of the interior second order Poisson equation as well; and indeed, this constitutes one of main reasons for having a second order gravitational theory in the first place. With the observational confirmation of the relativistic corrections to the Newtonian limit that the theory then yields, the overwhelming consensus in the community is that the correct theory of gravity has already been found, at least at the classical level. Despite this consensus (which has so far not been eroded even though the standard Newton–Einstein gravitational theory then requires the universe to contain enormous amounts of as yet unestablished non-luminous or dark matter), it should be noted that as of today there is in fact no known basic underlying principle which would require relativistic gravitational theory, or even its weak gravity limit for that matter, to actually be second order. (Indeed it is the very absence of any such underlying principle which has engendered problems such as the notorious cosmological constant problem.) There is thus some value in exploring other candidate covariant equations of motion for the gravitational field to see whether they might also fit observation, so that we can then address basic issues of principle such as the uniqueness of gravitational theory and identify what it is that the data actually mandate.

In the literature there historically have been many challenges to the standard Einstein theory. However, those challenges were based either on a nonacceptance of covariance and Einstein's special view of space and time in the first place, or on a desire to describe the gravitational field in a manner different from Einstein's choice of purely the space-time metric. All of these efforts have long since been turned back, and today it can safely be said that gravity is indeed a general covariant, purely metric based theory. Since general covariance requires only that the gravitational action be a general coordinate scalar, the general covariance principle is not, in and of itself, sufficient to single out the standard second order Einstein–Hilbert action from the infinite class of actions (of all orders) that one could possibly consider, with the issue of which particular action to use thus representing perhaps the only remaining open theoretical question in gravitational theory, at least at the classical level.

While the question of the correct relativistic theory has often been addressed in the past, it is remarkable how little attention has been given to the question of the correct non-relativistic limit, with it being taken almost as a given that the non-relativistic potential should be that of Newton. To the extent that modifications to the Newtonian potential have actually been sought from time to time, such studies have generally still taken the familiar $1/r$ potential to be the leading one at infinity. (Such proposed changes have also generally been ad hoc and not developed within a covariant gravitational formulation.) Since both the standard Newton theory and its Einstein generalization have been established through observations made predominantly on solar system size or smaller distance scales, it is not clear that one is immediately able to assume the validity of Newtonian gravity on larger distance scales, with the question of the leading gravitational potential term at infinity actually being an observational issue and not merely a theoretical one. Given the fact that the standard second order theory requires the presence of dark matter on both galactic and essentially every larger distance scale where the theory has been applied, while not seeming to require dark matter on any shorter ones, this need for dark matter could be considered as signaling an actual breakdown of the standard theory at the larger distances. Thus the question for observational astronomy is whether one can find another covariant theory of gravity which can differ from the standard one on large distance scales in a way which would resolve issues such as the galactic rotation curve issue (the most clearcut problem for the standard theory in the sense that it involves no dynamical assumptions or models, just data) without the need for dark matter, and yet at the same time still recover all of the familiar gravitational results on shorter distance scales.

As we shall see, all of these requirements and constraints are in fact achievable in the fourth order conformal invariant theory we consider in this paper, in a manner which will actually raise some basic questions regarding our general understanding of the standard relativistic theory and its familiar Newtonian limit. Specifically, we shall find that the conformal theory of gravity relates Newton's Law of Gravity to a fourth order rather than a second order interior Poisson equation. Since, as will become apparent, we are still able to recover all of the other standard features of non-relativistic Newtonian gravity in the weak field limit as well, we are therefore able to conclude that the second order Poisson equation (and hence its second order Einstein generalization) are not necessary ingredients for obtaining a viable non-relativistic phenomenology but only sufficient ones. We thus see that while the second order Poisson equation implies Newton's Law, it does not follow that Newton's Law implies the second order Poisson equation, with the data admitting of a much richer range of possibilities; and indeed, even if the conformal theory considered here were to fall by the wayside, the question of the uniqueness of the second order gravitational theory would still require addressing.

Motivated by a desire to have a theory of gravity which is based on a local invariance principle and dimensionless coupling constants (to thus both put gravity on the same footing as that enjoyed by the other fundamental interactions and resolve the freedom inherent in choosing the appropriate general covariant gravitational action), we have recently embarked [1–6] on a study of fourth order conformal gravity, and considered its viability as a candidate gravitational theory. (Fourth order conformal gravity is the theory which is based on invariance of the geometry under local conformal stretchings $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$ and which has $I_W = -\alpha \int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} = -2\alpha \int d^4x(-g)^{1/2}(R_{\lambda\mu}R^{\lambda\mu} - (R^{\alpha}{}_{\alpha})^2/3)$ as its only allowable gravitational action where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor and α is a purely dimensionless coefficient.) In particular, we have obtained [1] its complete and exact exterior vacuum solution in a static, spherically symmetric geometry. The exterior metric is found (up to an unobservable overall conformal factor) to be of the form

$$-g_{00} = 1/g_{rr} = B(r) = 1 - \beta(2 - 3\beta\gamma)/r - 3\beta\gamma + \gamma r - kr^2 \qquad (1)$$

where β, γ, and k are three appropriate dimensionful integration constants. This solution contains the familiar exterior Schwarzschild solution (thereby yielding the desired exterior Newtonian potential term) together with two additional potential terms. The kr^2 term represents a background de Sitter geometry (a vacuum solution in the present theory) which requires no cosmological term—indeed such a cosmological term is actually forbidden by the underlying conformal invariance of the theory—to thus provide a potential resolution of this longstanding problem [2]. The confining-type linear γr term is a new gravitational term and is a special feature of the fourth order theory. For small enough γ the conformal theory would thus appear to enjoy the static experimental successes of Einstein gravity all of which have been obtained on solar system or smaller distance scales. Further, it was originally suggested in [1] that the linear potential term might first manifest itself on galactic distance scales where it could then both compete with and be comparable to the galactic Newtonian term; it could thus, on the one hand, prevent galactic rotation curves from actually undergoing any Keplerian fall-off as a function of distance in the first place, while on the other, the balancing between the two potentials (one falling and one rising) could even produce a region of approximate flatness, to thus potentially bring rotation curves into agreement with observation without the apparent need for any dark matter. Moreover, this qualitative expectation has recently even been quantitatively borne out [6], with an acceptable fit to Begeman's NGC3198 rotation curve data [7] (so chosen because these (prototypical) data go out to the largest known number of surface brightness scale lengths) being found. The fit of [6] (which also provides comparable fitting to some other typical galaxies) is reproduced here as Figure 1, and it shows that at the very least the data will tolerate the presence of a linear potential in addition to the standard Newtonian one. Intriguingly, for the interplay between the two potentials to occur in Fig. 1, it was actually found that for this galaxy the parameter $1/\gamma$ should be of magnitude 3×10^{29} cm, a value which is of the order of the Hubble length. (For a discussion of some possible bounds on γ for laboratory sized objects see Wood and Nemiroff; Ref. 8.) Moreover, numerical simulation studies [9] have shown that this balancing of the linear and Newtonian potentials provides for the stability of galaxies without the need for any galactic dark matter halo, this being the other main galactic distance scale issue which the standard theory cannot apparently resolve without dark matter. With regard to cosmological implications of the theory, it has recently been shown [5] that the associated cosmology actually possesses no flatness problem (thus not requiring any inflationary epoch); and the fourth order theory is thus seen to be of some interest since it would appear to have the potential to eliminate the need for dark matter on both galactic and cosmological distance scales, and being a fully covariant theory, would thus warrant further consideration.

