REALIZABILITY OF COMBINATORIAL TYPES OF CONVEX POLYHEDRA OVER FIELDS

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It is shown that the minimal subfield of the field of real numbers over which all real combinatorial types of convex polyhedra can be realized is the field of all real algebraic numbers.

Let $S=\{S_0,S_1\}$ be a block-scheme, S_0 be a finite set, S_1 be a system of subsets of S_0; C a realization over $\mathbb{F}$ of the block-scheme S (where $\mathbb{F}$ is an ordered field) i.e., a polytope (convex polyhedron) in the space $\mathbb{F}^d$, such that the block-scheme formed by the set of its vertices and system of hyperfaces, as a system of subsets of the set of vertices, is isomorphic with S. A configuration α in the projective plane Π is called an ℓ-realization of the block-scheme S over the projective plane Π, if the block-scheme formed by the points of α and the lines of α, as subsets of the set of points, is isomorphic with S. The block-scheme of a polytope (configuration) will be called its combinatorial type. Polytopes (configurations) with given combinatorial type are called combinatorially equivalent. Some block-schemes are not combinatorial types. A given combinatorial type can have a C- (or ℓ-) realization over one field and not over another. M. Perles constructed a block-scheme of twelve points having a C-realization over $\mathbb{R}$ (for $d=8$) and not having a C-realization over $\mathbb{Q}$ (cf. [1]). In other words, he gave an example of a polytope with twelve vertices in $\mathbb{R}^8$, whose vertices cannot possibly be placed simultaneously at rational points (i.e., be realized in $\mathbb{Q}^8$) so as to preserve the combinatorial type.

A. M. Vershik suggested to the author proving the following improvement of this result:

<u>THEOREM.</u> The minimal subfield of the field $\mathbb{R}$ over which all combinatorial types of polytopes realizable over $\mathbb{R}$ are realizable is the field of all real algebraic numbers $\mathbb{A}$. In other words, 1) for any finite extension $\mathbb{F}$ of the field of rational numbers, there exists a polytope in $\mathbb{R}^d$, whose combinatorial type is not realizable over $\mathbb{F}$; 2) any combinatorial type of polytopes realizable over $\mathbb{R}$ is realizable over $\mathbb{A}$.

In its own right, the basic auxiliary assertion is Lemma 1 which was suggested by Vershik and which generalizes Perles' construction and refines the basic theorem of projective geometry.

In Sec. 1° we construct a series of block-schemes with the following property: they are ℓ-realizable over $\mathbb{R}$, but for each finite extension $\mathbb{K}\supset\mathbb{Q}$ one can find a block-scheme in this series which is not ℓ-realizable over $\mathbb{K}$. In Sec. 2, with the help of the Gale transformation we make the transition from configurations to polytopes, which gives a series of block-schemes with the same property but not for C-realizability. In Sec. 3 we prove

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that any block-scheme which is C-realizable over $\mathbb{R}$ is C-realizable over $\mathbb{A}$, which completes the proof of the theorem.

1°. The following lemma is a refinement of the fundamental theorem of projective geometry (cf. [4]) for projective spaces over finite extensions of $\mathbb{Q}$.

LEMMA 1. For any finite extension $\mathbb{F} \supset \mathbb{Q}$ there exists a block-scheme $S_{\mathbb{F}}$ such that $S_{\mathbb{F}}$ is ℓ-realizable in $P^2_{\mathbb{L}}$ if and only if $\mathbb{L} \supset \mathbb{F}$.

The collection of such schemes $\{S_{\mathbb{F}}\}$, where $\mathbb{F}$ runs through all subfields of $\mathbb{R}$, which are finite extensions of $\mathbb{Q}$, forms a lower series.

Proof. We shall call a block-scheme labeled if four points in general position, i.e., such that no triple of them lies in one block, are distinguished in it. By an ℓ-realization of a labeled block-scheme in the plane Π we mean any configuration in which the incidences indicated in the scheme hold and possibly some others, but the four distinguished points are in general position. To prove the lemma, we prove the

Proposition. For any polynomial with integer coefficients f, one can find a labeled block-scheme R_f such that R_f is realizable in $P^2_{\mathbb{L}}$, if and only if f has a root in $\mathbb{L}$.