While the conformal theory can thus potentially address some outstanding issues of observational astrophysics and even recover the Schwarzschild limit on distance scales $r \ll 1/\gamma$ while never containing the Einstein equations at all, nonetheless many other questions and challenges remain for the theory before it could possibly replace the standard Einstein one. In this paper we address and resolve three crucial such concerns regarding the theory's non-relativistic limit, concerns now made somewhat urgent by the apparent quality and general structure of the fit of Fig. 1 which suggest that the fourth order theory may be more than simply a possible logical option. By studying not the exterior but rather the interior structure of the theory, first we shall explore the connection of the theory to Poisson theory; and by utilizing the dynamical mass generation mechanism now standard in treatments of the other fundamental interactions, we shall resolve two further questions which are raised by the fact that in a conformal invariant theory the matter energy-momentum tensor is kinematically traceless. Specifically, we first have to explain why there are non-zero particle masses at all, since the tracelessness condition immediately excludes kinematical or mechanical masses, and strict unbroken conformal symmetry requires no mass scales. However, if the scale symmetry is spontaneously broken it is then possible to generate mass scales, and interestingly, we find that this

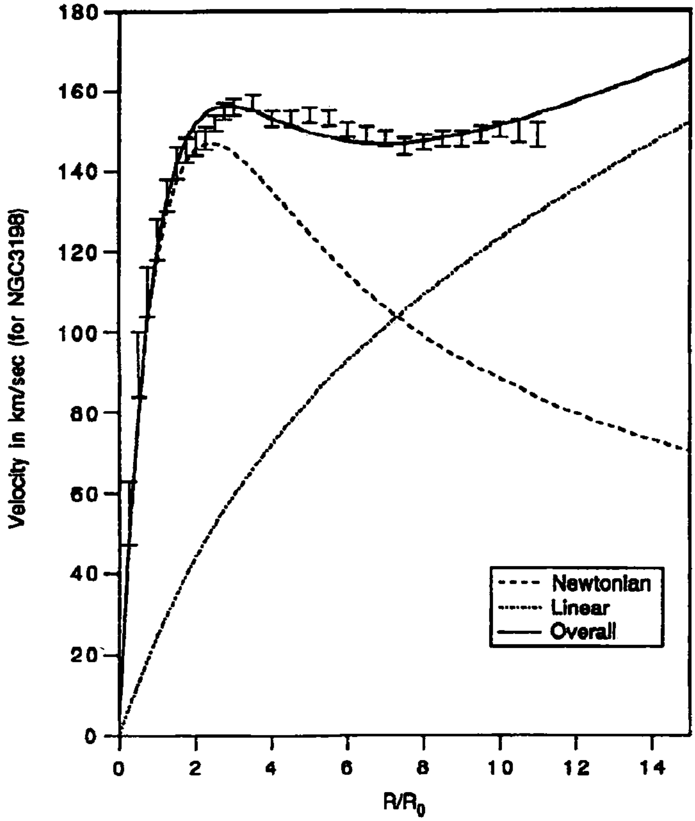

Figure 1. Comparison of the calculated rotational velocity curve for NGC3198 with Begeman's data shown with his quoted error bars. The velocity is plotted as a function of distance from the center of the galaxy with the distance expressed in units of the exponential scale length R_0 (=2.72 kpc) of eq. (36). The full curve shows the overall theoretical velocity prediction (in km/s) of [6] which integrates the Newtonian and linear potentials over the observed luminous matter distribution, while the two indicated dotted curves show the rotation curves that separate Newtonian and linear potentials would produce. No dark matter is assumed.

can be done while maintaining the tracelessness condition on the matter energy-momentum tensor. However, this in turn then creates a further problem since the familiar perfect fluid energy-momentum tensor of ordinary matter needed for the standard Euler hydrostatic theory is not in fact traceless; and one of the major results of this paper will be the deriving of the Euler equation even in the presence of a traceless energy-momentum tensor. In the process our analysis will actually challenge the relevance of the second order Poisson equation to gravitational dynamics as well as some longstanding notions regarding the nature of gravitational sources, namely that they should effectively be describable by mechanical, point delta functions and that they should make no asymptotic contributions to the geometry. As we shall see, this purely mechanical, Newtonian picture of matter is not compatible with the underlying conformal structure of the theory. Nor for that matter is it even mandated by observation. Indeed, in a conformal theory all mass should be dynamical and all particles should be extended and not pointlike. We believe the results we present in this paper not merely to be of relevance for the viability of the fourth order theory, but also to be of a very general nature and to constitute a quite strong challenge to the conventional gravitational wisdom independent of the correctness or otherwise of the fourth order theory itself.

The present paper is organized as follows. In Section 2 the fourth order Poisson equation is derived in the conformal theory as an exact, and not merely as a perturbative, property of the field equations of conformal gravity, and its exact interior and exterior solutions are presented. Additionally, the weak gravity limit of the conformal theory and the implications of the theory for the structure of gravitational sources are presented. Section 3 is devoted to a discussion of mass generation in theories with a traceless energy-momentum tensor and to a discussion of macroscopic perfect fluids within the framework of the fourth order theory. In Section 4 a derivation of the standard Newton–Euler hydrostatic equation is provided, a critical comparison between the second and fourth order Poisson equations is presented, and the constraints actually imposed on the general structure of gravitational theory by the currently available observational data are identified. In particular, it is shown that the observational evidence for the validity of the second order gravitational Poisson equation is actually not as definitive as it is commonly thought to be. The paper concludes with a review of the main outstanding open questions for the fourth order theory.

2. THE FOURTH ORDER POISSON EQUATION

To formulate the theory we first identify the functional derivative $(-g)^{-1/2}\delta I_{\rm W}/\delta g_{\mu\nu}$ as the specific gravitational rank two tensor $-2\alpha W^{\mu\nu}$ of the conformal theory (this then being the analog here of the Einstein tensor $G^{\mu\nu}$ of the standard theory), so that the gravitational equations of motion take the form [1]

$$
\begin{aligned}
W_{\mu\nu} = {} & \frac{1}{2}\, g_{\mu\nu}(R^{\alpha}{}_{\alpha})^{;\beta}{}_{;\beta} + R_{\mu\nu}{}^{;\beta}{}_{;\beta} - R_{\mu}{}^{\beta}{}_{;\nu;\beta} - R_{\nu}{}^{\beta}{}_{;\mu;\beta} \\
& - 2R_{\mu\beta}R_{\nu}{}^{\beta} + \frac{1}{2}\, g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3} g_{\mu\nu}(R^{\alpha}{}_{\alpha})^{;\beta}{}_{;\beta} \\
& + \frac{2}{3}\,(R^{\alpha}{}_{\alpha})_{;\mu;\nu} + \frac{2}{3}\, R^{\alpha}{}_{\alpha}R_{\mu\nu} - \frac{1}{6}\, g_{\mu\nu}(R^{\alpha}{}_{\alpha})^{2} = \left(\frac{1}{4\alpha}\right) T_{\mu\nu}
\end{aligned}
\tag{2}
$$

with $T^{\mu\nu}$ being the conformal energy-momentum tensor. Because of the conformal invariance of the theory both $W^{\mu\nu}$ and $T^{\mu\nu}$ are kinematically traceless thus giving the energy-momentum tensor a structure not possessed by a standard Newtonian point particle.

In order to study the geometry associated with a static, spherically symmetric matter distribution in the conformal theory, we found it very convenient in [1] to write the standard coordinate metric

$$ds^2 = b(\rho)dt^2 - a(\rho)d\rho^2 - \rho^2 d\Omega \tag{3}$$

in a slightly different form. We note first that under the general coordinate transformation

$$\rho = p(r), \qquad B(r) = \frac{r^2 b(r)}{p^2(r)}, \qquad A(r) = \frac{r^2 a(r) p'^2(r)}{p^2(r)} \tag{4}$$

with $p(r)$ so far arbitrary, the standard coordinate metric takes the form

$$ds^2 = \frac{p^2(r)}{r^2}\,[B(r)dt^2 - A(r)dr^2 - r^2 d\Omega]. \tag{5}$$

If we now fix $p(r)$ according to

$$-\frac{1}{p(r)} = \int \frac{dr}{r^2 (a(r)b(r)\,)^{1/2}} \tag{6}$$

we then find that the metric assumes the form

$$ds^2 = \frac{p^2(r)}{r^2}\left[B(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega\right] \tag{7}$$

and is just as general as it was, with the two metric functions in eq. (3) having been traded for two other functions in eq. (7). However, the resulting metric is now conformal to a standard metric in which $g_{rr} = -1/g_{00}$.

Since we are in a conformal theory, the gravitational rank two tensor $W^{\mu\nu}$ defined above in eq. (2) transforms conformally according to

$$W^{\mu\nu} \to \Omega^{-6}(x)W^{\mu\nu} \tag{8}$$

and thus we can evaluate its dependence on the metric coefficients simply by evaluating it in the geometry associated with the metric

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega. \tag{9}$$

Thus in the conformal theory all dynamical information is contained in the simple metric of eq. (9), so that while we have to deal with a much higher derivative theory than the Einstein one, the underlying conformal invariance of the theory enables us to sharply reduce the number of independent metric coefficients.

Because of the trace and Bianchi identities of the theory, in the static case of interest the gravitational rank two tensor $W^{\mu\nu}$ has only one independent component, W^{rr}, which for the line element of eq. (9) takes the form [1]

$$\begin{aligned}\frac{W^{rr}}{B(r)} = \frac{B'B'''}{6} - \frac{B''^2}{12} - \frac{BB''' - B'B''}{3r} \\ - \frac{BB'' + B'^2}{3r^2} + \frac{2BB'}{3r^3} - \frac{B^2}{3r^4} + \frac{1}{3r^4}.\end{aligned} \tag{10}$$

The complete and exact vacuum ($W^{rr} = 0$) solution to this theory is then given by the line element of eq. (1). With regard to the non-vacuum structure of the theory we note that, even though it is not independent of W^{rr}, the component W^{00} turns out to be very useful. Using the methods of [1] and [3] W^{00} is found to be of the form

$$\begin{aligned}W^{00} = -\frac{B''''}{3} + \frac{B''^2}{12B} - \frac{B'''B'}{6B} - \frac{B'''}{r} - \frac{B''B'}{3rB} \\ + \frac{B''}{3r^2} + \frac{B'^2}{3r^2B} - \frac{2B'}{3r^3} - \frac{1}{3r^4B} + \frac{B}{3r^4}.\end{aligned} \tag{11}$$

Combining the above two equations then leads to the remarkably compact relation

$$\frac{3(W^0{}_0 - W^r{}_r)}{B} = B'''' + \frac{4B'''}{r} = \frac{(rB)''''}{r} = \nabla^4 B. \tag{12}$$

We stress that eq. (12) is an exact relation and not a linearized perturbative one, even though some such form can be found in linearized approximations to fourth order theories. Additionally we note that in the standard second order Einstein theory no combination of any of the components of the Einstein tensor $G^{\mu\nu}$ yields the second order Laplacian as an exact expression. Rather, the Laplacian form is only obtained in the linearized version of that theory. Here in the conformal theory no recourse to perturbation theory is needed. [While the nice simple form of eq. (12) is obtained in the static, spherically-symmetric case, the gravitational rank two tensor $W^{\mu\nu}$ does not reduce to such a compact Laplacian (or d'Alambertian in general) form in an arbitrary geometry, and its structure even in the time dependent spherically-symmetric case is already far more complicated [3]. Nonetheless, just like the situation in the standard second order Einstein theory, for the standard weak gravity applications we only need explore the relativistic static, spherically-symmetric case for appropriately chosen fundamental sources (such as the spherically-symmetric stars in a disk shaped galaxy), take the ensuing non-relativistic weak gravity limit and add up the associated potentials in the standard linearized way. Thus all of the weak gravity implications of the fourth order theory are derivable from the dynamics associated with the fourth order Laplacian of eq. (12)].

In the presence of a static, spherically-symmetric source we define the convenient function

$$f(r) = 3(T^0{}_0 - T^r{}_r)/4\alpha B(r) \tag{13}$$

so that eq. (2) yields (after implementing the transformation of eq. (8) on the conformal energy-momentum tensor $T^{\mu\nu}$ as well) the compact relation

$$\nabla^4 B(r) = f(r) \tag{14}$$

which we recognize as a fourth order Poisson equation, with our exterior metric of eq. (1) immediately emerging as the solution to the associated fourth order Laplace equation. (While it will not prove necessary for us to specify the detailed dynamical structure of $f(r)$ in our treatment of eq. (14) below, we note in passing that for a standard perfect fluid the quantity $T^0{}_0 - T^r{}_r$ is given by $-(\rho+p)$ which for slow moving sources nicely reduces to the energy density ρ.) For a spherically-symmetric source of radius R eq. (14) can be integrated completely to yield

$$\begin{aligned} B(r) &= -\frac{1}{8\pi}\int d^3\mathbf{r}' f(r')\,|\mathbf{r}-\mathbf{r}'| \\ &= -\frac{1}{12r}\int dr' f(r') r' [\,|r'+r|^3 - |r'-r|^3] \end{aligned} \tag{15}$$

with the last equality following since the angular integration can be performed analytically. (There is also an uninteresting particular integral solution of the form $w - kr^2$ which we do not display.) Exterior to the source the solution then takes the form

$$B(r > R) = -\frac{1}{6}\int_0^R dr' f(r')\,[3r'^2 r + r'^4/r] \tag{16}$$

while in the interior the solution is given by

$$\begin{aligned} B(r < R) = &-\frac{1}{6}\int_0^r dr' f(r')\,[3r'^2 r + r'^4/r] \\ &-\frac{1}{6}\int_r^R dr' f(r')\,[3r'^3 + r^2 r']. \end{aligned} \tag{17}$$