To prove the proposition, we use the technique of geometric representation of algebraic equations developed in [2]. Namely, we fix in the abstract projective plane Π an ordered collection B of four points in general position. Any such collection ("basis") determines an algebraic structure of "natural field Π":

On the pencil of lines $\mathcal{L}$ incident with the first of the points of B, with the help of the construction there are defined operations of multiplication, addition, and their inverses. Lines of $\mathcal{L}$ which are incident with the three remaining points of the basis are given the symbols $0, I, \infty$. Now to each $f \in \mathbb{Z}[x]$ one can associate a construction $\mathcal{B}_f$ which assigns to an element X of the natural field an element $f(X)$, where f is understood formally as a fixed sequence of additions and multiplications. The given construction is a finite sequence of unions and intersections, starting from a system of generators consisting of B and a line X in $\mathcal{L}$. In the course of the construction there arises a labeled block-scheme T_f of points and lines. We add to the scheme T_f an incidence guaranteeing the coincidence of the line $f(X)$ and the line 0. The block-scheme R_f obtained is also, according to [2], the geometric representation of the equation $f(X)=0$.

R_f has the following properties which are important for us:

a) any realization of R_f as a labeled block-scheme is uniquely determined by a fixation of the points of B in general position and the line X in the distinguished pencil;

b) if R_f is realized, then the line corresponding to X in the natural field, defined by the points corresponding to B, is a root of the formal polynomial f in this natural field, and conversely;

c) if some basis B is fixed in the projective plane Π and the line X is a root of f as a formal polynomial in the corresponding natural field, then application of the algorithm $\mathcal{B}_f$ gives a realization R_f.

We shall show that R_f has the properties needed. If Π is a plane over a field, any natural field of it is isomorphic with the ground field [3]. Let R_f be realized so that the points of B are in general position. Then in the natural field defined by them, in accord with b) there is a root of the formal polynomial f. Consequently, if the ground field contains $\mathbb{Q}$, then by virtue of the isomorphism, there is a root of the polynomial f in it, as a polynomial $f\in\mathbb{Z}[x]$.

Converse. Suppose there is a root of $\mathbb{F}$ in f. We take some four points of $P^2_{\mathbb{F}}$ in general position, and in the corresponding natural field we take a line X which is the image of a root of f under the isomorphism. After this, by c) we get a realization R_f. The proposition is proved.

Now we take $\mathbb{F}\supset\mathbb{Q}$ to be a finite extension. Then $\mathbb{F}=\mathbb{Q}(\alpha)$, α being a primitive element of $\mathbb{F}$ over $\mathbb{Q}$. Let $p=\mathrm{Irr}(\alpha,\mathbb{Z})$. By hypothesis, R_p can be realized in $P^2_{\mathbb{L}}$ as a labeled block-scheme if and only if $\mathbb{L}\supset\mathbb{Q}(\beta)$, β being a root of p, but $\mathbb{Q}(\beta)=\mathbb{Q}(\alpha)=\mathbb{F}$. We shall show that any two realizations of the labeled block-scheme R_p are combinatorially equivalent, and consequently, are realizations in the usual sense of some block-scheme S_p, which also satisfies the conditions of the lemma. Let C_1,C_2 be two realizations of the labeled block-scheme R_p in planes $\Pi_1=P^2_{\mathbb{L}_1}$, $\Pi_2=P^2_{\mathbb{L}_2}$, $\mathbb{L}_1\mathbb{L}_2\supset\mathbb{Q}$; $B_1 B_2$ be the bases corresponding to C_1,C_2 for Π_1 and Π_2, and N_1,N_2 be the corresponding natural fields of Π_1,Π_2. Let α_1,α_2 be roots of the formal polynomial p in the natural fields N_1, N_2 defined by C_1, C_2 according to property b) of the proof of the proposition. Let F_1,F_2 be the imbeddings of $\mathbb{Q}(f)=\mathbb{F}$ in N_1, N_2 generated by α_1 and α_2. P_1, P_2 are the imbeddings of $P^2_{\mathbb{F}}$ in Π_1, Π_2 generated by these imbeddings. By construction, all points of C_1 and C_2 lie: those of C_1 in P_1, and those of C_2 in P_2. Since P is irreducible, there exists an isomorphism $\sigma:\mathbb{F}_1\to\mathbb{F}_2$ carrying α_1 into α_2. The triple B_1,B_2,σ determines a semilinear map $\Lambda:P_1\to P_2$, which is a collineation [4] and which carries B_1 into B_2, the line α_1 into the line α_2, and consequently [by property c) of the proof of the proposition] carries C_1 into C_2, as configurations in P_1, P_2. It is easy to verify that the correspondence generated by Λ between the points of C_1 and C_2 extends to an isomorphism of block-schemes of points and lines of C_1 into C_2 as configurations in Π_1 and Π_2.