Equations (16) and (17) are exact relations and represent the main new results of this work. We see that our solution thus recovers both the r and $1/r$ terms of our previous exterior solution of eq. (1) in the $r > R$ region as it should. Furthermore, we may identify the coefficients of these potential terms as system dependent moments of the source distribution according to

$$\beta(2 - 3\beta\gamma) = \frac{1}{6}\int_0^R dr' f(r') r'^4, \qquad \gamma = -\frac{1}{2}\int_0^R dr' f(r') r'^2. \tag{18}$$

(The solution of eq. (1) also entails a specific form for the parameter w of the particular integral solution in the exterior vacuum region outside of the source; while should there actually be any matter distribution at large distances outside the source, the second integral in eq. (17) would then generate effective constant and quadratic potential terms to thus dynamically determine the two particular integral terms in that case and make them of the order of magnitude of the scale of the matter distribution.) We thus establish the complete consistency of the interior and exterior structure, with the exterior solution of eqs. (1) and (16) immediately yielding us Newtonian and linear potential terms which can then be added source by source in the standard weak gravity manner to yield the associated non-relativistic predictions of the theory. In particular this will enable us (Sections 3 and 4 below) to recover the standard Newton–Euler phenomenology on distance scales where the linear potential term is negligible.

As a somewhat unexpected outcome we thus see that even though the Green's function of the fourth order theory is linear in the distance r, after integrating we find that we recover not only the linear potential term but also the $1/r$ Newtonian term in eq. (16) even though the second order Laplacian operator ∇^2 is not present anywhere in the fourth order theory. Thus, as we noted earlier, we confirm that while a second order Poisson equation is sufficient to generate a $1/r$ potential, it is not in fact necessary, with Newton's Law of Gravity obtaining in the fourth order theory as well.

While the solution of eq. (16) shows that the Newtonian term will be obtainable for any spherically-symmetric matter distribution, we note that its strength is related to the fourth moment of $f(r)$ rather than to the second one, the case in the familiar second order Einstein theory. Since this fourth moment would vanish for a delta function source, we see that in order to yield a Newtonian potential in the fourth order theory the source must be extended rather than pointlike. While this violates our standard second order intuition (but not any observational information incidentally),[3] it is not all that surprising since our experience with dynamical mass generation in the other fundamental interactions (which we recall motivated our choice of locally conformal invariant gravity in the first place) indicates that we should generally anyway expect elementary particles to actually be extended soliton or bag-like objects rather than pointlike ones, with the only new feature here being that curvature must now play a role in producing such structures.

With regard to our derivation of the exterior solution of eq. (16), we would like to emphasize that the Newtonian term emerges as the short distance limit of the theory rather than as its long distance one. The Newtonian piece is effectively an additional piece which is left over after the leading large distance contribution of the $-|\mathbf{r}-\mathbf{r}'|/8\pi$ Green's function of the ∇^4 operator is extracted out. It should then not come as too much of a surprise that the Newtonian term is intimately related to the internal structure of gravitational sources. The fact that the Newtonian potential is explicitly associated with the short distance behavior of the theory stands

[3] We are not aware of any experiment which has demonstrated the existence of Newtonian point particles, or more specifically which has actually shown that the only moment of the matter distribution which is sizeable is the second one. (The fourth moment which is central to the conformal theory according to eq. (16) plays no role in the second order theory, and hence its presupposed insignificance—if sources could be approximated by delta functions, that is—has never been tested). With regard additionally to the fact that the second moment integral of eq. (16) leads to a non-asymptotically flat geometry, we note (notwithstanding our experience with second order theories) that not only are we not aware of any observation which has shown that isolated sources yield an asymptotically flat geometry, we are not even sure how such an observation could be performed since the universe does not appear to be asymptotically flat.

in sharp contrast to the type of behavior generally sought in treatments of fourth order theories wherein attempts are made to reduce the fourth order equations of motion to those of Einstein gravity on long enough distance scales thereby realizing Newton's Law as a long distance aspect of the theory.[4] Having now established the presence of a fourth order Poisson equation in the conformal theory and shown its connection to Newton's Law, we turn next to a discussion of mass generation in the presence of conformal symmetry.[5]

3. MASS GENERATION AND PERFECT MATTER FLUIDS

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4 With regard to our fourth order solution we should also point out that it entails a conceptual departure from the standard connection between potentials and Green's functions familiar in second order theories. Specifically, for second order theories it is generally the case that the structure of the potential (the solution to the second order Poisson equation) is the same as that of the Green's function which is used to integrate the Poisson equation in the first place, and indeed our whole perturbative picture of exchange of particles as a mechanism for producing interactions between particles is based on this connection. With the fourth order theory we suddenly find that the Newtonian potential has nothing to do with an exchange of the quantum particle associated with a $1/r$ potential, but rather, it is a byproduct of exchanging the quantum particle associated with a linear potential between extended rather than pointlike sources. We thus see a clear demarcation between the Coulomb and Newtonian potentials while also exposing the limitations of the canonical perturbation theory picture of free propagators and point vertices.

5 We note in passing that it has been claimed in the literature, on the basis of perturbative studies, that fourth order theories do not in fact possess a good interior Newtonian limit [10–13]. Beyond the detailed non-perturbative analysis of this point which we make in this paper, we note that this claim was based on the unproven assertion that it is impossible to obtain the standard Newtonian term using a positive definite source $f(r)$ in the fourth order Poisson equation of eq. (14), an equation which also happens to appear in the perturbative treatments made by these authors, and not just in the exact treatment given here. To this end we note only that the particular positive definite source $f(r) = -2p\delta(r)/r^2 - (3q/2)[\nabla^2 - (r^2/12)\nabla^4][\delta(r)/r^2]$ provides a specific counterexample to the claim. The positivity of $f(r)$ may be exhibited by writing the q-dependent part of the source as the $\epsilon \to 0$ limit of $6q\epsilon(9r^4 - 3\epsilon^4 - 10\epsilon^2 r^2)/\pi(r^2+\epsilon^2)^5$, a limit in which $f(r)$ traps a singularity at the origin; with explicit calculation showing that in this limit the source does in fact yield the requisite Newtonian potential by providing contributions to the second and fourth moment integrals in eq. (18) according to $\beta(2 - 3\beta\gamma) = q$ and $\gamma = p$ (to thus incidentally show that the strengths of the linear and the Newtonian terms are in principle independent in the absence of more detailed dynamical information). While it remains to be seen to what degree this source may represent the interior of an elementary particle or possibly even the completely unknown singularity structure of an elementary particle within its own Schwarzschild radius (a region where it is not even known whether the energy density is in fact positive definite anyway - only the integral over the entire particle is known to be positive), its only purpose here is to show that the formal claim made by the

In order to discuss bulk matter fluids we need only consider a fermion field $\psi(x)$ to generically represent normal matter and a scalar field $S(x)$ to produce a symmetry breaking mass scale. For these fields the most general covariant, conformal invariant matter action is given by

$$I_{\mathrm{M}} = -\int d^4x(-g)^{1/2}[S^\mu S_\mu/2 + \lambda S^4 - S^2 R^\mu{}_\mu/12 + i\bar{\psi}\gamma^\mu(x)(\partial_\mu + \Gamma_\mu(x))\psi - hS\bar{\psi}\psi] \tag{19}$$

where $\Gamma_\mu(x)$ is the fermion spin connection and h and λ are dimensionless coupling constants. With this action the matter field equations of motion take the form

$$i\gamma^\mu(x)[\partial_\mu + \Gamma_\mu(x)]\psi - hS\psi = 0 \tag{20}$$

and

$$S^\mu{}_{;\mu} + SR^\mu{}_\mu/6 - 4\lambda S^3 + h\bar{\psi}\psi = 0 \tag{21}$$

while the conformal energy-momentum tensor is given by

$$T_{\mu\nu} = i\bar{\psi}\gamma_\mu(x)[\partial_\nu + \Gamma_\nu(x)]\psi + 2S_\mu S_\nu/3 - g_{\mu\nu}S^\alpha S_\alpha/6 - SS_{\mu;\nu}/3 + g_{\mu\nu}SS^\alpha{}_{;\alpha}/3 - S^2(R_{\mu\nu} - g_{\mu\nu}R^\alpha{}_\alpha/2)/6 - g_{\mu\nu}\lambda S^4. \tag{22}$$

Thus when the scalar field acquires a non-vanishing vacuum expectation value the fermion acquires a mass parameter $m = hS$.

Because of the conformal invariance of the theory, should the solutions to the theory yield a spacetime dependent value for $S(x)$, we are always able to remove this spacetime dependence by a conformal transformation and bring $S(x)$ to a space-time independent constant (i.e. the physics is in this constant being non-zero), so with no loss of generality we may set $S(x) = S_0$ in the above equations of motion so that $T_{\mu\nu}$ then simplifies to

$$T_{\mu\nu} = i\bar{\psi}\gamma_\mu(x)[\partial_\nu+\Gamma_\nu(x)]\psi - g_{\mu\nu}hS_0\bar{\psi}\psi/4 - S_0^2(R_{\mu\nu} - g_{\mu\nu}R^\alpha{}_\alpha/4)/6 \tag{23}$$

which makes its tracelessness manifest.

With this last form for the energy-momentum tensor it is now possible to establish the perfect fluid form needed for bulk matter. Specifically, if we momentarily restrict the theory to flat spacetime, the quantization of the theory is straightforward and yields one particle plane wave eigenstates

authors of [10–13] is technically incorrect.

$|k\rangle$ of four momentum $k^\mu = \{E_k = (\mathbf{k}^2 + m^2)^{1/2}, \mathbf{k}\}$ (the mass being given by $m = hS_0$). For these states we obtain [2] the matrix elements

$$\langle k| \int d^3x T_{00} |k\rangle = E_k - m^2/4E_k \tag{24}$$

$$\langle k| \int d^3x T^i{}_i |k\rangle = \mathbf{k}^2/E_k + 3m^2/4E_k = -\langle k| \int d^3x T^0{}_0 |k\rangle \tag{25}$$

so that the tracelessness property is still manifest. In eqs. (24) and (25) we note that the energy is reduced from the usual one particle E_k term by the vacuum contribution $-m^2/4E_k$ while the usual kinematic $\mathbf{k}^2/E_k$ pressure is augmented by a corresponding vacuum contribution. These vacuum contributions (which are akin to the Poincaré stresses of a completely electrodynamical electron) thus appear when the mass is dynamical and serve to maintain the tracelessness of $T_{\mu\nu}$, to thus show that the tracelessness of $T_{\mu\nu}$ does not in fact necessitate masslessness even while it does entail the vanishing of kinematic masses, to thus resolve the mass problem in conformal theories.

Following the general averaging prescription developed in Ref. [14] we find that an incoherent averaging over the above plane wave states then yields a perfect fluid, two in fact. One is a completely standard kinematic fluid coming from the averaging over the fermion $i\bar{\psi}\gamma_\mu(x)(\partial_\nu + \Gamma_\nu(x))\psi$ kinetic energy term, viz. the kinematic $T^{\rm kin}_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$ where

$$\rho = \frac{1}{\pi^2} \int dk k^2 E_k, \qquad p = \frac{1}{3\pi^2} \int dk \frac{k^4}{E_k}, \tag{26}$$

while the other one (which we denote by $T^{\rm sc}_{\mu\nu}$) is due to the self consistent $-g_{\mu\nu} hS_0 \bar{\psi}\psi/4$ term. Since $T^{\rm sc}_{\mu\nu}$ serves to maintain tracelessness in the fermion sector it must take the form $T^{\rm sc}_{\mu\nu} = -g_{\mu\nu}(3p - \rho)/4$ where, to repeat, ρ and p are the standard kinematic energy density and pressure of $T^{\rm kin}_{\mu\nu}$, so that the full $T^{\mu\nu}$ reduces to

$$T_{\mu\nu} = (\rho + p)\,(U_\mu U_\nu + g_{\mu\nu}/4) - S_0^2(R_{\mu\nu} - g_{\mu\nu} R^\alpha{}_\alpha/4)/6 \tag{27}$$

Even though an interplay between the two fluids is seen to be necessary to maintain the tracelessness of the energy-momentum tensor, it is crucial to note (see also Refs. 5,15) that no such interplay is actually needed in order to enforce its covariant conservation. Specifically, the fermion kinetic energy term contribution $T^{\rm kin}_{\mu\nu}$ is covariantly conserved all by itself in the $S = S_0$ gauge simply because of the fermion equation of motion.