2°. LEMMA 2. For any finite extension $\mathbb{F}\supset\mathbb{Q}$ ($\mathbb{F}$ being an ordered field), there exists a block-scheme $U_{\mathbb{F}}$ such that $U_{\mathbb{F}}C$ is realizable over $\mathbb{L}$ if and only if $\mathbb{L}\supset\mathbb{F}$.

Proof. By a g-realization of a block-scheme G is meant a Gale diagram in the spaces $\mathbb{K}^3$ ($\mathbb{K}$ being an ordered field) such that the block-scheme formed by the points, simplices containing $\mathbb{O}$ in the relative interior, and the incidences between them is isomorphic with G. Let a be an ℓ-realization of the block-scheme S in $P^2_{\mathbb{K}}$. We shall consider $P^2_{\mathbb{K}}$ as the projective space of lines incident with zero in $\mathbb{K}^3$. We note on the lines of $\mathbb{K}^3$ pairs of antipodal points of $\mathbb{K}^3$ corresponding to the vertices of a. The collection of points a^g in $\mathbb{K}^3$ so obtained will be considered as a Gale diagram. Let S^g be the combinatorial type of Gale diagrams containing a^g. If $S=S_{\mathbb{F}}$ ($S_{\mathbb{F}}$ is from the proof of Lemma 1), it is easy to verify that the ℓ-realizability of $S_{\mathbb{F}}$ over $\mathbb{L}$ is equivalent with the g-realizability of $S_{\mathbb{F}}^{g}$ over $\mathbb{L}$. Then by the standard passage from Gale diagrams to convex polytopes, we get a block-scheme $S_{\mathbb{F}}^{c}$ whose C-realizability over $\mathbb{L}$ is equivalent with the ℓ-realizability of $S_{\mathbb{F}}$ over $\mathbb{L}$. Now we set $U_{\mathbb{F}} = S_{\mathbb{F}}^{c}$. The $U_{\mathbb{F}}$ so constructed satisfy the condition of the lemma being proved.

3°. LEMMA 3. Any block-scheme S, which is C-realizable over $\mathbb{R}$, is C-realizable over $\mathbb{A}$.

Proof. Let S be a block-scheme, $S_0 = \overline{1:n}$ be the set of its elements. Let $V = \{x_i\}_{i \in S_0}$ $x_i \in \mathbb{R}^d$ be a collection of points of $\mathbb{R}^d$, indexed by S_0. Let x_{ij} be the j-th coordinate of the point x_i. The condition that V is the set of vertices of a polytope, which is a C-realization of S under the isomorphism $i \to x_i$, can be written in terms of the equality to zero and conditions on the signs of determinants composed of coordinates of points of V. This means that V is the set of vertices of a polytope which is a C-realization of S under the isomorphism $i \to x_i$ if and only if the matrix of coordinates of $V \{x_{ij}\}_{i \in S_0, j \in \overline{1:d}}$, as a point of $\mathbb{R}^{dn}$, lies in a certain semialgebraic variety $M(S)$. In this variety, if it is not empty, one can necessarily find points with algebraic coordinates.

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