(This can also be seen directly from eq. (22) when $S(x)$ is a constant since the Einstein tensor obeys the Bianchi identities.) Consequently, the kinematic ρ and p obey none other than the standard covariant conservation equations

$$(\rho + p)U^{\mu}{}_{;\nu}U^{\nu} + p_{\nu}(g^{\mu\nu} + U^{\mu}U^{\nu}) = 0 \tag{28}$$

and also, because of eq. (26), the standard equations of state, the presence of the spontaneous mass generation mechanism notwithstanding. Equations (26) and (28) are thus completely insensitive to whether mass is dynamical or kinematical, and the motions of particles in ordinary hydrostatic or hydrodynamic fluids are unaffected by the mechanism by which the particles acquire their masses in the first place. The self-consistent $T^{sc}_{\mu\nu}$ thus plays no role at all in ordinary non-relativistic physics, which of course is why its neglect in the literature has not created difficulty. The only place where $T^{sc}_{\mu\nu}$ does appear is in the vacuum energy, with gravity being the only interaction which is sensitive to its presence. Moreover, in the standard second order theory this same vacuum energy leads to a huge cosmological term, and it is a virtue of the conformal gravity theory that in it the cosmological term is in fact constrained and controlled in an acceptable way [2]. We thus establish the key and quite remarkable feature that not only is the full energy-momentum tensor covariantly conserved (as would be expected in a covariant theory), but in fact it turns out that two independent pieces of it are separately covariantly conserved also. We find that the standard kinematic non-traceless perfect fluid piece is covariantly conserved all on its own, and thus it does not exchange energy and momentum with the self consistent piece. It is the absence of any exchange of this type will which lead us to the standard Euler fluid equation, as we now see.

4. DISCUSSION AND GENERAL COMMENTS

In the standard Newtonian theory the second order Poisson equation is not a fundamental equation, but only a restatement of the Newtonian Law of Gravity in the presence of many sources. Specifically, to calculate the overall potential $\phi(\mathbf{r})$ of a set of sources in the non-relativistic weak gravity limit one can either sum up all of the Newtonian potentials directly one by one, viz.

$$\phi(\mathbf{r}) = \sum \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{29}$$

or alternatively, one can integrate up the second order Poisson equation

$$\nabla^2\phi(\mathbf{r}) = g(\mathbf{r}) \tag{30}$$

with appropriate source $g(\mathbf{r})$. The advantage of using the Poisson equation is that it directly exploits the symmetry of the problem, so that for a spherically-symmetric static source for instance, eq. (30) yields exterior and interior solutions

$$\phi(r > R) = -\frac{1}{r}\int_0^R dr' g(r') r'^2 \tag{31}$$

and

$$\phi(r < R) = -\frac{1}{r}\int_0^r dr' g(r') r'^2 - \int_r^R dr' g(r') r' \tag{32}$$

respectively.

Regarding these solutions several comments are in order. Firstly, eq. (31) shows that no matter how the spherically-symmetric source function $g(r)$ behaves as a function of the radial distance r, exterior to the source a test particle only sees a $1/r$ potential. It is not necessary that the source be a delta function in order for the Poisson equation of eq. (30) to yield an exterior $1/r$ potential (even though it is of course sufficient). Thus no matter how extended a source and no matter how convoluted a function of r the source function $g(r)$ is, the exterior potential is always Newtonian. Moreover, no matter how accurate any measurements of the exterior potential may be, as long as the potential $\phi(r)$ is only observed in the $r > R$ region, it is impossible to reconstruct $g(r)$ in the $r < R$ region. Observation thus never establishes a delta function nature for gravitational sources, and the issue is simply not addressable.

In the exterior region observation not only yields a $1/r$ potential, but also determines its coefficient, and observationally it is found in the weak binding limit that this coefficient grows as indicated in eq. (29), i.e. directly as the number of sources, so that the gravitational potential of a weakly bound macroscopic system is an extensive function of the number of fundamental microscopic constituents that it contains. Thus given eq. (29), the appropriate source to choose for eq. (30) is the number density of the sources. However, since the energy density of a macroscopic collection of weakly bound sources also grows with the number density, in the second order theory one can then take the source $g(r)$ to be the energy density, an unnecessary but acceptable replacement which unfortunately fixes in one's mind the idea that it is the energy density rather than the number density which is the key quantity (and all the more so in fact since in the second order Einstein theory it is the energy density rather than the number density which emerges as the source of the gravitational field).

As regards the interior solution of eq. (32), we should point out that despite its name it is in fact still an exterior solution, i.e. we are evaluating eq. (29) at points $r < R$ by summing over the exterior potentials due to the individual fundamental gravitational sources (typically nuclei or atoms) inside the material. We are thus interior to the bulk material but exterior to its individual fundamental sources. No knowledge of the gravitational potential inside of the individual protons, nuclei or atoms is required, and according to eqs. (29) and (32) the macroscopic potential is again proportional to the total number of fundamental microscopic sources in the weak binding limit. Now of course the Poisson equation is also assumed to hold within the fundamental sources as well and it would have dynamical consequences there. However, those consequences have never been explored either theoretically or experimentally since gravity is not the major dynamical force inside fundamental particles. Thus to sum up, the non-relativistic theory never appeals to anything other than the exterior Newtonian potential, and the second order Poisson equation as it has so far been used contains no additional dynamical information, it merely puts Newton's Law into a convenient form for weakly bound bulk matter, and as long as the sources are putting out $1/r$ potentials that is all that matters in the weak field limit, and we are not aware of any dynamical information regarding the internal structure of microscopic sources which has been obtained by gravitational means.

Returning now to the fourth order theory, we see directly that the gravitational potential put out by gravitational sources such as nuclei or atoms in the conformal case is given by eq. (1) or eq. (16) since that is the exact exterior solution of the theory. Moreover, this remains true no matter how complicated the source function $f(r)$ may be in the microscopic nuclear interior. Then for weakly bound bulk matter we may sum over the potentials of the individual nuclei to thus obtain none other than eq. (29) (whenever the linear potential term is insignificant that is),[6] with the weak gravity potential thus growing as the total number of nuclei just

[6] By the same reasoning we see that the parameter γ for macroscopic matter should also be an extensive function of the number of fundamental sources in a system in the weak binding and weak gravity limit. Since its magnitude for a weakly bound N-particle system of protons would then typically be of order $N\gamma_p$ where γ_p would be the appropriate parameter for an individual proton, a value for γ for a galaxy of order the inverse Hubble radius leads to a fantastically small value for γ_p of order 10^{-96} cm^{-1}. Such a value would then make the contribution of the linear potential to solar and terrestrial distance gravity completely insignificant, and much, much smaller than any bounds on the linear potential coming from the requirement that it not spoil the standard Newton–Einstein successes on those distances. For instance, the resulting value of order 10^{-39} cm^{-1} for the γ parameter of the Sun yields a potential for the planet Mercury of order 10^{-27}, while the Schwarzschild correction which produces the perihelion advance is of order 10^{-15}. (In passing we note that even if we were to take

as required by observation. Hence we see that even though the coefficient of the Newtonian term is related to the fourth moment of $f(r)$ in eq. (16), the coefficient is still linear in the number density of weakly bound sources, since in the macroscopic weak gravity limit we are simply insensitive to the structure of $f(r)$ inside of the fundamental microscopic sources. Now of course, in the interior of a nucleus we would have to deal directly with the explicit structure of the source function $f(r)$ within each of the individual protons within the nucleus, and also we would have to take into account the explicit two-body strong coupling nuclear forces between the protons in order to evaluate the fourth moment integral. Thus the weak binding limit approximation would simply not be relevant, and the fourth moment integral of $f(r)$ for a nucleus would not immediately be related to the number density of its nucleons. Similarly, for compact macroscopic sources we would again be unable to assume weak binding and would again need to evaluate the full eqs. (16) and (17) without approximation. For the moment however the strong binding case is intractable, just like its second order counterpart. For weak binding though, where there are data, one need not deal with the details of the fourth moment integral at all; all that is needed is to sum over the exterior eqs. (1) and (16), (i.e. for weakly bound bulk matter the potential is obtained by summing incoherently over all the microscopic protons (for simplicity) of the system with each one contributing a term $B(r) = 1 - 2\beta_p/r$ to the sum whenever all the γ dependent terms in eq. (1) may be ignored), with the theory thus leading to gravitational forces which grow as the total number of sources in precisely the kinematic regime where they are observationally found to do so.

Finally, as regards the interior gravitational potential of weakly bound bulk matter composed of fundamental static protons, we need only integrate the potential of eq. (1) [or equivalently that of eq. (16)] over a spherical distribution of weakly bound proton sources each of mass m_p and Schwarzschild radius $2\beta_p$. This then gives the standard weak gravity expression for the metric coefficient $B(r) = -g_{00}$, viz.

$$\frac{B'}{B} \simeq B' = \frac{2\beta_p}{r^2}\int_0^r 4\pi r^2 n(r) dr \qquad (33)$$

a value for γ for the Sun as large as the value we have suggested for a galaxy, the linear potential contribution to the perihelion would still not outperform the Schwarzschild term, and would actually fall within the current observational uncertainty.) With a γ of order 10^{-39} cm^{-1} for the Sun, the linear potential produces a solar surface gravity $\gamma R_\odot$ of order 10^{-28}, a value which is far smaller than the Newtonian surface gravity $\beta(2 - 3\beta\gamma)/R_\odot \sim 10^{-6}$, and so here also the linear term is not of any significance.

where $n(r)$ is the number density of protons. (Phenomenologically for a bulk source with N protons and total mass M, we may may identify Newton's constant G according to $2MG = 2N\beta_p$, i.e. according to $m_pG = \beta_p$, if we neglect the weak bulk source binding energy, that is.)[7] For static, spherically-symmetric bulk matter, the perfect fluid covariant conservation condition of eq. (28) yields

$$-2p'/(\rho + p) = B'/B \tag{34}$$

where we recall and stress again that the energy density and pressure are the kinematic ones of eq. (26). For weak binding and weak gravity then where p is the standard ram pressure and where the energy density required for eq. (26) may be taken to be given by $\rho = n(r)m_p$, we see that eqs. (33) and (34) reduce to the standard ones, so that the theory nicely recovers the standard Newtonian Euler hydrostatic theory [and in particular its Chandrasekhar limit which explicitly uses a Fermi sphere containing the modes of eq. (26)] just as required even while not possessing any second order Poisson equation at all, and even while the full energy-momentum tensor of the theory is traceless.

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[7] While we have introduced Newton's constant here in order to make contact with the conventional gravitational discussion, we note that G itself is not in fact observable gravitationally, with only the product MG being measurable. The standard Cavendish value for G thus represents only the choice of units to measure mass, and in this sense G is a derivative concept in the same way as Boltzmann's constant serves only to define temperature units in the observable product kT. Even without a fundamental G in our theory, gravity is still universal because the fundamental parameter α defined in the Weyl action I_W and present in the source function $f(r)$ of eq. (13) serves to couple gravity universally to gravitating matter in exactly the same way as the fine structure constant couples electromagnetism universally to electromagnetically participating matter. Through the coupling constant α gravity is universally coupled to matter for all gravitational field strengths, both strong and weak. The Cavendish-type experiments show only that gravitational effects are extensive in the number of particles and universal in the weak gravity limit, a fact that can be described with or without a fundamental G as we noted above. Since the Einstein theory then elevates G to a fundamental element of the gravitational action, it thereby insists that gravity is coupled universally through a fundamental G even for strong gravitational fields, an effect for which there is currently no experimental guidance. Since our theory couples gravity through α and not through G at all, its strong gravity predictions will thus in principle differ from those of the standard theory, though this issue would not appear to be explorable in the immediate future. (Given these remarks, it is now not at all clear as to whether the Planck length should play a central role in quantum gravity, with the expectation of a huge Planck density cosmological term perhaps no longer being warranted.)

It is also of interest to consider the weak gravity limit when the linear term is not negligible, such as in a galaxy, a system which provides an example of a non-spherically-symmetric distribution of stellar sources. With each star putting out a potential

$$V(r) = -\beta/r + \gamma r/2 \tag{35}$$

according to eqs. (1) and (16), we may integrate this potential over a thin exponential disk of stars with scale length R_0 and surface brightness

$$\Sigma(R) = \Sigma_0 e^{-R/R_0} \tag{36}$$

and $N = 2\pi\Sigma_0 R_0^2$ particles, to straightforwardly obtain [6] circular orbit rotational velocities of the form

$$v^2(r) = \frac{N\beta r^2}{2R_0^3}\left[I_0\left(\frac{r}{2R_0}\right)K_0\left(\frac{r}{2R_0}\right) + \left(\frac{\gamma R_0^2}{\beta} - 1\right)I_1\left(\frac{r}{2R_0}\right)K_1\left(\frac{r}{2R_0}\right)\right]. \tag{37}$$

Given the asympototic behaviour of the modified Bessel functions, we find that at infinity

$$rV'(r) \to \frac{N\beta}{r} + \frac{N\gamma r}{2} - \frac{3N\gamma R_0^2}{4r} \tag{38}$$

to thus nicely demonstrate the extensive property of both the Newtonian and the linear potentials in the weak gravity limit. (In passing we note that the fit of Fig. 1 is nothing more than a fit of eq. (37) as applied to the explicit luminous matter distribution of NGC3198 given in [16] and [17], and we refer the reader to Mannheim's paper [6] for further details.)

Now that we have established that the conformal theory does indeed have a good weak field limit, we can address our original question of what it is that the data actually mandate, and to do this we must discuss both the non-relativistic limit and its relativistic corrections. With regard first to the non-relativistic limit, we note first that the only data which unambiguously establish a Newtonian law at all are those solely on solar and smaller distance scales. There is no definitive evidence at all that the gravitational potential is a pure $1/r$ potential on galactic or larger distance scales, as the need for dark matter clearly indicates. Moreover, only if the potential can be shown to be strictly $1/r$ on all distance scales (something which has yet to be done observationally) would it then be possible to extract out a second order Poisson structure. If the potential shows some deviation from $1/r$ on large distance scales, some other equation would be needed. One might at first expect that in such a case the Poisson equation might be modified so that it would contain both ∇^2 and ∇^4 type terms and so on with appropriately chosen coefficients. The beauty of our discussion of the fourth order Poisson equation is that in fact no ∇^2 term is necessary at all, with the Newtonian term still being generated in the integration of eq. (14). In fact there is even a general moral here. If we take an even higher order Poisson equation we would generate even more terms in the potential, with ∇^6 for instance yielding r^3, r and $1/r$ terms on integration. The Newtonian potential is thus in principle divorced from the second order Poisson equation, and the issue of what order or combination of orders of the operator ∇^2 is needed depends on measurements made on all distance scales and not merely on those made on solar and smaller ones alone.

As we can see, there is still a great deal of observational ignorance in our ability to ascertain the gravitational potential on all distance scales, and this issue can only be resolved by observation. Since we do not yet know what order non-relativistic Poisson equation is mandated, finding the correct relativistic theory purely from observation is not yet possible. Both the second order and fourth order Poisson equations have clearcut and distinct relativistic generalizations, and yet both yield the same Schwarzschild relativistic corrections of eq. (1) on solar and smaller distance scales. Moreover, both of these two covariant generalizations automatically obey the equivalence principle, with geodesic motion in a background external field occurring in both the cases. In fact, as is discussed in detail in [15], geodesic motion has nothing to do with the detailed structure of the gravitational field equations, rather it follows strictly from covariance. Indeed, in the absence of any pressure, the covariant conservation equation for $T^{\mu\nu}$ given by eq. (28) is the geodesic equation. Thus while the gravitational field equations serve to fix the background gravitational field, the response of a test particle to the field is strictly geodesic. Since the conformal theory is fourth order, it is necessary to comment on why test particles actually obey second order geodesic equations of motion at all. The answer to this is that the same conformal invariance which requires the gravitational sector of the theory to uniquely be described by the fourth order Weyl action I_W actually requires all the other fundamental interactions to be second order as the conformal matter action of eq. (19) makes manifest. In fact conformal invariance actually provides a rationale for why the other fundamental interactions should in fact be second order in the first place, a fact for which there is actually no other known explanation at the present time.

The covariant conservation of the conformal invariant energy-momentum tensor is then automatically a second order rather than a fourth order condition on the motion of the matter fields. With the recognition that the other fundamental interactions are constrained by conformal invariance to be second order, we thus see a basic in principle difference between the electromagnetic and gravitational Poisson equations. For electromagnetism, the second order electrostatic Poisson equation follows from Gauss's Law, i.e. from the fundamental conformal invariant second order Maxwell equations of motion of the theory. The second order gravitational Poisson equation has no such similar rationale, being just a phenomenology until it is derived from some fundamental footing such as a covariant set of gravitational equations; and as we have seen, if one does not impose conformal invariance for the gravitational sector as well to make an unambiguous choice, there are potentially quite a few covariant theories (covariant generalizations of possibly all ∇^{2n}) that one could write down which could have a good Newtonian limit, and phenomenologically all of them would in principle at least seem to need exploring.

There are two key regimes where the second and fourth order theories do differ. One is strong gravity, though there are as of yet no calculations or clearcut data for that matter; the other, as indicated by eq. (1), is on large enough distance scales; and it is perhaps only with the detection of dark matter (in just the right amount) or with a demonstration of its absence that one might eventually be able to discriminate between the theories.[8] Since a linear potential would deviate from a Newtonian one more and more as we go to larger and larger distance scales, we would

[8] As we noted and discussed in our earlier papers, there are some other outstanding open questions for the conformal gravity theory which still need to be dealt with. At the classical level the theory still awaits a calculation which would enable it to address the crucial non-static gravitational radiation reaction question relevant to the decay of the orbit of the binary pulsar PSR 1913+16 (that there must be some orbit decay in the theory follows since every relativistically invariant theory automatically contains retardation—getting the precise numerical prediction of the theory however requires a highly detailed and sophisticated calculation). Additionally, at the quantum mechanical level the theory awaits a non-perturbative treatment of the ghosts and conformal anomalies which appear in a linearization of theory around flat spacetime, a limit which however, as can be seen from eq. (1), may not be all that relevant to the theory, with the restoring linear potential possibly even confining the ghosts and removing them from the physical spectrum in the true vacuum altogether. As regards the ghost question, we note additionally that in first order perturbation theory the fourth order theory yields a fourth order box operator, to thus yield corrections to the metric which grow with distance (just like the linear potential) and which hence become indefinitely large at large distances. Since the ghost states would appear on shell at low momenta, we see that their presence in the theory is inferred in precisely

expect the conformal theory to differ from the Einstein one not only on galactic distance scales, but essentially on every larger one as well. There is thus potentially a wealth of testing and of comparing with the standard theory to be done in gravitational lensing, dynamics of clusters of galaxies, large scale structure, large scale streaming and velocity flows, galaxy formation and fluctuations in the cosmic microwave background, and in cosmology and nucleosynthesis (as indicated in Ref. 5). This is of course a mammoth undertaking, which given the copious need for dark matter in the standard theory on all these scales, may eventually prove definitive. For the moment however, it appears to us that despite the fact that the fourth order theory possesses a linear potential, it nonetheless cannot be excluded on any currently known observational grounds.

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REFERENCES

1. Mannheim, P. D., and Kazanas, D. (1989). *Astrophys. J.* **342**, 635.
2. Mannheim, P. D. (1990). *Gen. Rel. Grav.* **22**, 289.
3. Kazanas, D., and Mannheim, P. D. (1991). *Astrophys. J. Suppl. Ser.* **76**, 431.
4. Mannheim, P. D., and Kazanas, D. (1991). *Phys. Rev. D***44**, 417.
5. Mannheim, P. D. (1992). *Astrophys. J.* **391**, 429.
6. Mannheim, P. D. (1993). *Astrophys. J.* **419**, 150.
7. Begeman, K. G. (1989). *Astron. Astrophys.* **223**, 47.
8. Wood, K. S., and Nemiroff, R. J. (1991). *Astrophys. J.* **369**, 54.
9. Christodoulou, D. M. (1991). *Astrophys. J.* **372**, 471.
10. Buchdahl, H. A. (1962). *Nuovo Cimento* **23**, 141.
11. Pechlaner, E., and Sexl, R. (1966). *Commun. Math. Phys.* **2**, 165.
12. Deser, S., Tsao, H.-S., and van Nieuwenhuizen, P. (1974). *Phys. Rev. D***10**, 3337.
13. Havas, P. (1977). *Gen. Rel. Grav.* **8**, 631.

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the kinematic region where first order perturbation theory becomes untrustworthy. (Nonetheless, the ghosts are still free to contribute at very short distances far off the mass shell, to thus maintain the power counting renormalizability of the fourth order theory.) Thus, if anything, the ghosts are signaling only that flat spacetime is not a good limit to the theory, something which would anyway be expected given the structure of the exact, non-perturbative solution of eq. (1).

14. Mannheim, P. D., and Kazanas, D. (1988). *Gen. Rel. Grav.* **20**, 201.
15. Mannheim, P. D. (1993). *Gen. Rel. Grav.* **25**, 697.
16. Wevers, B. M. H. R., van der Kruit, P. C., and Allen, R. J. (1986). *Astron. Astrophys. Suppl. Ser.* **66**, 505.
17. Kent, S. M. (1987). *Astronom. J.* **93**, 816.